Pacific Journal of Mathematics

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Vol. 99, No. 1 May 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 1, 1982

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If X_n is a simple random walk on the nonnegative integers with transition probabilities $P_{ij}^{(k)}=Pr\{X_{n+k}=j \mid X_n=i\}$, then $P_{ij}^{(k)}$ has an integral representation in terms of a family of orthogonal polynomials and the associated probability distribution function $F(x)$ for these polynomials. The relationship between the distribution F, the family of polynomials and the random walk X_n is studied. Necessary and sufficient conditions for the support of F to be contained in [0, 1] are given.

1. Preliminaries. Throughout this paper X_n will be a random walk on the integers with transition matrix $P = (P_{ij}) = (Pr\{X_{n+1} = j \mid X_n = i\})$. We shall say that X_n is a "simple random walk" if $P_{ij} = 0$ whenever $|i - j| > 1$, and in this case we set $q_n = P_{nn-1}$, $r_n = P_{nn}$, and $p_n = P_{nn+1}$. We shall concentrate on simple random walks, X_n, whose state space is the non-negative integers which we shall henceforth denote by N_0. Following Karlin and McGregor (1959) we find an integral representation for the transition probabilities $P_{nm}^{(k)}$.

Suppose then that X_n is a simple random walk on N_0. For each state $n \in N_0$, we associate a polynomial $Q_n(x)$, of degree n, defined recursively by

$$(1.1)\qquad \begin{aligned} &Q_{-1}(x) = 0,\ Q_0(x) = 1\ , \quad \text{and} \\ &xQ_n(x) = q_nQ_{n-1}(x) + r_nQ_n(x) + p_nQ_{n+1}(x)\ , \quad n \geqq 0\ . \end{aligned}$$

By the following theorem, due to Favard (1935), we see that the family of polynomials $\{Q_n(x)\}$ defined by (1.1) is orthogonal with respect to a probability distribution $F(x)$.

THEOREM 1.1. *Suppose that the family of polynomials $\{R_n(x)\}$ is defined recursively by*

$$(1.2)\qquad \begin{aligned} &R_0(x) = 1,\ R_1(x) = x - c_0\ , \quad \textit{and} \\ &R_{n+1}(x) = (x - c_n)R_n(x) - \lambda_nR_{n-1}(x)\ , \quad n \geqq 1\ , \end{aligned}$$

where c_n is real and $\lambda_{n+1} > 0$ for $n \geqq 0$. Then there is a (probability) distribution function $F(x)$, such that the polynomials $\{R_n(x)\}$ are orthogonal with respect to $F(x)$. That is, $\int_{-\infty}^{\infty} R_n(x)R_m(x)dF(x) = 0$ whenever $n \neq m$.

THEOREM 1.2. *If $\{R_n(x)\}$ is a family of orthogonal polynomials, with $R_n(x)$ having degree n, and normalized to be monic, then relation (1.2) holds for any three consecutive polynomials, where c_n is real and $\lambda_{n+1} > 0$, for $n \geqq 0$.*

Note that Theorem 1.2 is the converse to Favard's theorem (Theorem 1.1). For a proof and related results see Szegö (1939).

If we write $Q(x) = [Q_0(x)Q_1(x)Q_2(x)\cdots]^t$, then (1.1) is equivalent to $xQ(x) = PQ(x)$, so that $x^kQ(x) = P^kQ(x)$. In other words, $x^kQ_n(x) = \sum_{m=0}^{\infty} P_{nm}^{(k)}Q_m(x)$. Note that for a simple random walk X_n, only a finite number of terms on the right side are non-zero. Multiplying both sides of this equation by $Q_m(x)$, and exploiting the orthogonality of the family we have

$$P_{nm}^{(k)} = \pi_m \int_{-\infty}^{\infty} x^k Q_n(x) Q_m(x) dF(x) , \tag{1.3}$$

where $1/\pi_m = \int_{-\infty}^{\infty} Q_m^2(x)dF(x)$. It can easily be shown, from (1.1), that $\pi_0 = 1$ and $\pi_m = (p_0p_1 \cdots p_{m-1})/(q_1q_2 \cdots q_m)$.

The support of F, Supp F, is defined by Supp $F = \{x: F(x+h) \neq F(x-h) \text{ for all } h \neq 0\}$. If we set $n = m = 0$ in (1.3) we see that the support of F is contained in $[-1, 1]$ since $P_{nm}^{(k)}$ is a probability. Thus,

$$P_{nm}^{(k)} = \pi_m \int_{-1}^{1} x^k Q_n(x) Q_m(x) dF(x) . \tag{1.4}$$

Note that $F(x)$ is uniquely determined since the support of F is contained in a finite interval; hence, the moments uniquely determine the measure.

Karlin and McGregor (1959) took an alternative approach to the above development. That approach was to consider (1.4) as a spectral representation of the linear operator P (our transition matrix) acting on an appropriate Hilbert space. For this reason Karlin and McGregor refer to $F(x)$ in (1.4) as the spectral measure function, SMF, for the random walk X_n. Using this approach Karlin and McGregor (1959) obtain a representation similar to (1.4) if our simple random walk X_n is on $\{0, 1, 2, \cdots N\}$, or $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$. In the former case (1.1) defines only a finite family of orthogonal polynomials, and the SMF F, in (1.4), has only a finite number of support points. In the latter case the representation takes on the form

$$P_{ij}^{(n)} = \pi_j \int_{-1}^{1} x^n \sum_{\alpha,\beta=1}^{2} Q_i^{(\alpha)}(x) Q_j^{(\beta)}(x) dF_{\alpha\beta}(x) , \tag{1.5}$$

where the $F_{\alpha\beta}(x)$ are distributions on $[-1, 1]$, the polynomials $Q_i^{(\alpha)}(x)$

are defined by

$$Q_{-1}^{(1)}(x) = 0,\ Q_0^{(1)}(x) = 1,\ Q_{-1}^{(2)}(x) = 1,\ Q_0^{(2)}(x) = 0\ , \quad \text{and}$$
$$xQ_i^{(\alpha)} = q_i Q_{i-1}^{(\alpha)}(x) + r_i Q_i^{(\alpha)}(x) + p_i Q_{i+1}^{(\alpha)}(x)\ , \quad (\alpha = 1, 2)\ .$$

For $n \geqq 0$ π_n is the same as above and

$$\pi_{-n} = (q_0 q_{-1} q_{-2} \cdots q_{-n+1})/(p_{-1} p_{-2} \cdots p_{-n})\ .$$

The following results contain some of the essential facts about orthogonal polynomials; for proofs and related results, see Szegö (1939).

Suppose that $F(x)$ is a probability distribution on $[a, b]$ and that $Q_n(x)$ is the corresponding family of orthogonal polynomials. Then

THEOREM 1.3. *The zeros of $Q_n(x)$ are real and distinct, and are* located in the interior of the interval $[a, b]$.

THEOREM 1.4. *Let $x_1 < x_2 < \cdots < x_n$ be the zeros of $Q_n(x)$; also let $x_0 = a$ and $x_{n+1} = b$. Then each interval (x_k, x_{k+1}), $0 \leqq k \leqq n$, contains at least one zero of $Q_m(x)$, $m > n$.*

THEOREM 1.5. *In the open interval (x_k, x_{k+1}), between two consecutive zeros of $Q_n(x)$, the function $F(x)$ cannot be constant.*

If $F(x)$ is a probability distribution on $[-1, 1]$, then one can easily show that there is a family of polynomials $\{Q_n(x)\}$, satisfying (1.1), which are orthogonal with respect to $F(x)$. Furthermore, we may assume that p_n and q_{n+1} are positive for $n \geqq 0$. We may also assume that $Q_n(1) = 1$. This follows from the fact that the zeros of $Q_n(x)$ are in $(-1, 1)$ so $Q_n(1) > 0$ for $n \geqq 0$. Since $Q_n(1) = 1$ we have $q_n + r_n + p_n = 1$. Thus, it easily follows that if $F(x)$ has an infinite number of support points, then

THEOREM 1.6. *$F(x)$ is the* SMF *for some simple random walk X_n on N_0* iff

$$r_n/\pi_n = \int_{-1}^{1} xQ_n^2(x)dF(x) \geqq 0 \quad \text{for} \quad n \geqq 0\ .$$

If $F(x)$ has only a finite number of support points, then the simple random walk X_n above, is just on $\{0, 1, 2, \cdots, N\}$.

COROLLARY 1.7. *If $F(x)$ is a probability distribution of $[0, 1]$, then F is the* SMF *for some simple random walk.*

2. The support of the SMF. A sequence of numbers $u = (u_n; n \geqq 0)$ is a renewal sequence if there exists a sequence $f = (f_n; n \geqq 1)$ such that $u_0 = 1$ and $u_n = \sum_{k=1}^{n} f_k u_{n-k}$ $(n \geqq 1)$ where $f_n \geqq 0$ and $\sum_{n=1} f_n \leqq 1$. An important characterization of renewal sequences is: A sequence (u_n) is a renewal sequence iff $u_n = Pr(X_n = i \mid X_0 = i)$, $n \geqq 0$, for some Markov chain X and some state i. For a proof and related results see Kingman (1972). A renewal sequence (u_n) has period d if and only if $d = \text{G.C.D.}\{n: u_n > 0\}$. If $d = 1$, (u_n) is said to be aperiodic. If u has period d, then ${}^d u = (u_0, u_d, u_{2d}, \cdots)$ is an aperiodic renewal sequence, see Kingman (1972). Thus, it is clear that aperiodic renewal sequences play the major role in the theory of renewal sequences.

EXAMPLE 2.1. Consider the simple random walk Y_n on Z with transition probabilities $P_{ii+1} = p_i$, $P_{ii-1} = q_i = 1 - p_i$, and $P_{ij} = 0$ otherwise (where $0 < p_i < 1$). Note that $v_n = Pr(Y_n = 0 \mid Y_0 = 0)$ defines a renewal sequence with period 2. If we start this random walk at an even integer and then take two steps at a time (i.e., we only consider Y_{2n}), then we have a "simple" random walk X_n on the even integers. (To denote the dependence on the original random walk we shall use the notation ${}^2Y_n$). In this case $u_n = Pr(X_n = 0 \mid X_0 = 0) = Pr(Y_{2n} = 0 \mid Y_0 = 0)$ defines an aperiodic renewal sequence, and in fact $u = {}^2v$.

An important type of renewal sequence is the Kaluza sequence, which is defined to be any sequence $u = (u_n; n \geqq 0)$ such that $0 \leqq u_n \leqq u_0 = 1$ and $u_n^2 \leqq u_{n-1}u_{n+1}$ for $n \geqq 1$. Kingman (1972) shows that any Kaluza sequence is always a renewal sequence. By the Schwarz inequality it is clear that

$$u_n = \int_0^1 x^n dG(x) \tag{2.1}$$

is a Kaluza sequence for any probability distribution $G(x)$ on $[0, 1]$. In the above example, if $p_i \equiv p$, $q_i \equiv q$, then $u_n = \binom{2n}{n}(pq)^n$ is a Kaluza sequence. Kingman (1972) raises the question of whether this renewal sequence is of the form (2.1). Letac (1977) notes that in the case of $p = q = 1/2$

$$u_n = \binom{2n}{n} \Big/ 2^{2n} = \int_0^1 (\cos \pi t)^{2n} dt = \int_0^1 x^n dG(x)$$

where $dG(x)$ is the measure carried from Lebesque measure on $[0, 1]$ by the map $t \mapsto \cos^2 \pi t$.

We shall show that the renewal sequence generated in Example 2.1 is always of the form (2.1), even when $p_i \not\equiv p$. Furthermore, any renewal sequence of the form (2.1) corresponds to a random walk of the form ${}^2Y_n$. Following Karlin and McGregor (1959) we shall say that a simple random walk is symmetric if $r_n \equiv 0$. The reason for this terminology is that $r_n \equiv 0$ iff the SMF $F(x)$ is symmetric about $x = 0$.

THEOREM 2.1. *Suppose that* F *is the* SMF *of the simple random walk* X_n *(on* N_0*). Then,* $\operatorname{Supp} F \subset [0, 1]$ iff X_n *is probabilistically the same as* ${}^2Y_n$, *where* Y_n *is a symmetric random walk on* $\{n/2 : n \in N_0\}$ *which starts at an integer,* i.e., $Y_0 = k$, $k \in N_0$.

Proof. Suppose that $\operatorname{Supp} F \subset [0, 1]$. Exploiting Theorem 1.6, we define the (symmetric) probability distribution $G(x)$ on $[-1, 1]$ by

$$G(x) = \begin{cases} 1/2\ [1 + F(x^2)] & \text{if} \quad x \geqq 0 \\ 1/2\ [1 - F(x^2)] & \text{if} \quad x < 0\ . \end{cases}$$

G is the SMF of a symmetric random walk, Y_n, which we may assume to be on $\{n/2 : n \in N_0\}$ taking steps of size $+1/2$ and $-1/2$. Exploiting (1.4) and the symmetry of G we obtain $Pr\{{}^2Y_n = 0 \mid {}^2Y_0 = 0\} = 2\int_0^1 x^{2n} dG(x) = \int_0^1 y^n dF(y)$. Now ${}^2Y_n$ is a simple random walk on N_0; hence, ${}^2Y_n$ has a SMF $H(x)$. Thus, $Pr\{{}^2Y_n = 0 \mid {}^2Y_0 = 0\} = \int_{-1}^1 x^n dH(x) = \int_0^1 x^n dF(x)$. However, the moments uniquely determine the measure in this case, so $H(x) = F(x)$. Furthermore, the orthogonal polynomials corresponding to $H(x)$ and $F(x)$ are the same. Hence, the representations given by (1.4) are the same; that is X_n is probabilistically the same as ${}^2Y_n$.

On the other hand, suppose that X_n is probabilistically the same as ${}^2Y_n$, where Y_n is as given in the statement of the theorem and with Y_n having SMF $G(x)$. Since $G(x)$ is symmetric about $x = 0$ we have

$$Pr\{{}^2Y_{n+m} = 0 \mid {}^2Y_m = 0\} = \int_{-1}^1 x^{2n} dG(x) = 2\int_0^1 x^{2n} dG(x)\ .$$

If $F(x)$ is the SMF for X_n, then

$$Pr\{X_{n+m} = 0 \mid X_m = 0\} = \int_{-1}^1 x^n dF(x) = 2\int_0^1 x^{2n} dG(x)\ .$$

Setting $y = x^2$ and $H(y) = 2G(\sqrt{y}) - 1$, for $y \geqq 0$, we obtain

$$\int_{-1}^1 x^n dF(x) = \int_0^1 y^n dH(y)\ .$$

Again, however, the moments uniquely determine the measure, so that $F(x) = H(x)$, and hence $\operatorname{Supp} F \subset [0, 1]$.

As a consequence of Theorem 2.1 we have

COROLLARY 2.2. *A renewal sequence* $u = \langle u_n \rangle$ *is of the form* (2.1) iff $u_n = w_{2n}$, *where* $w = \langle w_n \rangle$ *is the renewal sequence associated with some symmetric random walk on* N_0.

In the case of a simple random walk on Z we obtain from (1.5) $P_{00}^{(n)} = \int_{-1}^{1} x^n dF_{11}(x)$. Hence, the argument of Theorem 2.1 can be extended to this case. Thus, in Corollary 2.2 we may replace N_0 by Z so that the question raised by Kingman (1972) and answered by Letac (1977) is a special case of this result.

Exploiting Theorem 2.1 we now develop necessary and sufficient conditions for $\operatorname{Supp} F \subset [0, 1]$, in terms of the coefficients q_n, r_n, and p_n in (1.1). In theory one could compute the zeroes of $Q_n(x)$ and calculate the support of the SMF, F, from them. This procedure is rarely practical because of the complexity of the polynomials, although in some special case $dF(x)$ can actually be calculated (see for example Maki (1967), Karlin and McGregor (1958) or (1959)). In theory one could also use Hausdorff's criteria, in terms of the moments, to test for $\operatorname{Supp} F \subset [0, 1]$. Using the fact that $m_n = \int_{-1}^{1} x^n dF(x) = P_{00}^{(n)}$, we can compute the moments by calculating the probabilities of the various paths. For example $m_3 = r_0^3 + 2r_0p_0q_1 + p_0r_1q_1$. It is clear, however, that this procedure is also not practical. We now give a tractable procedure for determining whether $\operatorname{Supp} F \subset [0, 1]$.

We shall use the following notation for finite continued fractions:

$$\frac{a_1|}{|b_0} = a_1/b_0, \text{ and recursively, for } n \geqq 2 \tag{2.2}$$

$$\frac{a_n|}{|b_{n-1}} - \frac{a_{n-1}|}{|b_{n-2}} - \cdots - \frac{a_1|}{|b_0} = a_n \div \left[b_{n-1} - \left(\frac{a_{n-1}|}{|b_{n-2}} - \cdots - \frac{a_1|}{|b_0} \right) \right].$$

To avoid notational difficulties, in the next theorem, we shall assume that our original random walk is on the non-negative even integers, $E_0 = \{2n: n \in N_0\}$. We shall use $p_n = P_{2n2n+2}$, $r_n = P_{2n2n}$, and $q_n = P_{2n2n-2}$.

THEOREM 2.3. *Suppose that* $F(x)$ *is the* SMF *for the simple random walk* X_n, *on* E_0, *corresponding to the family of orthogonal polynomials,* $\{Q_n(x)\}$, *defined by* (1.1). *Suppose also that* $p_n + r_n + q_n = 1$ *with* $p_n, q_{n+1} > 0$ *for* $n \geqq 0$ $(q_0 = 0)$. *Then,* $\operatorname{Supp} F \subset [0, 1]$ iff *for* $n \geqq 1$

$$0 < h_n = \frac{a_n|}{|1} - \frac{a_{n-1}|}{|1} - \cdots - \frac{a_2|}{|1} - \frac{a_1|}{|1 - p_0} < 1 \tag{2.3}$$

where $a_{2n} = p_n$ *and and* $a_{2n-1} = q_n$.

[*Note*: We could drop the left inequality since $h_n < 1$ and $a_{n+1} > 0$ imply that $h_{n+1} = a_{n+1}/(1 - h_n) > 0$.]

Proof. We define $\bar{p}_n$ and $\bar{q}_n$ for $n \geqq 0$ by

$$(2.4) \quad \bar{q}_0 = 0,\ \bar{p}_1 = p_0,\ \bar{q}_{2n} = h_{2n-1},\ \bar{p}_{2n+1} = h_{2n}\ , \quad \text{and} \quad \bar{p}_n + \bar{q}_n = 1\ .$$

If (2.3) is satisfied, then $0<\bar{p}_n, \bar{q}_n<1$ for $n\geqq 1$. Thus, we may define a symmetric random walk $\bar{X}_n$, on N_0 with transition probabilities $\bar{P}_{nn+1} = \bar{p}_n$ and $\bar{P}_{nn-1} = \bar{q}_n$. If we start at an even integer, then ${}^2\bar{X}_n$ is a random walk on E_0 with transition probabilities $\bar{\bar{P}}_{2n2n+2} = \bar{p}_{2n}\bar{p}_{2n+1}$, $\bar{\bar{P}}_{2n2n-2} = \bar{q}_{2n}\bar{q}_{2n-1}$, and $\bar{\bar{P}}_{2n2n} = 1 - \bar{\bar{P}}_{2n2n-2} - \bar{\bar{P}}_{2n2n+2}$. From the definition of h_n and continued fractions we have $\bar{p}_{2n}\bar{p}_{2n+1} = (1 - \bar{q}_{2n})\bar{p}_{2n+1} = (1 - \bar{q}_{2n})p_n/(1 - \bar{q}_{2n}) = p_n$. Thus, $\bar{\bar{P}}_{2n2n+2} = p_n$. Similarly, $\bar{\bar{P}}_{2n2n-2} = q_n$ and $\bar{\bar{P}}_{2n2n} = r_n$. Thus, ${}^2\bar{X}_n$ is probabilistically the same as our original random walk X_n. By Theorem 2.1 Supp $F \subset [0, 1]$.

On the other hand, if Supp $F \subset [0, 1]$, then there is a symmetric random walk Y_n, on N_0, such that ${}^2Y_n$, on E_0, is probabilistically the same as X_n. Let $\hat{p}_n$ and $\hat{q}_n$ represent the transition probabilities of Y_n. Note that $\hat{r}_n = 0$, $\hat{q}_0 = 0$ and $\hat{p}_n + \hat{q}_n = 1$ since Y_n is symmetric and ${}^2Y_n$ is probabilistically the same as X_n.

By induction we shall show that $\hat{p}_n = \bar{p}_n$ and $\hat{q}_n = \bar{q}_n$, where $\bar{p}_n$ and $\bar{q}_n$ are given by (2.4). From above we know that $\hat{p}_0 = 1 = \bar{p}_0$ and $\hat{q}_0 = 1 = \bar{q}_0$. Assume then that $\hat{p}_n = \bar{p}_n$ and $\hat{q}_n = \bar{q}_n$ for $n \leqq m$. If m is even, then

$$p_{m/2} = Pr\{X_{k+1} = m + 2 \mid X_k = m\} = Pr\{{}^2Y_{k+1} = m + 2 \mid {}^2Y_k = m\}$$
$$= \hat{p}_m\hat{p}_{m+1} = \bar{p}_m\hat{p}_{m+1}\ .$$

Thus, $\hat{p}_{m+1} = p_{m/2} \div \bar{p}_m = p_{m/2} \div (1 - \bar{q}_m) = \bar{p}_{m+1}$ from (2.3) and (2.4). Similarly, if m is odd, then $\hat{q}_{m+1} = q_{(m+1)/2} \div \bar{q}_m = \bar{q}_{m+1}$. Since $\hat{q}_n + \hat{p}_n = 1 = \bar{q}_n + \bar{p}_n$, we have in either case $\hat{p}_{m+1} = \bar{p}_{m+1}$ and $\hat{q}_{m+1} = \bar{q}_{m+1}$. Thus, $\hat{p}_n = \bar{p}_n$ and $\hat{q}_n = \bar{q}_n$. It is clear that $0 < \hat{p}_n, \hat{q}_n < 1$ for $n \geqq 1$, so from (2.4) we see that $0 < h_n < 1$ for $n \geqq 1$.

In many cases the continued fractions in (2.3) will not be difficult to compute since $h_{n+1} = a_{n+1}/(1 + h_n)$.

EXAMPLE 2.2. Suppose that $p_0 = 1/2 = r_0$ and for $n \geqq 1$ $r_n = 1/2$ and $p_n = q_n = 1/4$. From (2.3) we see that $h_n = 1/2$ for $n \geqq 1$. Thus, by Theorem 2.3, we see that the SMF corresponding to this random walk is supported by [0, 1].

EXAMPLE 2.3. Suppose that $p_0 = 1/3$ and $r_0 = 2/3$, and for $n \geqq$

1, $r_n = p_n = q_n = 1/3$. From (2.3) $h_1 = 1/2$, $h_2 = 2/3$, but $h_3 = 1$. Thus, the SMF corresponding to this random walk is not supported by $[0, 1]$.

3. Some special cases. Consider a family of polynomials $\{R_n(x)\}$, defined by (1.2), which are orthogonal with respect to $F(x)$. Let X be the set of zeros; i.e., $X = \{x: R_n(x) = 0 \text{ for some } n \geqq 1\}$. From Blumenthal (1898) we have

THEOREM 3.1. *Suppose that c_n and λ_n converge to c and λ (finite) respectively. Then the set X is dense in $[\sigma, \tau]$ where $\sigma = c - 2\lambda^{1/2}$ and $\tau = c + 2\lambda^{1/2}$; hence, the* Supp F *is dense in $[\sigma, \tau]$. Furthermore, dF consists only of a countable number of atoms outside of $[\sigma, \tau]$.*

For a proof and related results see Chihara (1968).

COROLLARY 3.2. *If X_n is a simple random walk on N_0 with* SMF *F, then dF is purely atomic if $r_n \to 1$.*

Proof. The corresponding family of orthogonal polynomials $\{Q_n(x)\}$ is defined by (1.1). If we normalize $Q_n(x)$ to be monic, then the normalized polynomials satisfy (1.2) with $c_n = r_n$ and $\lambda_n = p_{n-1}q_n$. Since $q_n + r_n + p_n = 1$, we see that $\lambda_n \to 0$. so $\sigma = \tau = 1$.

Still assuming that X_n has SMF F we have

THEOREM 3.3. *If* Supp $F \subset [0, 1]$ *and $r_n \to 0$, then dF is purely atomic.*

Proof. We may assume that $q_n + r_n + p_n = 1$ since normalizing $Q_n(x)$ so that $Q_n(1) = 1$ does not affect r_n. For $n \geqq N(\varepsilon)$ $r_n < \varepsilon$. By Theorem 2.3 we have $h_{2n} = p_n/(1 - h_{2n-1}) < 1$ so $p_n + h_{2n-1} < 1$ and $h_{2n-1} = q_n/(1 - h_{2n-2}) > q_n$.

Thus, for $n \geqq N(\varepsilon)$, $1 - \varepsilon < p_n + q_n < p_n + h_{2n-1} < 1$. Hence, $0 < h_{2n-1} - q_n < \varepsilon$; but $h_{2n-1} - q_n = q_n[h_{2n-2}/(1 - h_{2n-2})]$. Also $p_{n-1} = h_{2n-2}(1 - h_{2n-3}) < h_{2n-2} < [h_{2n-2}/(1 - h_{2n-2})]$. Therefore, $0 < \lambda_n = q_n p_{n-1} < q_n[h_{2n-2}/(1 - h_{2n-2})] < \varepsilon$ for $n \geqq N(\varepsilon)$. Hence $\lambda_n \to 0$ and dF is purely atomic.

For general families of orthogonal polynomials $\{R_n(x)\}$ defined by (1.2) with Supp $F \subset [a, b]$ we have

COROLLARY 3.4. *If $c_n \to a$ or $c_n \to b$, then dF is purely atomic.*

Proof. Starting with (1.2) we set $Q_n(x) = R_n(y)/R_n(b)$ where $x = (y - a)/(b - a)$. This new family of polynomials $\{Q_n(x)\}$ is orthogonal on [0, 1] with SMF F translated to [0, 1]. The family $\{Q_n(x)\}$ satisfies (1.1) with $q_n = \lambda_n t_{n-1}/(b-a)t_n$, $r_n = (c_n - a)/(b-a)$, and $p_n = t_{n+1}/(b-a)t_n$ where $t_n = R_n(b)$ which is positive by Theorem 1.3. Since $Q_n(1) = 1$ we have $q_n + r_n + p_n = 1$ so we may apply the previous two results to this family. If $c_n \to a$ or $c_n \to b$ then $r_n \to 0$ or $r_n \to 1$. Thus, the original measure dF is purely atomic.

Note that if $\operatorname{Supp} F = [0, 1]$ and p_n, q_n and r_n converge to p, q and r respectively, then a necessary condition for F to be absolutely continuous is that $r = 1/2$ and $p = q = 1/4$. This follows immediately from Theorem 3.1.

ACKNOWLEDGEMENT. The author wishes to thank Professor Daniel Maki for his helpful suggestions and discussions concerning this paper.

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Received February 4, 1980.

FLORIDA INTERNATIONAL UNIVERSITY
MIAMI, FL 33199

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 99, No. 1 May, 1982