Pacific Journal of Mathematics

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Vol. 99, No. 1 May 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 1, 1982

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Locological spaces are introduced. The G-locology for a subset R of a group G leads to the symmetric G-topology of R. The connected components of R correspond to ideals of any normal finite dimensional G-graded nonassociative algebra A which, for A an idempotent Lie algebra with set R of roots, are the central primitive idempotents of A.

0. **Introduction.** The underlying ideas in this paper are that "ideals" in a Lie algebra or graded nonassociative algebra A correspond to "open sets" in the set R of roots of A; and "direct sums" correspond to "disjoint unions of open sets."

The first section is devoted to making these ideas precise, in the language of *locologies* and topologies for R.

The second section is devoted to the development of a theory of decompositions of idempotent nonassociative algebras 1 as sums $1 = E_1 + \cdots + E_n$ of pairwise orthogonal central primitive idempotents; and to showing for idempotent Lie algebras that the central primitive idempotents correspond to the connected components $R_1, \cdots, R_n$ of R discussed in Section 1.

The third section is devoted to relating the open set structure of R to the ideal structure of a Lie algebra L not assumed to be idempotent, taking as starting point Theorem 1.21.

1. **Locological spaces and root locologies.** Let R be a set, k a set with a specified point $0 \in k$ called the *origin* of k, H a collection of functions from R into k. Suppose that H contains the *zero function* which maps all elements of R into 0. Suppose, furthermore, that for each $a \in R$, $x(a) \neq 0$ for some $x \in H$. For $X \subset H$, let $R(X) = \{a \in R \mid x(a) = 0 \text{ for all } x \in X\}$. Then the collection $\mathscr{C} = \{R(X) \mid X \subset H\}$ of subsets of R contains R and ϕ; and is closed under intersections since

$$R(\bigcup_{i \in I} X_i) = \bigcap_{i \in I} R(X_i) .$$

We call $R(X)$ the *locus of zeros* of X. The collection $\mathscr{C}$ is a locology for R in the sense of the following definition.

DEFINITION 1.1. A *locology* for a set R is a collection $\mathscr{C}$ of subsets of R such that

(1) $\phi \in \mathscr{C}$ and $R \in \mathscr{C}$;

(2) $\mathscr{C}$ is closed under intersections, that is, $\mathscr{S} \subset \mathscr{C}$ implies $\bigcap_{S \in \mathscr{S}} S \in \mathscr{C}$.

A *locological space* is a set R together with a locology $\mathscr{C}$ for R. □

If, in the above example, H also separates the points of R, we can imbed R in the set $F(H, k)$ of functions from H to k by regarding $a \in R$ as the function $a: H \to k$ such that $a(x) = x(a)$ for $x \in H$. Thus, $R(X)$ so imbedded is $R(X) = \{a \in R \mid a(x) = 0$ for all $x \in X\}$. Let us suppose furthermore that k is a group with product $+$ (not necessarily commutative) and identity equal to the origin 0. Then the sets $R(X)$ satisfy the following conditions, $a + b$ and $-a$ denoting pointwise product and inverse of $a, b \in R$ and $a \in R$ respectively.

(1) if $a, b \in R(X)$, then $a + b \in R(X)$ if $a + b \in R$, $a - b \in R(X)$ if $a - b \in R$, and $(-a) + b \in R(X)$ if $(-a) + b \in R$;

2. if $a \in R(X)$ and $-a \in R$, then $-a \in R(X)$.

Thus, $R(x)$ is closed and symmetric in the G-locology for R in the sense of the following definition, G being the group $G = F(R, k)$.

DEFINITION 1.2. Let R be subset of a group G with product $ab (a, b \in G)$. Then a subset S of R is *G-closed* if $(S^2 \cup SS^{-1} \cup S^{-1}S) \cap R \subset S$, and S is *symmetric* if $S^{-1} \cap R \subset S$. Here, $ST = \{ab \mid a \in S, b \in T\}$, $S^2 = SS$, $S^{-1} = \{a^{-1} \mid a \in S\}$ for $S, T \subset G$. The collection $\mathscr{C}$ of G-closed (respectively symmetric G-closed) subsets of R is called the *G-locology* (respectively *symmetric G-locology*) of R. □

The G-locology (respectively symmetric G-locology) for a subset R of a group G obviously satisfies the axioms for a locology for R.

We now assume that R is an arbitrary locological space with locology $\mathscr{C}$. The elements of $\mathscr{C}$ are called the *closed* sets of R, their complements the *open* sets of R. Note that R and ϕ are both open and closed. For any subset S of R, $\mathscr{C}_S = \{A \cap S \mid A \in \mathscr{C}\}$ is a locology for S, called the *relative locology* on S. The closed and open sets of S are called the *relatively closed* and *open* sets of S respectively. If S is closed, $\mathscr{C}_S = \{A \in \mathscr{C} \mid A \subset S\}$. The *closure* of a subset S of R is the intersection $\bar{S}$ of all closed sets of R containing S. Note that $\bar{S}$ is closed, contains S and is contained in every closed set containing S. We say that a subset S of R is *connected* if $S = S_1 \cup S_2$ where S_1 and S_2 are disjoint and relatively closed in S implies that $S = S_1$ or $S = S_2$.

PROPOSITION 1.3 *Let S be connected. Then $\bar{S}$ is connected.*

Proof. For A, B closed, $\bar{S} \subset A \cup B$ and $\bar{S} \cap A \cap B = \phi$, we must show that $\bar{S} \subset A$ or $\bar{S} \subset B$. But this follows from the fact that A and B are closed and, since S is connected, $S \subset A$ or $S \subset B$. □

For $x \in R$, $C(x)$ is the union of all connected subsets of R which contain x.

THEOREM 1.4. *For $x \in R$, $C(x)$ is closed and connected and contains x. For $x, y \in R$, either $C(x) = C(y)$ or $C(x) \cap C(y) = \phi$.*

Proof. Since $\{x\}$ is connected, $C(x)$ contains x. Suppose that $C(x) \subset A \cup B$ and $C(x) \cap A \cap B = \phi$ with A, B closed. We may assume with no loss of generality that $x \in A$. Then every connected set S containing x is contained in A, so that $C(x) \subset A$. Thus, $C(x)$ is connected. Since $\overline{C(x)}$ is connected, $\overline{C(x)} = C(x)$ and $C(x)$ is closed. Suppose that $C(x) \cap C(y) \cdot \ni z$. Then $C(z) \supset C(x)$, $C(z) \supset C(y)$, whence $C(x) = C(z) = C(y)$. □

The above theorem shows that the sets $C(x)$ are the maximal connected subsets of R. We call $C(x)$ the *connected component* of R containing x.

COROLLARY 1.5. *R can be decomposed as a disjoint union $R = \bigcup_{i \in I} R_i$ where the $R_i (i \in I)$ are the connected components of R.* □

The connected components R_i of R are closed.

COROLLARY 1.6. *Suppose that $R = \bigcup_{i \in I} R_i$ (disjoint union) where R_i is nonempty, open and connected for all i. Then*

(1) *the R_i are the connected components of R;*

(2) *each open and closed subset S of R is a union $S = \bigcup_{i \in I} R_i$ of certain of the R_i; and every such union is open and closed.*

In particular, the collection $\mathscr{D}$ of open and closed subsets of R is closed under unions and intersections and is therefore a topology for R.

Proof. We first prove part of (2), namely, that each union $S = \bigcup_{i \in I} R_i$ of a subcollection $R_i (i \in I)$ of the R_i is open and closed. Since the R_i are open, S is open since $S^c = \bigcap_{i \in I} R_i^c$ is closed—as the intersection of closed sets. Similarly, $S^c = \bigcup_{j \notin I} R_j$ is open. Thus, S is also closed. Taking $I = \{i\}$, we have shown in particular that each R_i is open and closed, as is its complement R_i^c in R. This having been shown, we now note that for (1), it suffices to show that $C = R_i$ for any connected set C containing R_i. This follows easily from the connectedness of C and the fact that R_i, R_i^c are closed and disjoint, $C \subset R_i \cup R_i^c$ and $C \cap R_i$ is nonempty. For the remaining direction of (2), it suffices to show that whenever $R_i \cap S \neq \phi$, S contains R_i. This follows directly from the fact that S, S^c are closed and disjoint, R_i is connected, $R_i \subset S \cup S^c$ and $R_i \cap S \neq \phi$. □

We now specialize our considerations to a fixed subset R of a group G. We regard R as locological space with the G-locology for R, and refer to R with this locology as a *G-locological space.* For $S \subset R$, we denote the complement of S in R by S^c. We say that S is *G-open* if S^c is G-closed.

THEOREM 1.7. *Let S be a G-closed set of R. Then*

$$(SS^c \cup S^cS \cup S^{-1}S^c \cup S^cS^{-1} \cup S(S^c)^{-1} \cup (S^c)^{-1}S) \cap R \subset S^c$$

Proof. Let $a \in S$, $b \in S^c$. Then we have $b = a^{-1}(ab) = (ba)a^{-1} = a(a^{-1}b) = (ba^{-1})a = (ab^{-1})a^{-1} = a(b^{-1}a)^{-1}$. Let d be any one of the elements $ab, ba, a^{-1}b, ba^{-1}, ab^{-1}, b^{-1}a$. Since S is closed, $b \notin S$ and $b \in (S^{-1}d \cup dS^{-1} \cup Sd \cup dS \cup d^{-1}S \cup Sd^{-1})$, it follows that $d \notin S$. Thus, $d \in R$ implies $d \in S^c$. □

In general, the collection $\mathscr{D}$ of open and closed sets in a locological space R is not closed under finite unions and intersections. For example, if R is the disjoint union of nonempty sets A, B, C, D, then $\mathscr{D} = \{\phi, R, A, B, A^c, B^c\}$ where the closed sets of R are $\phi, R, A, B, C, D, A^c, B^c, (A \cup B)^c$. However, $\mathscr{D}$ is closed under finite unions and intersections for G-locological spaces R.

THEOREM 1.8. *Let $\mathscr{D}$ be the collection of subsets S of R which are both G-open and G-closed. Let $S, T \in \mathscr{D}$. Then*

(1) *for $a \in S, b \notin S$, we have $ab \notin R$, $a^{-1}b \notin R$, $ab^{-1} \notin R$;*

(2) *$S \cup T$ and $S \cap T$ are in $\mathscr{D}$.*

Proof. (1) follows from Theorem 1.7 because, since S and S^c are both closed, we have $(SS^c \cup S^{-1}S^c \cup SS^{c^{-1}}) \cap R \subset S \cap S^c = \phi$. For (2), it suffices to prove that $S \cup T$ is closed and open for all $S, T \in \mathscr{D}$, since $S \in \mathscr{D}$ implies $S^c \in \mathscr{D}$ and $(S \cap T)^c = S^c \cup T^c$. Moreover, $S \cup T$ is clearly open, since S and T are open. We claim that $S \cup T$ is closed. Thus, let $a, b \in S \cup T$. Then one of the following cases result:

(1) $(a, b \in S)$ or $(a, b \in T)$;

(2) $(a \in S, a \notin T, b \in T, b \notin S)$ or $(b \in S, b \notin T, a \in T, a \notin S)$.

In case (1), $\{ab, a^{-1}b, ab^{-1}\} \cap R \subset S \cup T$. In case (2), the same is true by the first assertion of the theorem which we have already proved. □

COROLLARY 1.9. *Let $S, T \in \mathscr{D}$ and let $a \in S$, $b \in T$. Then either $a, b \in S \cap T$ or R contains none of the elements ab, $a^{-1}b$, ab^{-1}.*

Proof. Suppose that $S \cap T$ does not contain both of a, b. Then either $a \in S$ and $b \notin S$ or $a \notin T$ and $b \in T$. In either case, $ab \notin R$, $a^{-1}b \notin R$ and $ab^{-1} \notin R$ by Theorem 1.8. □

COROLLARY 1.10. *If $\mathscr{D}$ is finite, then $R = R_1 \cup \cdots \cup R_n$ (disjoint union) where the R_i are the minimal nonempty elements of $\mathscr{D}$ (respectively, the minimal nonempty symmetric elements of $\mathscr{D}$).*

Proof. Let the R_i be the connected components of R in the topology $\mathscr{D}$ for R (respectively, in the topology $\mathscr{D}_1 = \{S \in \mathscr{D} \mid S$ is symmetric$\}$ for R). □

DEFINITION 1.11. The *open components* (respectively the *symmetric open components*) of R are the minimal nonempty elements of $\mathscr{D}$ (respectively $\mathscr{D}_1$). □

COROLLARY 1.12. *Let $\mathscr{D}$ be finite and express R as the disjoint union $R = R_1 \cup \cdots \cup R_n$ of its open (respectively symmetric open) components. Then a subset S of R is closed if and only if $S \cap S_i$ is closed for $1 \leqq i \leqq n$.*

Proof. If S is closed, then $S \cap R_i$ is closed since R_i is closed for $1 \leqq i \leqq n$. Suppose, conversely, that $S \cap R_i$ is closed for $1 \leqq i \leqq n$. Let $a, b \in S = S \cap R_1 \cup \cdots \cup S \cap R_n$. If $a, b \in S \cap R_i$ for some i, then $\{ab, a^{-1}b, ab^{-1}\} \cap R \subset S \cap R_i$ since $S \cap R_i$ is closed $(1 \leqq i \leqq n)$. Thus, suppose that $a \in S \cap R_i$, $b \in S \cap R_j$ with $i \neq j$. Then $a \in R_i$ and $b \notin R_i$, so that $\{ab, a^{-1}b, ab^{-1}\} \cap R = \phi$ by Theorem 1.8, since R_i is open and closed. It follows that $(S^2 \cup S^{-1}S \cup SS^{-1}) \cap R \subset S$ and S is closed. □

The above corollary determines the locology of R in terms of the locology of its open components $R_1, \cdots, R_n$ for $\mathscr{D}$ finite.

COROLLARY 1.13. *For $\mathscr{D}$ finite, the set of connected components of R (in the G-locology) is the union of the sects of connected components of the open (respectively symmetric open) components $R_1, \cdots, R_n$ of R.* □

For the remainder of this section, we specialize to G-locological spaces R where R is the set of roots of a G-graded nonassociative algebra A, G being a group. Here a *nonassociative algebra* is a vector space A over a field k and a product $xy \in A$ $(x, y \in A)$ which is bilinear in the sense that

(1) $(x + y)z = xz + yz$ $(x, y, z \in A)$;

(2) $x(y + z) = xy + xz$ $(x, y, z \in A)$;

(3) $(cx)y = c(xy) = x(cy)$ $(x, y \in A, c \in k)$.

A *subalgebra* of A is a subspace B of A such that $B^2 \subset B$; and an ideal of A is a subspace B of A such that $AB \subset B$ and $BA \subset B$. Here, BC is the span of $\{bc \mid b \in B, c \in C\}$ and $B^2 = BB$. A *G-graded nonassociative algebra*, G being a group, is a nonassociative algebra A together with a *G-grading* of A, that is, a collection $\{A_a \mid a \in G\}$ of subspaces of A indexed by G such that

(1) $A = \sum_{a \in G} A_a$ (direct sum of subspaces);

(2) $A_a A_b \subset A_{ab}$ for all $a, b \in G$.

The *set of roots* of A with respect to the G-grading of A is $R = \{a \in G \mid a \neq 1,\ A_a \neq 0\}$ where 1 is the identity of G and 0 is the null space of A. The elements of R are called *roots*. We let $H = A_1$, $A_S = \sum_{a \in S} A_a$ and $H_S = \sum_{a \in S} H_a H_{a^{-1}} + H_{a^{-1}} H_a$ for $S \subset R$.

We let $\langle B \rangle$ be the subalgebra of A generated by B for any subset B of A.

DEFINITION 1.14. We say that the G-graded nonassociative algebra A is *normal* if

(1) for each $a \in G$ and $S \subset G$, $A_a A_S = 0 = A_S A_a$ implies that $A_a \langle A_S \rangle \subset \langle A_S \rangle$ and $\langle A_S \rangle A_a \subset \langle A_S \rangle$;

(2) $A_1 \langle A_S \rangle \subset \langle A_S \rangle$, $\langle A_S \rangle A_1 \subset \langle A_S \rangle$ for all $S \subset G$;

(3) $A_1(A_a A_{a^{-1}}) \subset A_a A_{a^{-1}}$ for all $a \in G$;

(4) $A_S B \subset B$ and $BA_R \subset B$ and $A_S \subset B$ imply that $\langle A_S \rangle \subset B$ for all $S \subset G$.

Note that graded Lie algebras and associative algeras are normal.

THEOREM 1.15. *Let A be normal and let S be a subset of R. Then*

(1) *for S closed, H_{S^*} is an ideal of H and $\langle A_S \rangle = A_S + H_{S^*}$ where $S^* = S \cap S^{-1}$;*

(2) *for S open and symmetric, $\langle A_S \rangle$ is an ideal of A and $\langle A_S \rangle = A_S + A_S^2$;*

(3) *for S open and closed, $\{RS \cup RS^{-1} \cup SR \cup S^{-1}R\} \cap R \subset S$, $\{S^c S \cup S^c S^{-1} \cup SS^c \cup S^{-1} S^c\} \cap R = \phi$ and $\{RS \cup RS^{-1} \cup SR \cup S^{-1}R\} \cap$*

$\{RS^c \cup R(S^c)^{-1} \cup S^cR \cup (S^c)^{-1}R\} \cap R = \phi$;

(4) *for S open, closed and symmetric, $\langle A_S \rangle = A_S + H_{S^*}$ is an ideal of A.*

Proof. For (1), suppose that S is closed. By normality, H_{S^*} is an ideal of $A_1 = H$. Clearly, $A_S H_{S^*} \cup H_{S^*} A_S \subset A_S$. Finally, $A_S A_S \subset A_S + H_{S^*}$ since $S^2 \cap R \subset S$. The first part of (3) follows from Theorem 1.7 for S open and closed, since $(S^2 \cup SS^{-1} \cup S^{-1}S) \cap R \subset S$; and the second and third parts follow from the first applied to both S and S^c. Clearly, (4) follows from (1) and (2). For (2), assume that S is open and symmetric and let $B = A_S + A_S^2$. Let $a \in S^c$. Since S is symmetric, $a^{-1} \notin S$. By (3), $(S^cS \cup SS^c) \cap R = \phi$. Thus, $A_a A_S = 0 = A_S A_a$. By normality, therefore, $(A_1 + A_a)\langle A_S \rangle \subset \langle A_S \rangle$ and $\langle A_S \rangle (A_1 + A_a) \subset \langle A_S \rangle$ for all $a \in S^c$. Thus, $\langle A_S \rangle$ is an ideal of A. It now remains only to show that $\langle A_S \rangle = B$, that is, that $B = A_S + A_S^2$ is a subalgebra of A. For this, it suffices, by normality, to show that $A_S B \cup BA_S \subset B$; for then $\langle A_S \rangle \subset B$ by normality, since $A_S \subset B$, so that $\langle A_S \rangle = B$. Since $B = A_S + A_S^2$, to show $A_S B \cup BA_S \subset B$ reduces to showing that $A_S A_S^2 \cup A_S^2 A_S \subset A_S + A_S^2$. Therefore, consider $D = A_a(A_b A_c)$ where $a, b, c \in S$. If $a + b + c \in S$ or $a + b + c \notin R$, then $D \subset B$. Thus, assume that $a+b+c \in S^c$. Since S^c is closed, $a \notin S^c$, and $a = (a + b + c) - (b + c)$, we have $b + c \notin S^c$. But then either $b + c \in S$, in which case $D \subset A_S^2$; or $b + c \notin R$, in which case $D = A_a(0) = 0$. Thus, in all cases, $D \subset B$. Thus, $A_S A_S^2 \subset B$. Similarly, $A_S^2 A_S \subset B$, and it follows that $\langle A_S \rangle \subset B$, therefore $\langle A_S \rangle = B$. □

DEFINITION 1.16. If $A^2 = 0$, A is *abelian*. If A has no ideals other than A and 0, A is *simple*. □

COROLLARY 1.17. *For A simple and nonabelian and normal, $H_S = H$ for every nonempty symmetric open set S of R.*

Proof. By Theorem 1.15, $A_S + A_S^2$ must equal A, so that $H = H_S$. □

COROLLARY 1.18. *Let A be normal and let S, T be open and closed sets of R. Then $A_{S \cap T} + H_{(S \cap T)^*}$ and $\langle A_S \rangle \cap \langle A_T \rangle = A_{S \cap T} + H_{S^*} \cap H_{T^*}$ are ideals of A.*

Proof. This follows directly from Theorem 1.8 and 1.15. □

Some of our observations can now be summarized as follows. The proof is straight forward.

THEOREM 1.19. *Let A be finite dimensional and normal, let $R_1, \cdots, R_n$ be the open components of R, let $A_i = A_{R_i} + H_{R_i}^*$ $(1 \leqq i \leqq n)$ and let I be the sum of all ideals of A which are contained in H. Then*

(1) *the A_i are ideals of A $(1 \leqq i \leqq n)$ and $A = H + A_1 + \cdots + A_n$;*

(2) *I is an ideal of A contained in H and $IA_a = 0 = A_a I$ for all $a \in R$;*

(3) *$\bar{A} = \bar{H} \oplus \bar{A}_1 \oplus \cdots \oplus \bar{A}_n$ (direct sum) where $\bar{A} = A/I$, $\bar{H} = H + I/I$ and $\bar{A}_i = A_i + I/I (1 \leqq i \leqq n)$.* □

Finally, we specialize to the context of a finite dimensional Lie algebra L over a field k with split Cartan subalgebra H. Let G be the group $G = F(H, k)$ with a product $a + b (a, b \in G)$ defined by $(a + b)(h) = a(h) + b(h)$ $(h \in H)$. Then the Cartan decomposition $L = \sum_{a \in G} L_a$ is a G-grading for L such that $H = L_0$. Let R be the corresponding set of roots with the G-locology, so that $L = H + \sum_{a \in R} L_a$.

Corollary 1.18 and Theorem 1.19 can now be refined as follows.

COROLLARY 1.20. *Let $S, T \in \mathscr{D}$. Then*

(1) *$[L_S, L_T] \subset L_{S \cap T} + H_{S \cap T_*}$ where $T_* = T \cup (-T)$;*

(2) *for S and T symmetric, $a \in S$, $b \in T$, we have $[L_a, L_b] = [H_a, L_b] = [L_a, H_b] = [H_a, H_b] = 0$ unless $a, b \in S \cap T$.*

Proof. Since $(S + T) \cap R \subset (R + S) \cap (R + T) \cap R \subset S \cap T$ by Theorem 1.7, we have $[L_S, L_T] \subset L_{S \cap T} + H_{S \cap T_*}$. Suppose next that S and T are symmetric, $a \in S$ and $b \in T$. If $a + b = 0$ or $a - b = 0$, then $a, b \in S \cap T$ by symmetry. Thus, suppose that $a + b \neq 0$ and $a - b \neq 0$. Then $a + b \notin R$, $a - b \notin R$ and $-a + b \notin R$ unless $a, b \in S \cap T$, by Corollary 1.9. Since $[H_a, L_b] = [[L_a, L_{-a}], L_b] = [[L_a, L_b], L_{-a}] + [L_a, [L_{-a}, L_b]]$, it follows that $[H_a, L_b] = 0$ unless $a, b \in S \cap T$ or $a, -b \in S \cap T$; that is, unless $a, b \in S \cap T$. And since $[H_a, H_b] = [[H_a, L_b], L_{-b}] + [L_b, [H_a, L_{-b}]]$, it follows that $[H_a, H_b] = 0$ unless either $a, b \in S \cap T$ or $a, -b \in S \cap T$; that is unless $a, b \in S \cap T$. □

COROLLARY 1.21. *Let $R_1, \cdots, R_n$ be the symmetric open components of R and let $L_i = L_{R_i} + H_{R_i} (1 \leqq i \leqq n)$. Then $L = H + L_1 + \cdots + L_n$, $[L_i, L_i] \subset L_i$, $[L_i, L_j] = 0$ for $1 \leqq i, j \leqq n$ and $i \neq j$ and $L^\infty = L_1 + \cdots + L_n$.*

Proof. Since $R = R_1 \cup \cdots \cup R_n$ (disjoint union of symmetric open and closed sets), this follows directly from Corollary 1.20 and the fact proved in Winter [4] that $L^\infty = \sum_{a \in R} [L_a, L_{-a}] + \sum_{a \in R} L_a$. $\square$

Before turning to the next section, we mention that the set $\mathscr{D}$ of open and closed (respectively symmetric open and closed) sets of a G-locology for R determine a topology $\langle \mathscr{D} \rangle$ for R as defined below. Our use of this topology has been restricted to the case where $\mathscr{D}$ is finite, in which case $\mathscr{D} = \langle \mathscr{D} \rangle$. That $\langle \mathscr{D} \rangle$ is, in general, a topology for R is evident.

DEFINITION 1.22. The set $\langle \mathscr{D} \rangle$ of unions of subsets of $\mathscr{D}$ is called the *G-topology* (respectively *symmetric G-topology*) for R. $\square$

2. Idempotent nonassociative algebras and Lie algebras.

In this section, all nonassociative algebras are finite dimensional.

DEFINITION 2.1. In a nonassociative algebra A, an *idempotent* is a subalgebra E of A such that $E = E^2 \neq 0$. If $E \supsetneqq E_1$, E_1 is *proper* in E. If $E_1 E_2 = 0 = E_2 E_1$, E_1 and E_2 are *orthogonal*. If an idempotent E cannot be written as $E = E_1 + E_2$ where E_1 and E_2 are proper orthogonol idempotents in E, then E is a *primitive* idempotent. The *identity* of A is $1_A = A^{(\infty)} = \bigcap_{i=1}^{\infty} A^{(i)}$; where $A^{(1)} = A^2$ and $A^{(i+1)} = A^{(i)2}$ for all i. An idempotent E of A is *central* if either $1_A = E$ or $1_A = E + F$ where E and F are orthogonol idempotents. If $A = A^2 \neq 0$, A is an *idempotent algebra*. And A is *primitive* if A is a primitive idempotent of A.

Note that $1_A = 0$ if and only if A is *solvable* in the sense that $A^{(i)} = 0$ for some i. For A nonsolvable, 1_A is an idempotent of A and 1_A contains every idempotent E of A. If $A = A^2 \neq 0$, then $A = 1_A$, in which case A is an idempotent algebra. If E is a central idempotent of A, we have $1_A E = E 1_A = E$, since $1_A = E + F$ where $(E + F)E = E(E + F) = E$.

It is possible to align our language even more closely with the classical theory of idempotents by noting that each central idempotent E of A determines a unique minimal central idempotent, called $1_A - E$, such that $1_A - E$ and E are orthogonal and such that $1_A = E + (1_A - E)$. For if $1_A = E + F = E + G$ where F and G are central idempotents orthogonal to A, then $1_A = L_A^2 = E + FG = E + F \cap G = E + (F \cap G)^{(\infty)} = E + H$ where H is the central idempotent $(F \cap G)^{(\infty)}$ contained in $F \cap G$.

THEOREM 2.2. *A nonassociative algebra A has only finitely many central primitive idempotents $E_1, \cdots, E_n$. They are pairwise orthogonol and their sum is $1_A = E_1 + \cdots + E_n$. Every central* *idempotent E of A is the sum $E = \sum_{EE_i \neq 0} E_i$ of those E_i not orthogonol to E. In particular, A has only finitely many central idempotents.*

Proof. We claim first that any central idempotent E of A can be written as $E = E_1 + \cdots + E_m$ where the E_i are pairwise orthogonol central primitive idempotents. We use induction on the dimension of E. If E is primitive (as when E has dimension 1), we take $E = E_1$. Otherwise, we can write $E = F + G$ where F and G are proper orthogonol idempotents. Since E is central, so are F and G. By induction, we may write both F and G, and therefore also E, as sum $E = E_1 + \cdots + E_m$ of pairwise orthogonol central primitive idempotents, as claimed. Since either $1_A = E$ or $1_A = E + F$ where $[E, F] = 0$ and F is a central idempotent, we can write $F = E_{m+1} + \cdots + E_n$ and $1_A = E_1 + \cdots + E_n$ where the E_i are pairwise orthogonal central primitive idempotents for $1 \leqq i \leqq n$. Let P be any central primitive idempotent. Then $P = 1_A P = P 1_A = PE_1 + \cdots + PE_n = E_1 P + \cdots + E_n P$ and $PE_i \cup E_i P \subset P \cap E_i$ for all i. Thus, $PE_i \neq 0$ for some i, say $i = 1$, without loss of generality. We claim that $P = E_1$, since $PE_1 \neq 0$. We have $P = P^{(\infty)} = P_1 + \cdots + P_n$ where $P_j = (P \cap E_j)^{(\infty)}$. Since $P_i^2 = P_i$ and $P_j P_i = 0 = P_i P_j$ for $i \neq j$, $P = P_j$ for some j. Thus, $P \subset E_j$. Since $PE_1 \neq 0$, we have $j = 1$ and $P \subset E_1$. If $P = 1_A$, then $1_A = P = E_1$, and we are done. Otherwise, write $1_A = P + Q$ where P and Q are orthogonol central idempotents. Then $E_1 = E_1 1_A = E_1 P + E_1 Q = P + E_1 \cap Q = P + P'$ where $P' = (E_1 \cap Q)^{(\infty)}$. Thus, $P' = 0$ and $E_1 = P$; for otherwise P' is an idempotent orthogonol to P and E_1 is not primitive. □

THEOREM 2.3. *Let G be the connected component of the identity of the automorphism group* $\operatorname{Aut} A$ *of a nonassociative algebra A. Then G and its Lie algebra $\dot{G}$ stabilize each central idempotent of A. If the characteristic is 0, the central idempotents are stable under the derivations of A. And if A is a Lie algebra of characteristic 0, the central idempotents are ideals of A.*

Proof. The subgroup H of elements of G which stabilize each central idempotent of A is closed. Furthermore, G permutes the central idempotents of A. Since there are only finitely many, by Theorem 2.2, $G: H$ is finite. But then H is open, since H and its finitely many cosets are closed. Thus, H is open and closed, so that $G = H$ by the connectedness of G. Thus, the central idempotents of A are stable under G, therefore under $\dot{G}$. In characteristic 0, $\dot{G} = \operatorname{Der} A$, where $\operatorname{Der} A$ is the algebra of derivations of A. If A is a Lie algebra of characteristic 0, we therefore have $\operatorname{ad} A \subset$

Der $A \subset \dot{G}$, so that the central idempotents of A are ad A-stable, that is, they are ideals of A. □

COROLLARY 2.4. *Let the central idempotents of A be $E_1, \cdots, E_n$. Then for any idempotent ideal I of 1_A, $I = I_1 + \cdots + I_n$ where I_i is an idempotent of $E_i (1 \leqq i \leqq n)$. If A is a Lie algebra, these I_i can be taken to be ideals of 1_A.*

Proof. $I = 1_A I = \sum_{i=1}^n E_i I \subset \sum_{i=1}^n E_i \cap I \subset I$ and $I = \sum_{i=1}^n I_i$ where $I_i = (E_i \cap I)^{(\infty)}$. Note that I_i is an ideals of 1_A if A is a Lie algebra. □

COROLLARY 2.5. *Suppose that L is a Lie algebra. Then the Cartan subalgebras H of $1_L = L^{(\infty)}$ are the subalgebras $H = H_1 + \cdots + H_n$ where the central primitive idempotents are $E_1, \cdots, E_n$ are H_i is a Cartan subalgebra of E_i for $1 \leqq i \leqq n$. For each such H, $H_i = E_i \cap H$ for $1 \leqq i \leqq n$.*

Proof. Each such H is a Cartan subalgebra of 1_L, since $(1_L)_0(\text{ad}\, H) = \sum_{i=1}^n (E_i)_0(\text{ad}\, H) = \sum_{i=1}^n (E_i)_0(\text{ad}\, H_i) = \sum_{i=1}^n H_i = H$. Conversly, let H be a Cartan subalgebra of $1_L = L^{(\infty)}$. Let $H_i = E_i \cap (H + \sum_{i \neq j} E_j)$ for $1 \leqq i \leqq n$, and note that $H \subset H_1 + \cdots + H_n$ since $H \subset E_1 + \cdots + E_n$. We may conclude that $H_i \subset (E_i)_0(\text{ad}\, H_i) \subset (E_i)_0(\text{ad}\, H) = E_i \cap H \subset H$ for $1 \leqq i \leqq n$, so that $H = H_1 + \cdots + H_n$. But then $H_i = E_i \cap H = (E_i)_0(\text{ad}\, H) = (E_i)_0(\text{ad}\, H_i)$ and H_i is a Cartan subalgebra of E_i for $1 \leqq i \leqq n$. □

Note that the Cartan subalgebra H, in the above theorem, is split if and only if H_i is split for $1 \leqq i \leqq n$. In the proofs of Theorems 2.6 and 3.3, we make use of $[H_i, H_j] = 0$ for $i \neq j$ to conclude that $R(X_i \cup X_j) = R(H_1 \cup \cdots \cup H_n) = R(H_1 + \cdots + H_n) = R(H)$.

THEOREM 2.6. *Let H be a split Cartan subalgebra of an idempotent Lie algebra L, and let $R = R_1 \cup \cdots \cup R_n$ be the decomposition of the set R of roots of H into its connected components R_i $(1 \leqq i \leqq n)$ in the symmetric G-locology for R where $G = F(H, k)$. Then*

(1) *R_i is open and closed for $1 \leqq i \leqq n$;*

(2) *the ideals $E_i = \langle L_{R_i} \rangle = L_{R_i} + H_{R_i}$ $(1 \leqq i \leqq n)$ are the central primitive idempotents of L so that $L = E_1 + \cdots + E_n$, $[E_i, E_j] = 0$ for $i \neq j$;*

(3) *L is primitive if and only if R is connected.*

Proof. Let $E_1, \cdots, E_m$ be the central primitive idempotents of

L and $H_i = H \cap E_i$ $(1 \leqq i \leqq m)$. By Theorem 2.2 and Corollary 2.5, $L = E_1 + \cdots + E_m$, H_i is a split Cartan subalgebra of $E_i (1 \leqq i \leqq m)$ and $H = H_1 + \cdots + H_m$. Let $X_i = \bigcup_{j=1}^{m} H_j - H_i$ and $R_i = R(x_i)$ $(1 \leqq i \leqq m)$. We claim that the R_i, which are closed, are also open; and that the R_i are, in fact, the connected components of R. Note first that $R_i \cap R_j = R(X_i \cup X_j) = R(H) = \phi$ for $i \neq j$. Next, let $a \in R$, so that $0 \neq L_a(\mathrm{ad}\, H) = \sum (E_i)_a(\mathrm{ad}\, H_i)$ and $0 \neq (E_i)_a(\mathrm{ad}\, H_i)$ for some i. Then $0 = (E_i)_a(\mathrm{ad}\, H_j)$ since $[E_i, E_j] = 0$, so that $a(H_j) = 0$ for $i \neq j$. Thus, $a \in R(X_i) = R_i$. It follows that $R = R_1 \cup \cdots \cup R_m$ (disjoint union of closed sets). Furthermore, $a(H_i) \neq 0$, and we see easily that R_i therefore is also $R_i = R - R(H_i)$, an open set $(1 \leqq i \leqq m)$. Moreover, we see that $R_i = \{a \in R \mid (L_i)_a(\mathrm{ad}\, H) \neq \{0\}\}$ $(1 \leqq i \leqq m)$. Since $R_i \cap R_j = \phi$ for $i \neq j$, it follows that E_i contains L_{R_i} and $E_i \cap L_{R_j} = 0$ for $1 \leqq i, j \leqq m$ and $i \neq j$. Since R_i is open, closed and symmetric, $F_i = L_{R_i} + H_{R_i}$ is an ideal of L $(1 \leqq i \leqq m)$. Since $E_i \supset \langle L_{R_i} \rangle = F_i$, since $F_i^2 = F_i$ $(1 \leqq i \leqq m)$ and since $L = L^2 = L_R + H_R = F_1 + \cdots + F_m$, the F_i are central idempotents of L. It follows easily from Theorem 2.2 that $E_i = F_i$, so that $E_i = L_{R_i} + H_{R_i}$ $(1 \leqq i \leqq m)$. For (1) and (2), it now remains only to show that R_i is connected. Thus, suppose that $R_i = S \cup T$ (disjoint union) where S, T are relatively closed and symmetric in R_i. Since S and T are relatively closed and symmetric in R_i, and disjoint, S and T are relatively open in R_i. It follows that, in the Lie algebra $L_i = L_{R_i} + H$, S and T are open, closed and symmetric. Thus, $[L_S, L_T] = 0$ by Corollary 1.2, since $S \cap T = \phi$. It follows that $E_i = L_{R_i} + H_{R_i} = E + F$ where $E = L_S + H_S$, $F = L_T + H_T$, $E^2 = E$, $F^2 = F$, $EF = 0$. Since E_i is primitive, $E_i = E$ or $F_i = F$ and $T = \phi$ or $S = \phi$. It follows that R_i is connected $(1 \leqq i \leqq m)$. In particular $m = n$. Now (3) follows from (1) and (2), and all assertions have been established. □

COROLLARY 2.7. *For a Lie algebra L with split Cartan subalgebra H and set R of roots, if L is semisimple (characteristic 0) or classical (characteristic $p > 0$), then the connected components R_i of R in the symmetric G-locology are the irreducible root systems of R in the sense of Bourbaki* [1]. □

In the proof of Theorem 2.6, it is actually shown that the R_i are open and closed in the locology $\{R(x) \mid X \subset H\}$ which, a priori, is a coarser locology than the symmetric G-locology. On the other hand, the R_i are also the connected components of R in the symmetric G-topology of R.

3. Ideal structure and locology of a Lie algebra and its root spaces. In this section, we consider a finite dimensional Lie algebra

L with split Cartan subalgebra H and corresponding set R of roots with the symmetric G-locology of 1.2, 1.20.

THEOREM 3.1. *Let* $L = L_1 + \cdots + L_n$ *(sum of ideals) where* $[L_i, L_j] = 0$ *for* $1 \leqq i, j \leqq n$ *and* $i \neq j$. *Then*

(1) $H = H_1 + \cdots + H_n$ *and* $R = R_1 \cup \cdots \cup R_n$ *(disjoint) where* $H_i = H \cap L_i$ *and* $R_i = \{a \in R \mid (L_i)_a(\mathrm{ad}\, H) \neq 0\}$ *for* $1 \leqq i \leqq n$;

(2) R_i *is open and closed,* H_i *is a Cartan subalgebra of* L_i *and* $L_i = H_i + L_{R_i}$ *for* $1 \leqq i \leqq n$;

(3) $L^\infty = \sum L_i^\infty$, $L_i^\infty = L_{R_i} + H_{R_i}$ *and* $[L, L_i^\infty] = L_i^\infty$ *for* $1 \leqq i \leqq n$.

Proof. As in the proof of Theorem 2.6, we see that $H = H_1 + \cdots + H_n$, $R = R_1 \cup \cdots \cup R_n$ (disjoint), R_i is open and closed and H_i is a Cartan subalgebra of L_i for $1 \leqq i \leqq n$. For $a \in R_i$, we have $a \notin R_j$ and therefore $(L_j)_a(\mathrm{ad}\, H) = 0$ for $i \neq j$. It follows that the decomposition of L_i under $\mathrm{ad}\, H$ is $L_i = H_i + \sum_{a \in R_i} L_a = H_i + L_{R_i}$. Clearly $L^\infty = L_1^\infty + \cdots + L_n^\infty$, since $[L_i, L_j] = 0$ for $i \neq j$. Since $L_i \supset L_{R_i}$ and $[L, L_i^m] = L_i^{m+1}$ for all m, we have $L_i \supset L_{R_i}$, $L_i^2 = [L, L_i] \supset L_{R_i}, \cdots$. Thus $L_i^\infty \supset L_{R_i}$. Since $L_{R_i} + H_{R_i}$ is and ideal of L_i and $L_i/(L_{R_i} + H_{R_i})$ is nilpotent, we also have $L_i \subset L_{R_i} + H_{R_i}$, so that $L_i = L_{R_i} + H_{R_i}$ for $1 \leqq i \leqq n$. That $[L, L_i^\infty] = L_i^\infty$ is clear since $L = L_1 + \cdots + L_n$ and $[L_i, L_j] = 0$ for $i \neq j$.

The following theorem is proved in Winter [3] and, under a stronger hypothesis, in Winter [2].

THEOREM 3.2. *Let* L *be a Lie algebra,* I *and ideal of* L. *Suppose that either the characteristic* p *of* L *is* 0 *or* $(\mathrm{ad}_I I)^p \subset \mathrm{ad}_I I$. *Then* $I_0(\mathrm{ad}\,(H \cap I))$ *is a Cartan subalgebra of* I *for every Cartan subalgebra* H *of* L. □

THEOREM 3.3. *Let* I *be an ideal of* L *and suppose that* $I_0(\mathrm{ad}\, H \cap I)$ *is a Cartan subalgebra of* I. *Let* $I = I_1 + \cdots + I_n$ *(sum of ideals) where* $[I_i, I_j] = 0$ *for* $1 \leqq i, j \leqq n$ *and* $i \neq j$. *Then*

(1) $H_I = H_1 + \cdots + H_n$ *and* $R_i = R_1 \cup \cdots \cup R_n \cup S$ *(disjoint) where* $H_I = H \cap I$, $R_i = \{a \in R \mid I_a(H_I) \neq 0\}$, $S = R_I(H_I)$ *and* $R_i = \{a \in R - S \mid I_{ia}(H_I) \neq 0\}$ *for* $1 \leqq i \leqq n$;

(2) R_i *is relatively open and closed in* $R_I - S$, $H_i + I_{iS}$ *is a Cartan subalgebra of* I_i *and* $I_i = (H_i + I_{iS}) + I_{R_i}$ *for* $1 \leqq i \leqq n$.

Proof. $I_0(\mathrm{ad}\, I_H) = H_I + I_S$ is a Cartan subalgebra of I by Theorem 3.2. We have $H_I = I_0(\mathrm{ad}\, H) = \sum_{i=1}^n I_{i0}(\mathrm{ad}\, H) = \sum_{i=1}^n H_i$. Letting $X_i = \bigcup_{j=1}^n H_j - H_i$ and $\hat{R}_i = R_I(X_i)$ for $1 \leqq i \leqq n$, we have $\hat{R}_i \cap \hat{R}_j = R_I(X_i \cup X_j) = R(H_1 \cup \cdots \cup H_n) = R(H_1 + \cdots + H_n) = R_I(H_I) = S$

for all $i \neq j$. Here, we use the fact that $[h_i, h_j] = 0$ $(h_i \in H_i)$ for all $i \neq j$ implies that $a(h_1 + \cdots + h_n) = a(h_1) + \cdots + a(h_n)$. Let R_i be the complement of S in $\widehat{R_i}$, so that $R_i \cap R_j = \phi$ for $i \neq j$. For $a \in R_I - S$, we have $0 \neq I_{ia}(H_I) = I_{ia}(\operatorname{ad} H_i)$ for some i; and therefore $a(H_j) = 0$ for $j \neq i$; and therefore $a(H_i) \neq 0$ and $a \in \widehat{R_i} - S = R_i$. It follows that $R_I = R_1 \cup \cdots \cup R_n \cup S$ (disjoint), with R_i relatively open and closed in $R_I - S$. It also follows that $I_i = I_{i0}(\operatorname{ad} H_I) + \sum_{a \in R_i} I_a = (H_i + I_{iS}) + I_{R_i}$. As in the proof of Theorem 3.1, $K = H_I + I_S$ is Cartan subalgebra of I implies that $K_i = K \cap I_i = H_i + I_{iS}$ is a Cartan subalgebra of I_i for $1 \leqq i \leqq n$. □

We can now improve Corollary 1.21 and use it and Theorem 3.3 to prove that if $H_\infty = H \cap L^\infty$ is a Cartan subalgebra of L, the connected components R_i of R in the symmetric G-locology are both open and closed. Whether this is true when H_∞ is not a Cartan subalgebra of L^∞ is an open question, the answer of which is probably negative.

THEOREM 3.4. *Let $R_1, \cdots, R_n$ be the connected components of R, in the symmetric G-locology, and let $L_i = L_{R_i} + H_{R_i}$ $(1 \leqq i \leqq n)$. Then $[L_i, L_i] \subset L_i$, $[L_i, L_j] = 0$ for $i \neq j$ and $L^\infty = L_1 + \cdots + L_n$.*

Proof. Choose a decomposition $R = R_1 \cup \cdots \cup R_n$ (disjoint) with n maximal satisfying all of the following conditions:

(1) The R_i are closed, nonempty, pairwise disjoint;

(2) every connected subset of R is contained in some R_i;

(3) the conclusion of the Theorem 3.4 holds.

We claim that the R_i are the connected components of R, that is, that each R_i is connected. If R_n is not connected, then $R_n = R'_n \cup R'_{n+1}$ (nonempty, closed, disjoint) and each connected subset of R_n is either in R'_n or in R'_{n+1}. In the context of the Lie algebra $L_n = L_{R_n} + H_{R_n}$, R'_n and R'_{n+1} are relatively closed and open, so that $L_n = L_a + L_b$ with $L_a^2 \subset L_a$, $L_b^2 \subset L_b$, $[L_a, L_b] = 0$ where $L_a = L_{R'_n} + H_{R'_n}$ and $L_b = L_{R'_{n+1}} + H_{R'_{n+1}}$. Thus, $R_1, \cdots, R_{n-1}, R'_n, R'_{n+1}$ satisfies conditions (1), (2), (3), a contradition. We must conclude that R_n (and, similarly, R_i for all i) is connected as asserted. Note that the assertion $L^\infty = L_1 + \cdots + L_n$ is verified as in Corollary 1.21. □

COROLLARY 3.5. *Suppose that $H_\infty = H \cap L^\infty$ is a Cartan subalgebra of L^∞. Then*

(1) *the connected components R_i $(1 \leqq i \leqq n)$ of R are both open and closed;*

(2) *H_{R_i} is a Cartan subalgebra of $L_{R_i} + H_{R_i} = L_i$ $(1 \leqq i \leqq n)$*

and $L^\infty = L_1 + \cdots + L_n$ *(sum of ideals of* L*) where* $[L_i, L_j] = 0$ *for* $i \neq j$.

Proof. We have (2) by Theorem 3.4 and the hypothesis. Thus, by Theorem 3.3, R_i is open and closed in $R_I - S = R - S = R - R(H_\infty) = R - \phi = R$. □

Finally, we note that Theorem 3.4 is in the direction of a converse to Theorem 3.1. It provides a decomposition $L^\infty = L_1 + \cdots + L_n$ where $L_i = L_{R_i} + H_{R_i}$ and the R_i are the connected components of R. It follows immediately that the same is true if the $R_1, \cdots, R_n$ are pairwise disjoint and every connected component of R is contained in one of the R_i as is the case when $R = R_1 \cup \cdots \cup R_n$ is disjoint union of open and closed sets (the situation which immerges in Theorem 3.1). Although it may not be possible to lift such a decomposition $L^\infty = L_1 + \cdots + L_n$ to a decomposition $L = \bar{L}_1 + \cdots \bar{L}_n$ of L (compare with the hypothesis of Theorem 3.1), the following lifting is possible when H is abelian.

THEOREM 3.6. *Let* H *be abelian and let* $L^\infty = L_1 + \cdots + L_n$ *with* $L_i = L_{R_i} + H_{R_i}$, $R = R_1 \cup \cdots \cup R_n$ *(disjoint) and* $[L_i, L_i] \subset L_i$, $[L_i, L_j] = 0$ *for all* $i \neq j$. *Then there is a Lie algebra* $\hat{L}$ *containing* L *as ideal and decomposition* $\hat{L} = \hat{L}_1 + \cdots + \hat{L}_n$ *(sum of ideals such that* $[\hat{L}_i, \hat{L}_j] = 0$ *for* $i \neq j$ *and* $\hat{L}_i \cap L = L_i$ $(1 \leqq i \leqq n)$.

Proof. L is ideal of $M = (\mathrm{Der}\, L) \oplus L$ (semidirect) where $[D, x] = D(x)$ for $D \in \mathrm{Der}\, L$, $x \in L$. Let $h \in H$ and define $D_i: L \to L$ so that D_i is linear, $D_i(H) = 0$, $D_i|_{L_{R_i}} = \mathrm{ad}\, h|_{L_{R_i}}$. $D_i(L_{R_j}) = 0$ for $i \neq j$. One easily verifies that $D_i \in \mathrm{Der}\, L$ $(1 \leqq i \leqq n)$. Since D_i depends on h, we use the notation $D_i = D_i(h)$. The span $\hat{H}_0$ of $\{D_i(h) \mid 1 \leqq i \leqq n, h \in H\}$ is a commutative subalgebra of $\mathrm{Der}\, L$ and we let $\hat{L} = \hat{H}_0 + L$ and $\hat{H} = \hat{H}_0 + H$. Clearly $\hat{H}$ is a Cartan subalgebra of $\hat{L}$. Let $\hat{H}_i = \{x \in \hat{L} \mid [x, L_j] = 0$ for all $i \neq j$ and $[x, H] = 0\}$. We claim that $\hat{H} = \hat{H}_1 + \cdots + \hat{H}_n$. Clearly, $\hat{H}_1 + \cdots + \hat{H}_n$ contains $\hat{H}_0$. Let $h \in H$ and $x = h - (D_1(h) + \cdots + D_n(h))$. Then $[x, L_i] = 0$ for $1 \leqq i \leqq n$. Furthermore, $[x, H_0] = 0$. Finally, $[x, \hat{H}_0] = 0$. It follows that x centralizes $\hat{L}$. In particular, $x \in \hat{L}_0(\mathrm{ad}\, \hat{H}) = \hat{H}$. It follows that $x \in \hat{H}_i$ for all i and that $h = x + D_1(h) + \cdots + D_n(h) \in \hat{H}_1 + \cdots + \hat{H}_n$. Thus, $H \subset \hat{H}_1 + \cdots + \hat{H}_n$, so that $\hat{H} \subset \hat{H}_1 + \cdots + \hat{H}_n$. Since $[\hat{H}_1, H] = 0$ and $[\hat{H}_i, L_j] = 0$ for $i \neq j$, we have $[\hat{H}_i, D_i(H)] = 0$ and $[\hat{H}_i, D_j(H)] = 0$ for $i \neq j$, so that $[\hat{H}_i, \hat{H}_0] = 0$. It follows that $\hat{H}_i \subset \hat{L}_0(\mathrm{ad}\, \hat{H}) = \hat{H}$ $(1 \leqq i \leqq n)$. Thus, $\hat{H} = \hat{H}_1 + \cdots + \hat{H}_n$. Let $\hat{L}_i = \hat{H}_i + L_i$ $(1 \leqq i \leqq n)$. It is then evident that $\hat{L} = \hat{L}_1 + \cdots + \hat{L}_n$ is a decomposition satisfying the asserted conditions. □

Clearly, the R_i in Theorem 3.6 are open and closed in R in the locology defined by $\hat{H}$.

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Received March 10, 1980. This reserch was supported in part by the National Science Foundation.

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ANN ARBOR, MI 48109

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 99, No. 1 May, 1982