ALTMAN’S CONTRACTORS AND FIXED POINTS OF MULTIVALUED MAPPINGS

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Let $P_i: D \subset X_1 \times \cdots \times X_n \to \text{CL}(Y_i)$ be multivalued mappings where $X_i$, $Y_i$ are Banach spaces and $\text{CL}(Y_i)$ is the set of all nonempty closed subsets of $Y_i$, $i = 1, \ldots, n$. We prove a theorem ensuring that $\theta \in P_i(x_1, \cdots, x_n)$ for some $(x_1, \cdots, x_n) \in D$ and deduce the fixed point theorems for multivalued mappings proved earlier by Czerwik, Nadler and Reich as corollaries. Besides, generalizations for multivalued mappings of the existence theorems proved by Altman using his theory of contractors are also obtained.

1. Introduction. In [2] we showed how the fixed point theorems of Altman [1] and Matkowski ([5], [6]) can be unified in the set-up of Banach spaces. The present paper studies further the relationship between Altman's theory of contractors and Matkowski's fixed point theorem and offers an existence theorem for the multivalued operator equation $\theta \in Px$ on subsets of a Banach space. We deduce as a corollary a comprehensive fixed point theorem proved by Czerwik [4] for multivalued mappings. Czerwik's fixed point theorem generalized the earlier fixed point theorems for set valued transformations on metric spaces obtained by Nadler [7], Covitz and Nadler [3] and Reich [9]. Apart from Czerwik's theorem, our main result obtains as corollaries generalizations to multivalued mappings of Altman's existence theorems and Matkowski's fixed point theorem. Section 3 gives the main result of the paper, while § 2 provides preliminaries basic to § 3.

2. Let $X$ be a Banach space. We employ the following notation of [7] and [8]:

\begin{align*}
(2.1) \quad & \text{CL}(X) = \{C : C \text{ is a nonempty closed subset of } X\} , \\
(2.2) \quad & N(\epsilon, C) = \{x \in X : \|x - c\| < \epsilon \text{ for some } c \in C\} , \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \epsilon > 0 , \quad C \in \text{CL}(X) . \\
(2.3) \quad & H(A, B) = \begin{cases} 
\inf \epsilon > 0 , & A \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A) , \\
\infty , & \text{if the infimum exists} , \\
\infty , & \text{otherwise} , \\
A, B \in \text{CL}(X) . 
\end{cases}
\end{align*}

The function $H$ is called the generalized Hausdorff distance for
CL (X) induced by the norm of X.

(2.4) \[ D(x, A) = \inf \{ \| x - a \| : a \in A \} . \]

The lemma given below is well-known and is used in § 3.

**Lemma 2.1.** Suppose \( A, B \in \text{CL} (X) \) and \( a \in A \). Then, for \( q > 0 \), there exists an element \( b \in B \) such that

\[ d(a, b) \leq H(A, B) + q . \]

A point \( x \in X \) is said to be a fixed point for the multivalued mapping \( f : X \to \text{CL} (X) \) if \( x \in f(x) \).

We follow the notation of [6].

Let \( (a_{ik}) \) be an \( n \times n \) nonnegative matrix. Define

(2.5) \[
a_{ik}^l = \begin{cases} a_{ik} \quad & i \neq k , \\ 1 - a_{ik} \quad & i = k , \\ \end{cases} \quad i, k = 1, \ldots, n .
\]

(2.6) \[
a_{ik}^{l+1} = \begin{cases} a_{i+1}^l a_{i+k+1}^l + a_{i+1}^l a_{i+k+1}^l \quad & i \neq k \\ a_{i+1}^l a_{i+k+1}^l - a_{i+1}^l a_{i+k+1}^l \quad & i = k \\ \end{cases} \quad l = 1, \ldots, n - 1 ; \quad i, k = 1, \ldots, n - l
\]

Matkowski ([5], [6]) proved the following

**Lemma 2.2.** Let \( a_{ik}^l > 0 \), \( i, k = 1, \ldots, n \). The system of inequalities

(2.7) \[
\sum_{k=1}^{n} a_{ik}^l r_k < r_i , \quad i = 1, \ldots, n ,
\]

has a solution \( r_i > 0 \), \( i = 1, \ldots, n \) if and only if the following inequalities hold:

(2.8) \[
a_{il}^l > 0 , \quad l = 1, \ldots, n ; \quad i = 1, \ldots, n + 1 - l .
\]

Using this lemma he obtained the following fixed point theorem.
(Actually Matkowski proved this theorem in the setting of complete metric spaces.)

**Theorem 2.1.** Let \( X_i \) be Banach spaces and \( T_i : X_i \times \cdots \times X_n \to X_i \), \( i = 1, \ldots, n \) be mappings such that

(2.9) \[
\| T_i(x_1, \ldots, x_n) - T_i(y_1, \ldots, y_n) \| \leq \sum_{k=1}^{n} a_{ik}^l \| x_k - y_k \| , \quad i = 1, \ldots, n , \quad x_k, y_k \in X_k , \quad k = 1, \ldots, n ,
\]

where \( a_{ik}^l > 0 \), \( i, k = 1, \ldots, n \). If the numbers \( a_{ik}^l \), \( l = 1, \ldots, n \);
Considerations of some of the fundamental problems of numerical analysis and operator theory led Altman [1] to the concept of contractors.

**Definition 2.1** [p. 8, [1]]. Let $P: D(P) \subset X \to Y$, $D(P)$ being the domain of $P$, be a nonlinear operator from a Banach space $X$ to a Banach space $Y$ and $\Gamma(x): Y \to X$ be a bounded linear operator associated with $x \in X$. We say that $P$ has a contractor $\Gamma(x)$ if there is a positive number $q < 1$ such that

\[
|P(s + \Gamma(x)y) - p_x - y| \leq q\|y\|,
\]

where $x \in D(P)$, $y \in Y$ are defined by the particular problem.

**Definition 2.2** [p. 10, [1]]. A contractor $\Gamma(x)$ is called regular if (2.10) is satisfied for all $y \in Y$ and $D(P) = \Gamma(x)(Y)$.

**Definition 2.3** [p. 6, [1]]. An operator $P: D(P) \subset X \to Y$ is closed on $D(P)$ if $x_n \in D(P)$, $x_n \to x$ and $Px_n \to y$ imply $x \in D(P)$ and $Px = y$.

Altman proved the following theorem:

**Theorem 2.2** [p. 13, Theorem 5.1, [1]]. Suppose that the closed nonlinear operator $P: D(P) \subset X \to Y$ has a bounded contractor $\Gamma$ such that

\[
\begin{align*}
(2.11) \quad & x + \Gamma(x)y \in D(P), \quad \text{whenever} \quad x \in D(P), \quad y \in Y; \\
(2.11) \quad & \|P(x + \Gamma(x)y) - Px - y\| \leq q\|y\|, \quad y \in Y, \quad 0 < q < 1; \\
(2.12) \quad & \|\Gamma(x)\| \leq B \quad \text{for all} \quad x \in D(P).
\end{align*}
\]

Then the equation $Px = y$ has a solution for $y \in Y$. When $\Gamma$ is regular, the assumption (2.11)(a) is readily satisfied and further, the solution is unique.
3. We generalize the notion of a closed operator to multivalued mappings as follows:

**DEFINITION 3.1.** A nonlinear multivalued operator \( P: D(P) \subseteq X \rightarrow \text{CL}(Y) \), \( X, Y \) being Banach spaces, is closed on \( D(P) \), if \( x_n \rightarrow x \), \( y_n \in P x_n \) and \( y_n \rightarrow y \) imply that \( x \in D(P) \) and \( y \in P x \).

Let \( X_i, Y_i, i = 1, \ldots, n \) be Banach spaces and \( b_{ik}, c_{ik} \geq 0 \), \( i, k = 1, \ldots, n \). Let \( a_{ik} = b_{ik} + c_{ik} \), \( i, k = 1, \ldots, n \), be positive and the numbers \( a_{ik} \) defined by (2.5) and (2.6) fulfill (2.8). Then, by Lemma 2.2, there exists a positive solution \( r_1, \ldots, r_n \) of the system of inequalities (2.7). We define

\[
q = \max_i \left( r_i^{-1} \sum_{k=1}^n a_{ik} r_k \right).
\]

Clearly, \( 0 < q < 1 \), and

\[
\sum_{k=1}^n a_{ik} r_k \leq q r_i, \quad i = 1, \ldots, n.
\]

**THEOREM 3.1.** Suppose that the closed nonlinear transformations \( P_i: D \subseteq X_i \times \cdots \times X_n \rightarrow \text{CL}(Y_i), i = 1, \ldots, n \) fulfill the following:

- There exist bounded linear operators \( \Gamma_i(x_i): Y_i \rightarrow X_i, x_i \in X_i \), \( i = 1, \ldots, n \) such that
  \[
  \| \Gamma_i(x_i) \| \leq B, \quad (x_1, \ldots, x_n) \in D, \quad i = 1, \ldots, n;
  \]

- (a) \( (x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) \in D \) whenever \( (x_1, \ldots, x_n) \in D \) and \( y_i \in Y_i, \quad i = 1, \ldots, n \);

- (b) \[
  H_i[P_i(x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n), P_i(x_1, \ldots, x_n) + y_i] \\
  \leq \sum_{k=1}^n b_{ik} \| y_k \| + \sum_{k=1}^n c_{ik} D_k[y_k, y_k - P_k(x_1, \ldots, x_n)] \\
  + c D[y_i, y_i] \\
  - \Gamma_i(x_i)P_i(x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) \\
  \text{for} \quad (x_1, \ldots, x_n) \in D, \quad y_i \in Y_i, \quad i = 1, \ldots, n,
\]

where \( a_{ik} = b_{ik} + c_{ik}, \quad i, k = 1, \ldots, n \) are positive and the numbers \( a_{ik} \) defined by (2.5) and (2.6) satisfy (2.8);

\[
c \text{ is a constant such that } 0 \leq cB < 1 - q.
\]

Then there exists \( (x_1, \ldots, x_n) \in D \) such that \( \theta_i \in P_i(x_1, \ldots, x_n), \theta_i = 1, \ldots, n \), where \( \theta_i \) is the zero element of the Banach space \( Y_i, i = 1, \ldots, n \).

**Proof.** Let \( (\hat{x}_1, \ldots, \hat{x}_n) \in D \) be an arbitrary element. Choose
\( \hat{y}_i \in P_i(\hat{x}_1, \ldots, \hat{x}_n) \), \( i = 1, \ldots, n \). We can assume without loss of generality that \( \| \hat{y} \| \leq r_i \), \( r_i \geq 1 \), \( i = 1, \ldots, n \), since the set of solutions to the system (2.7) is closed with respect to multiplication by positive scalars. Define

\[
(3.6) \quad x_i^1 = \hat{x}_i - \Gamma_i(\hat{x}_i)\hat{y}_i, \quad i = 1, \ldots, n.
\]

Replacing \( x_i \) by \( \hat{x}_i \) and \( y_i \) by \( -\hat{y}_i \) in (3.4)(b) and using (3.6) we get,

\[
(3.7) \quad H_i[P_i(x_1^1, \ldots, x_n^1), P_i(\hat{x}_1, \ldots, \hat{x}_n) - \hat{y}_i]
\]
\[
\leq \sum_{k=1}^n b_{ik}\|\hat{y}_k\| + \sum_{k=1}^n c_{ik}D_k[-\hat{y}_k, -\hat{y}_k - P_k(\hat{x}_1, \ldots, \hat{x}_n)]
\]
\[
+ cD_i[-\hat{y}_i, -\hat{y}_i - \Gamma_i(\hat{x}_i)P_i(x_1^1, \ldots, x_n^1)]
\]
\[
\leq \sum_{k=1}^n b_{ik}\|\hat{y}_k\| + \sum_{k=1}^n c_{ik}\|\hat{y}_k\| + cD_i[\theta_i, \Gamma_i(\hat{x}_i)P_i(x_1^1, \ldots, x_n^1)].
\]

As \( \hat{y}_i \in P_i(\hat{x}_1, \ldots, \hat{x}_n) \), \( \theta_i \in P_i(\hat{x}_1, \ldots, \hat{x}_n) - \hat{y}_i \), \( i = 1, \ldots, n \). So, for \( q > 0 \), by Lemma 2.1, there exists an element \( y_i^1 \in P_i(x_1^1, \ldots, x_n^1) \), \( i = 1, \ldots, n \) such that

\[
(3.8) \quad \| y_i^1 - \theta_i \| \leq H_i[P_i(x_1^1, \ldots, x_n^1), P_i(\hat{x}_1, \ldots, \hat{x}_n) - \hat{y}_i] + q.
\]

From (3.7) and (3.8) we have

\[
\| y_i^1 \| \leq \sum_{k=1}^n (b_{ik} + c_{ik})\|\hat{y}_k\| + c\|\Gamma_i(x_i)y_i^1\| + q
\]
\[
\leq \sum_{k=1}^n a_{ik}\|\hat{y}_k\| + cB\|y_i^1\| + q, \quad \text{by (3.3)}
\]

\[
(1 - cB)\|y_i^1\| \leq \sum_{k=1}^n a_{ik}r_k + q, \quad \text{by our assumption}
\]
\[
\leq qr_i + q, \quad \text{by (3.2)}
\]
\[
\leq 2qr_i, \quad \text{as} \quad r_i \geq 1, \quad i = 1, \ldots, n \quad \text{and} \quad 0 < q < 1.
\]

Hence

\[
(3.9) \quad \| y_i^1\| \leq \frac{2q}{(1 - cB)}r_i, \quad i = 1, \ldots, n.
\]

We shall now construct inductively sequences \( \{x_i^m\} \) and \( \{y_i^m\} \) \( i = 1, \ldots, n \) such that

\[
(3.10) \quad \text{(a)} \quad (x_1^m, \ldots, x_n^m) \in D, \quad y_i^m \in P_i(x_1^m, \ldots, x_n^m)
\]

\[
(3.10) \quad \text{(b)} \quad \| y_i^m\| \leq (m + 1)\left(\frac{q}{1 - cB}\right)^m r_i, \quad i = 1, \ldots, n.
\]

For \( m = 1 \), the above hypotheses are true, in view of (3.4)(a), (3.6) and (3.9). Assume the truth of (3.10)(a), (3.10)(b) for \( m - 1 \in N \), i.e.,
(3.11) (a) \( (x_i^{m-1}, \ldots, x_n^{m-1}) \in D \), \( y_i^{m-1} \in P_i(x_i^{m-1}, \ldots, x_n^{m-1}) \),

(3.11) (b) \( \| y_i^{m-1} \| \leq m \left( \frac{q}{1 - cB} \right)^{m-1} r_i \), \( i = 1, \ldots, n \).

Define

(3.12) \( x_i^m = x_i^{m-1} - \Gamma_i(x_i^m) y_i^{m-1}, \ i = 1, \ldots, n \).

In (3.4)(b), replacing \( x_i \) by \( x_i^{m-1} \), \( y_i \) by \( -y_i^{m-1} \) and using (3.12) we have

(3.13) \( H_i[P_i(x_i^m, \ldots, x_n^m), P_i(x_i^{m-1}, \ldots, x_n^{m-1}) - y_i^{m-1}] \)

\[ \leq \sum_{k=1}^n b_{ik} \| y_k^{m-1} \| + \sum_{k=1}^n c_{ik} D_k[-y_k^{m-1}, -y_k^{m-1} - P_k(x_1^{m-1}, \ldots, x_n^{m-1})] \]

\[ + cD_i[-y_i^{m-1}, -y_i^{m-1} - \Gamma_i(x_i^{m-1}) P_i(x_1^{m}, \ldots, x_n^m)] \]

\[ \leq \sum_{k=1}^n b_{ik} \| y_k^{m-1} \| + \sum_{k=1}^n c_{ik} \| y_k^{m-1} \| \]

\[ + cD_i[\theta_i, \Gamma_i(x_i^{m-1}) P_i(x_1^m, \ldots, x_n^m)] . \]

Since \( \theta_i \in P_i(x_i^{m-1}, \ldots, x_n^{m-1}) - y_i^{m-1} \), given \( q/(1 - cB)^{m-1} > 0 \), there exists \( y_i^m \in P_i(x_i^m, \ldots, x_n^m) \) such that

(3.14) \( \| y_i^m - \theta_i \| \leq H_i[P_i(x_i^m, \ldots, x_n^m), P_i(x_1^{m-1}, \ldots, x_n^{m-1}) - y_i^{m-1}] \)

\[ + \frac{q^m}{(1 - cB)^{m-1}} . \]

From (3.13) and (3.14) we get

\[ \| y_i^m \| \leq \sum_{k=1}^n (b_{ik} + c_{ik}) \| y_k^{m-1} \| + cB \| y_i^m \| + \frac{q^m}{(1 - cB)^{m-1}} \]

\[ (1 - cB)^{m-1} \| y_i^m \| \leq m \left( \frac{q}{1 - cB} \right)^{m-1} \sum_{k=1}^n a_{ik} r_k + \frac{q^m}{(1 - cB)^{m-1}} \]

\[ \leq m r_i \left( \frac{q}{1 - cB} \right)^{m-1} + \frac{q^m}{(1 - cB)^{m-1}} , \text{ by (3.2)} \]

i.e.,

\[ \| y_i^m \| \leq m \left( \frac{q}{1 - cB} \right)^m r_i + \left( \frac{q}{1 - cB} \right)^m r_i \]

\[ \leq (m + 1) \left( \frac{q}{1 - cB} \right)^m r_i , \text{ as } r_i \geq 1 , \ i = 1, \ldots, n . \]

Hence by induction (3.10)(b) holds for all \( m = 0, 1, 2, \ldots \). From (3.12), and (3.4)(a), it follows that \( (x_i^m, \ldots, x_n^m) \in D \). By construction, \( y_i^m \in P_i(x_i^m, \ldots, x_n^m) \). Hence by induction (3.10)(a) holds. By (3.5), \( 0 \leq q/(1 - cB) < 1 \) and it follows from (3.10)(a) that \( y_i^m \to \theta_i \), as \( m \to \infty , \ i = 1, \ldots, n \). From (3.12),
\[ \| x_i^{m+1} - x_i^m \| \leq \| \Gamma_i(x_i^m)y_i^m \| \leq B(m + 1)\left(\frac{q}{1 - cB}\right)^m r_i. \]

Hence \( \{x_i^m\} \) is a Cauchy sequence in \( X_i, \ i = 1, \ldots, n \). Therefore, \( x_i^m \rightarrow x_i, \ i = 1, \ldots, n \). As the operator \( P_i \) is closed, \( y_i^m \in P_i(x_i^m, \ldots, x_n^m) \), \( y_i^m \rightarrow \theta_i, \ x_i^m \rightarrow x_i, \ i = 1, \ldots, n \), imply that \( (\theta_1, \ldots, \theta_n) \in D \) and \( \theta_i \in P(x_i, \ldots, x_n), \ i = 1, \ldots, n. \)

We now deduce Czerwik's Theorem [4] as a corollary in the set-up of Banach spaces. Czerwik proved his theorem for multivalued mappings on complete metric spaces.

**Theorem 3.2** [Theorem, [4]]. Let \( X_i, \ i = 1, \ldots, n \) be Banach spaces and \( b_{ik}, c_{ik} \geq 0 \) for \( i, k = 1, \ldots, n \). Let \( a_{ik} = b_{ik} + c_{ik}, \ i, k = 1, \ldots, n \) be positive and let the numbers \( a_{ik} \) defined by (2.5) and (2.6) fulfill (2.8). Suppose that the transformations \( F_i: X_1 \times \cdots \times X_n \rightarrow \text{CL}(X_i), \ i = 1, \ldots, n \) fulfill

\[ H_i[F_i(x_1, \ldots, x_n), F_i(z_1, \ldots, z_n)] \leq \sum_{k=1}^n b_{ik} \| x_k - z_k \| + \sum_{k=1}^n c_{ik}D_h[x_k, F_h(x_1, \ldots, x_n)] + cD_i[z_i, F_i(z_1, \ldots, z_n)], \quad i = 1, \ldots, n; \]

for \( x_j, z_j \in X_j, \ j = 1, \ldots, n \), where \( c \) fulfills the condition \( 0 \leq c < 1 - q, \ q \) being defined by

\[ q = \max_i \left( r_i^{-1} \sum_{k=1}^n a_{ik}r_k \right). \]

Then the system \( (F_1, \ldots, F_n) \) has a fixed point, i.e., there exist points \( x_i \in X_i, \ i = 1, \ldots, n \) such that \( x_i \in F_i(x_1, \ldots, x_n), \ i = 1, \ldots, n. \)

**Proof.** That the above theorem follows as a corollary for the Theorem 3.1 can be seen if we put \( \Gamma_i(x_i) = I, \ (x_i \in X_i) \), the identity operator on \( X_i \) and \( P_i(x_1, \ldots, x_n) = x_i - F_i(x_1, \ldots, x_n) \ i = 1, \ldots, n \) in (3.4)(b) and observe that it reduces to

\[ H_i[x_i + y_i - F_i(x_1 + y_1, \ldots, x_n + y_n), x_i + y_i - F_i(x_1, \ldots, x_n)] \leq \sum_{k=1}^n b_{ik} \| y_k \| + \sum_{k=1}^n c_{ik}D_h[y_k, y_k - x_k + F_h(x_1, \ldots, x_n)] + cD_i[x_i + y_i, x_i + y_i - (x_i + y_i) + F_i(x_1 + y_1, \ldots, x_n + y_n)] \]

i.e.,

\[ H_i[F_i(x_1 + y_1, \ldots, x_n + y_n), F_i(x_1, \ldots, x_n)] \leq \sum_{k=1}^n b_{ik} \| y_k \| + \sum_{k=1}^n c_{ik}[y_k, y_k - x_k + F_h(x_1, \ldots, x_n)] + cD_i[x_i + y_i, F_i(x_1 + y_1, \ldots, x_n + y_n)]. \]
Taking \( x_i + y_i = z_i, \ i = 1, \ldots, n \), we have

\[
H_i[F_i(x_i, \ldots, x_n), F_i(z_i, \ldots, z_n)] 
\leq \sum_{k=1}^{n} b_{ik} \| x_k - z_k \| + \sum_{k=1}^{n} c_{ik} D_k[x_k, F_k(x_i, \ldots, x_n)] 
+ cD_i[z_i, F_i(z_i, \ldots, z_n)],
\]

which is nothing but condition (3.15). It can be similarly shown that (3.15) implies (3.4)(b) in this case. To prove that the operator \( F_i \), \( i = 1, \ldots, n \) is closed in the sense of Definition 3.1, observe that we have shown in the proof of Theorem 3.1 that \( x_i^{m} \rightarrow x_i \) and \( y_i^{m} \in P_i(x_1^{m}, \ldots, x_n^{m}) \), i.e., \( y_i^{m} \in x_i^{m} - F_i(x_1^{m}, \ldots, x_n^{m}) \) \( i = 1, \ldots, n \), and \( y_i^{m} \rightarrow 0 \) as \( m \rightarrow \infty \), i.e., \( x_i^{m} - y_i^{m} \rightarrow x_i, \ i = 1, \ldots, n \). It remains to show that \( x_i \in F_i(x_i, \ldots, x_n), \ i = 1, \ldots, n \).

\[
D_i[x_i, F_i(x_i, \ldots, x_n)] 
\leq \| x_i - x_i^{m} \| + D_i[x_i^{m}, F_i(x_i, \ldots, x_n)] 
\leq \| x_i - x_i^{m} \| + H_i[F_i(x_1^{m-1}, \ldots, x_n^{m-1}), F_i(x_i, \ldots, x_n)] 
\leq \| x_i - x_i^{m} \| + \sum_{k=1}^{n} c_{ik} D_k[x_k^{m-1}, F_k(x_1^{m-1}, \ldots, x_n^{m-1})] + cD_i[x_i, F_i(x_i, \ldots, x_n)] 
\leq \| x_i - x_i^{m} \| + \sum_{k=1}^{n} c_{ik} \| x_k^{m-1} - x_k \| 
+ \sum_{k=1}^{n} c_{ik} \| x_k^{m-1} - x_k \| + cD_i[x_i, F_i(x_i, \ldots, x_n)].
\]

As \( 0 < \epsilon < 1 \) and

\[
D_i[x_i, F_i(x_i, \ldots, x_n)] 
\leq \frac{1}{1 - \epsilon} \left[ \| x_i - x_i^{m} \| + \sum_{k=1}^{n} b_{ik} \| x_k^{m-1} - x_k \| + \sum_{k=1}^{n} c_{ik} \| x_k^{m-1} - x_k \| \right]
\]

it follows that \( D_i[x_i, F_i(x_i, \ldots, x_n)] = 0 \). Since \( F_i(x_i, \ldots, x_n) \) is a closed set, \( x_i \in F_i(x_i, \ldots, x_n). \)

Theorem 3.3 below is a generalization of Matkowski’s Theorem 2.1 to multivalued mappings.

**Theorem 3.3.** Let \( X_i, \ i = 1, \ldots, n \) be Banach spaces and \( a_{ik} \) \( i, k = 1, \ldots, n \) be positive and \( a_{ik} \) be defined by (2.5) and (2.6) and fulfill (2.8). Suppose that the transformations \( F_i: X_1 \times \cdots \times X_n \rightarrow \text{CL}(X_i), \ i = 1, \ldots, n \) satisfy

\[
H_i[F_i(x_i, \ldots, x_n), F_i(z_i, \ldots, z_n)] \leq \sum_{k=1}^{n} a_{ik} \| x_k - z_k \|
\]
for all $x_j, z_j \in X_j$, $j = 1, \ldots, n$. Then the operator $F = (F_1, \ldots, F_n)$ has a fixed point, i.e., there exist points $x_i \in X_i$, such that $x_i \in F_i(x_i, \ldots, x_n)$ for all $i = 1, \ldots, n$.

Proof. In Theorem 3.2, let $c_{ik} = 0$, $b_{ik} = a_{ik}$, $i, k = 1, \ldots, n$ and $c = 0$. □

The following is the multivalued version of Altman’s Theorem 2.2.

**Theorem 3.4.** Suppose a nonlinear closed operator $P: D(P) \subset X \rightarrow \text{CL}(Y)$ has a bounded contractor $\Gamma$ satisfying

\begin{align}
(3.16) & \quad (a) \quad x + \Gamma(x)y \in D(P), \text{ whenever } x \in D(P), \ y \in Y, \ D(P) \text{ being the domain of } P; \\
(3.16) & \quad (b) \quad H[P(x + \Gamma(x)y), Px + y] \leq q \| y \|, \\
& \quad x \in D(P), \ y \in Y, \ 0 < q < 1; \\
(3.17) & \quad \| \Gamma(x) \| \leq B, \ x \in D(P). 
\end{align}

Then there exists $x \in D(P)$ such that $\theta \in Px$, where $\theta$ is the zero element of $Y$.

Proof. For $n = 1$, Theorem 3.1 reduces to the above theorem for the choice $c_{ik} = 0$, $b_{ik} = a_{ik}$, $i, k = 1, \ldots, n$ i.e., $b_{11} = a_{11} = q < 1$, and $c = 0$. □

Besides, Theorem 3.1 yields as corollaries several fixed point theorems for single-valued mappings including the following theorem proved elsewhere (Theorem 2.1, [2]).

**Theorem 3.5.** Let $X_i, Y_i$, $i = 1, \ldots, n$ be Banach spaces and $T_i: D \subset X_1 \times \cdots \times X_n \rightarrow Y_i$, $i = 1, \ldots, n$ be closed non-linear operators. Suppose that there exist bounded linear operators $\Gamma_i(x_i): Y_i \rightarrow X_i$, $i = 1, \ldots, n$ such that

\begin{align}
(3.18) & \quad (a) \quad (x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) \in D \\
& \quad \text{whenever } (x_1, \ldots, x_n) \in D, \ y_i \in Y_i, \ i = 1, \ldots, n; \\
& \quad \| T_i(x_1 + \Gamma_i(x_i)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) - T_i(x_1, \ldots, x_n) - y_i \| \\
& \quad \leq \sum_{k=1}^{n} a_{ik} \| y_k \|,
\end{align}

where the nonnegative numbers $a_{ik}$, $i, k = 1, \ldots, n$ are defined by (2.5) and (2.6) and fulfill (2.8);
Then the system of operator equations
\begin{equation}
T_i(x_1, \ldots, x_n) = y_i, \quad i = 1, \ldots, n
\end{equation}
has a solution in $D$ for arbitrary $y_i \in Y_i$, $i = 1, \ldots, n$.

**Proof.** In Theorem 3.1, let $c_{ik} = 0$, $b_{ik} = a_{ik}$, $i, k = 1, \ldots, n$ and $c = 0$. Define $P_i(x_1, \ldots, x_n) = \{T_i(x_1, \ldots, x_n)\}$ for $(x_1, \ldots, x_n) \in D$, $i = 1, \ldots, n$. Clearly the assumptions of Theorem 3.1 are satisfied and hence the system of equations (3.20) has a solution in $D$. \hfill \Box

The above Theorem proved in [2] unified, in the setting of Banach spaces, Altman's extension of the contraction principle and Matkowski's fixed point theorem.

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Siegfried Graf, Realizing automorphisms of quotients of product σ-fields . . . 19
Alfred Washington Hales and Ernst Gabor Straus, Projective colorings . . . 31
Sandra Hayes, The weak Nullstellensatz for finite-dimensional complex spaces ................................................................. 45
Gerald Norman Hile and Murray Harold Protter, The Cauchy problem and asymptotic decay for solutions of differential inequalities in Hilbert space .................................................................................. 57
Robert D. Little, Projective space as a branched covering with orientable branch set ............................................................... 89
Jaroslav Mach, On the proximinality of Stone-Weierstrass subspaces . . . . 97
John C. Morgan, II, On product bases ............................................. 105
K. Balakrishna Reddy and P. V. Subrahmanyam, Altman’s contractors and fixed points of multivalued mappings ................................. 127
James Ted Rogers Jr., Decompositions of homogeneous continua .......... 137
Ahmed Ramzy Sourour, Characterization and order properties of pseudo-integral operators ......................................................... 145
Robert Moffatt Stephenson Jr., Pseudocompact and Stone-Weierstrass product spaces ..................................................................... 159
Bruce Stewart Trace, On attaching 3-handles to a 1-connected 4-manifold ................................................................................ 175
Akihito Uchiyama, The construction of certain BMO functions and the corona problem ................................................................. 183
Thomas Alva Whitehurst, An application of orthogonal polynomials to random walks ................................................................. 205
David J. Winter, Root locologies and idempotents of Lie and nonassociative algebras ................................................................... 215
William Robin Zame, The classification of uniform algebras on plane domains .......................................................................... 231