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RESOLUTION OF AMBIGUITIES IN THE EVALUATION OF CUBIC AND QUARTIC JACOBSTHAL SUMS

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If $p \equiv 1 \pmod{2k}$ is a prime, the Jacobsthal sum $\Phi_k(D)$ is defined by

$$\Phi_k(D) = \sum_{x=1}^{p-1} \left(\frac{x(x^k + D)}{p} \right) \quad (k = 2, 3, \dots).$$

It is shown how to evaluate $\Phi_2(D)$ and $\Phi_3(D)$ for any integer D .

1. Introduction. The Jacobsthal sum $\Phi_k(D)$ is defined for primes $p \equiv 1 \pmod{2k}$ by

$$(1.1) \quad \Phi_k(D) = \sum_{x=1}^{p-1} \left(\frac{x(x^k + D)}{p} \right), \quad k = 2, 3, \dots,$$

where $\left(\frac{\cdot}{p} \right)$ is the Legendre symbol, and D is an integer not divisible by p . It is well-known (see for example [8: p. 104]) that

$$(1.2) \quad \Phi_k(Dm^k) = \left(\frac{m}{p} \right)^{k-1} \Phi_k(D), \quad m \not\equiv 0 \pmod{p}.$$

In this paper, we show how to resolve the sign ambiguities in the evaluations of $\Phi_2(D)$ and $\Phi_3(D)$. (For a discussion of Jacobsthal sums see, for example, [7], [14], [1].)

2. $k = 2$. In this case $p \equiv 1 \pmod{4}$ and there are integers a and b such that

$$(2.1) \quad p = a^2 + b^2, \quad a \equiv 1 \pmod{4}, \quad b \equiv 0 \pmod{2},$$

with a and $|b|$ unique. Relation (1.2) gives in this case

$$(2.2) \quad \Phi_2(Dm^2) = \left(\frac{m}{p} \right) \Phi_2(D), \quad m \not\equiv 0 \pmod{p},$$

so that it suffices to consider $\Phi_2(D)$ for squarefree D . Choosing m such that $m^2 \equiv -1 \pmod{p}$ in (2.2), we have

$$(2.3) \quad \Phi_2(-D) = (-1)^{(p-1)/4} \Phi_2(D),$$

so that we may take D positive. Jacobsthal [7: pp. 240-241] has evaluated $\Phi_2(1)$. He has shown that

$$(2.4) \quad \Phi_2(1) = -2a,$$

and thus, by (2.2), for any D with $(D/p) = +1$, say $D \equiv E^2 \pmod{p}$,

one has

$$(2.5) \quad \Phi_2(D) = -\left(\frac{E}{p}\right)2a.$$

Thus it suffices to consider $\Phi_2(D)$ for quadratic nonresidues D . Emma Lehmer [8: p. 107] has shown that, if $(2/p) = -1$,

$$(2.6) \quad \Phi_2(2) = \mp 2b \quad \text{according as } b \equiv \pm 2 \pmod{8}$$

and it follows from the work of Brewer [2: p. 243] and (2.3), that, if $(3/p) = -1$,

$$(2.7) \quad \Phi_2(3) = \pm(-1)^{(p-1)/4}2b \quad \text{according as } a \equiv \pm b \pmod{3}.$$

Jacobsthal [7: p. 241] has shown, for an arbitrary D satisfying $(D/p) = -1$, that

$$(2.8) \quad \Phi_2(D) = \pm 2b,$$

and we begin in Theorem 1 by showing how to determine the correct sign in (2.8), when D is an odd prime q satisfying $(q/p) = -1$. Afterwards we illustrate how to prove the results for composite D .

Let q be an odd prime satisfying $(q/p) = -1$, so that $ab \not\equiv 0 \pmod{q}$. If $q \equiv 1 \pmod{4}$, there are unique positive integers r and s such that

$$(2.9) \quad q = r^2 + s^2, \quad r \equiv 1 \pmod{2}, \quad s \equiv 0 \pmod{2}.$$

Clearly r and s are not divisible by q . We define a set K , depending only on q , by

$$(2.10) \quad K = \left\{ k: -\frac{1}{2}(q-1) \leq k \leq \frac{1}{2}(q-1), \right. \\ \left. r(sk+r)^{(q-1)/4} - s(sk-r)^{(q-1)/4} \equiv 0 \pmod{q} \right\}.$$

Clearly $0 \notin K$. It is known that (see for example [4: p. 65])

$$(2.11) \quad q^{(p-1)/4} \equiv \pm a/b \pmod{p} \quad \text{according as } a \equiv \pm kb \pmod{q}$$

for some $k \in K$.

If $q \equiv 3 \pmod{4}$, we define K by

$$(2.12) \quad K = \left\{ k: -\frac{1}{2}(q-1) \leq k \leq \frac{1}{2}(q-1), \right. \\ \left. (k+i)^{(q+1)/4} - i(k-i)^{(q+1)/4} \equiv 0 \pmod{q} \right\}.$$

Again we have $0 \notin K$. Further

(2.13) $(-q)^{(p-1)/4} \equiv \pm a/b \pmod{p}$ according as $a \equiv \pm kb \pmod{q}$ for some $k \in K$.

We prove the following theorem.

THEOREM 1. *Let p be a prime congruent to 1 modulo 4 and define a and b by (2.1). Let q be an odd prime satisfying $(q/p) = -1$. Then, if $q \equiv 1 \pmod{4}$,*

(2.14) $\Phi_2(q) = \pm 2b$ according as $a \equiv \pm kb \pmod{q}$ for some $k \in K$; if $q \equiv 3 \pmod{4}$,

(2.15) $\Phi_2(q) = \pm(-1)^{(p-1)/4} 2b$, if $a \equiv kb \pmod{q}$ for some $k \in K$.

Proof. Emma Lehmer [10: p. 65] has proved that for $D \not\equiv 0 \pmod{p}$,

$$(2.16) \quad D^{(p-1)/4} \equiv \Phi_2(D)/\Phi_2(1) \pmod{p}.$$

Taking $D = q \equiv 1 \pmod{4}$ in (2.16), and appealing to (2.4) and (2.11), we obtain

$$\Phi_2(q) \equiv \begin{cases} +2b \pmod{p}, & \text{if } a \equiv kb \pmod{q} \text{ for some } k \in K, \\ -2b \pmod{p}, & \text{if } a \equiv -kb \pmod{q} \text{ for some } k \in K. \end{cases}$$

Since $\Phi_2(q) = \pm 2b$, by (2.8), and as $2b \not\equiv 0 \pmod{p}$, we obtain (2.14). The case $q \equiv 3 \pmod{4}$ is similar.

We illustrate Theorem 1 by giving $\Phi_2(q)$ for odd primes $q \leq 19$ satisfying $(q/p) = -1$; $\alpha(p) = (p - 1)/4$ with the upper signs and $(p + 3)/4$ with the lower signs.

| q | $\Phi_2(q)$ | k satisfying $a \equiv kb \pmod{q}$ |
|-----|-----------------------|---------------------------------------|
| 3 | $(-1)^{\alpha(p)} 2b$ | ± 1 |
| 5 | $\pm 2b$ | ∓ 1 |
| 7 | $(-1)^{\alpha(p)} 2b$ | $\mp 2, \mp 3$ |
| 11 | $(-1)^{\alpha(p)} 2b$ | $\mp 1, \mp 3, \mp 4$ |
| 13 | $\pm 2b$ | $\pm 1, \pm 2, \mp 6$ |
| 17 | $\pm 2b$ | $\pm 2, \mp 3, \pm 6, \pm 8$ |
| 19 | $(-1)^{\alpha(p)} 2b$ | $\pm 1, \mp 3, \pm 6, \mp 7, \pm 8$ |

The case $q = 3$ constitutes the result of Brewer (2.7).

We remark that these results can be combined to determine $\Phi_2(D)$ when D is composite and $(D/p) = -1$. We treat the case $D = 6 = 2 \times 3$. If $(6/p) = -1$, we have $(2/p) = +1$, $(3/p) = -1$, or $(2/p) =$

-1 , $(3/p) = +1$. In the former case, we have

$$(2.17) \quad \Phi_2(2) = -\left(\frac{2}{p}\right)_4 2a = -(-1)^{b/4} 2a,$$

and

$$(2.18) \quad \Phi_2(3) = \pm 2b \quad \text{according as } a \equiv \pm b \pmod{3}.$$

From (2.16) we obtain

$$(2.19) \quad \Phi_2(6) \equiv \frac{\Phi_2(2)\Phi_2(3)}{\Phi_2(1)} \pmod{p},$$

and so

$$\Phi_2(6) \equiv \begin{cases} (-1)^{b/4} 2b \pmod{p}, & \text{if } a \equiv b \pmod{3}, \\ (-1)^{b/4+1} 2b \pmod{p}, & \text{if } a \equiv -b \pmod{3}. \end{cases}$$

Hence, by (2.8), we have

$$(2.20) \quad \Phi_2(6) = \begin{cases} (-1)^{b/4} 2b, & \text{if } a \equiv b \pmod{3}, \\ (-1)^{b/4+1} 2b, & \text{if } a \equiv -b \pmod{3}. \end{cases}$$

The case when $(2/p) = -1$, $(3/p) = +1$ can be treated similarly.

3. $k = 3$. In this case $p \equiv 1 \pmod{6}$ and there are integers L and M such that

$$(3.1) \quad 4p = L^2 + 27M^2, \quad L \equiv 1 \pmod{3},$$

with L and $|M|$ unique. Clearly we have $L \equiv M \pmod{2}$. Relation (1.2) gives in this case

$$(3.2) \quad \Phi_3(Dm^3) = \Phi_3(D), \quad m \not\equiv 0 \pmod{p},$$

so that it suffices to consider $\Phi_3(D)$ for cubefree D . Clearly $\Phi_3(-D) = \Phi_3(D)$, so that we may take D positive. It follows from the work of von Schrutka [13: p. 258] (see also Chowla [3: p. 246], Whiteman [14: p. 96]) that

$$(3.3) \quad \Phi_3(1) = \begin{cases} L - 1, & \text{if } L \equiv M \equiv 0 \pmod{2}, \\ \frac{1}{2}(-L + 9M - 2), & \text{if } L \equiv M \equiv 1 \pmod{2} \\ & \text{and } L \equiv M \pmod{4}, \\ \frac{1}{2}(-L - 9M - 2), & \text{if } L \equiv M \equiv 1 \pmod{2} \\ & \text{and } L \equiv -M \pmod{4}. \end{cases}$$

From (3.2), $\Phi_3(k) = \Phi_3(1)$ for any cubic residue k modulo p , so that (3.3) gives unambiguously the value of $\Phi_3(k)$ for any cubic residue $k \pmod{p}$. Now 2 is a cubic residue \pmod{p} if and only if $L \equiv M \equiv 0$

(mod 2) [6: p. 68]. Thus we have

$$(3.4) \quad \Phi_3(1) = \Phi_3(2) = \Phi_3(4) = L - 1, \\ \text{if } 2 \text{ is a cubic residue (mod } p).$$

When 2 is not a cubic residue (mod p), so that $L \equiv M \equiv 1 \pmod{2}$, Emma Lehmer [8: p. 112] has proved that

$$(3.5) \quad \Phi_3(2) = \begin{cases} \frac{1}{2}(-L - 9M - 2), & \text{if } L \equiv M \pmod{4}, \\ \frac{1}{2}(-L + 9M - 2), & \text{if } L \equiv -M \pmod{4}, \end{cases}$$

and

$$(3.6) \quad \Phi_3(4) = L - 1.$$

For an arbitrary cubic nonresidue D , it is known that

$$(3.7) \quad \Phi_3(D) = \begin{cases} \left(\frac{1}{2}(-L - 9M - 2), & \text{if } L \equiv M \equiv 0 \pmod{2}, \right. \\ \text{or} \\ \left. \frac{1}{2}(-L + 9M - 2), \right. \end{cases}$$

$$(3.8) \quad \Phi_3(D) = \begin{cases} \left(L - 1 & \text{if } L \equiv M \equiv 1 \pmod{2} \text{ and} \right. \\ \text{or} \\ \left. \frac{1}{2}(-L - 9M - 2), & L \equiv M \pmod{4}, \right. \end{cases}$$

and

$$(3.9) \quad \Phi_3(D) = \begin{cases} \left(L - 1 & \text{if } L \equiv M \equiv 1 \pmod{2} \text{ and} \right. \\ \text{or} \\ \left. \frac{1}{2}(-L + 9M - 2), & L \equiv -M \pmod{4}. \right. \end{cases}$$

It is our purpose in Theorem 2 to show how to eliminate the ambiguities in (3.7), (3.8), and (3.9) when D is an odd prime q , which is a cubic nonresidue (mod p). (As q is a cubic nonresidue (mod p) we have $LM \not\equiv 0 \pmod{q}$ [9: p. 26].)

Our starting point is the congruence

$$(3.10) \quad q^{(p-1)/3} \equiv (\Phi_3(q) + 1)/(\Phi_3(1) + 1) \pmod{p},$$

which is given in [10: p. 66]. From (3.10) we obtain

$$(3.11) \quad \Phi_3(q) \equiv -1 + q^{(p-1)/3}(\Phi_3(1) + 1) \pmod{p}.$$

For $q \geq 5$, one of us [15: p. 282] has shown that there exists a set of integers \mathcal{L} (depending only on q) such that

$$(3.12) \quad q^{(p-1)/3} = \begin{cases} (L + 9M)/(L - 9M) \pmod{p}, & \text{if } L \equiv kM \pmod{q} \\ & \text{for some } k \in \mathcal{L}, \\ (L - 9M)/(L + 9M) \pmod{p}, & \text{if } L \equiv -kM \pmod{q} \\ & \text{for some } k \in \mathcal{L}. \end{cases}$$

It is shown in [15] that, if $q \equiv 1 \pmod{3}$,

$$(3.13) \quad \mathcal{L} = \{-\frac{1}{2}(q-1) \leq k \leq \frac{1}{2}(q-1) : (k^2 + 27)^{2(q-1)/3}(k + 3 + 6w)^{2(q-1)/3} \equiv w \pmod{q}\},$$

and, if $q \equiv 2 \pmod{3}$,

$$(3.14) \quad \mathcal{L} = \{-\frac{1}{2}(q-1) \leq k \leq \frac{1}{2}(q-1) : (k^2 + 27)^{(q-2)/3}(k + 3 + 6w)^{(q+1)/3} \equiv w \pmod{q}\},$$

where $w = \exp(2\pi i/3) = \frac{1}{2}(-1 + \sqrt{-3})$. In particular, we have (see [15: p. 283])

$$\begin{aligned} \mathcal{L} &= \{+1, -2\}, & \text{if } q = 5, \\ \mathcal{L} &= \{+2, -3\}, & \text{if } q = 7, \\ \mathcal{L} &= \{-1, -2, -3, +5\}, & \text{if } q = 11. \end{aligned}$$

Appealing to (3.3), (3.7), (3.8), (3.9), (3.11), and (3.12), we obtain

THEOREM 2. *Let p be a prime congruent to 1 modulo 6 and define L and M by (3.1). Let $q \geq 5$ be an odd prime, which is a cubic nonresidue \pmod{p} . Then*

$$\Phi_3(q) = \left\{ \begin{array}{l} L - 1, \text{ if } L \equiv M \equiv 1 \pmod{2}, L \equiv M \pmod{4} \text{ and} \\ \quad L \equiv -kM \pmod{q} \text{ for some } k \in \mathcal{L}, \\ \text{or} \\ L \equiv M \equiv 1 \pmod{2}, L \equiv -M \pmod{4} \text{ and} \\ \quad L \equiv kM \pmod{q} \text{ for some } k \in \mathcal{L}, \\ \frac{1}{2}(-L + 9M - 2), \\ \quad \text{if } L \equiv M \equiv 1 \pmod{2}, L \equiv -M \pmod{4} \text{ and} \\ \quad L \equiv -kM \pmod{q} \text{ for some } k \in \mathcal{L}, \\ \text{or} \\ L \equiv M \equiv 0 \pmod{2} \text{ and } L \equiv kM \pmod{q} \\ \quad \text{for some } k \in \mathcal{L}, \\ \frac{1}{2}(-L - 9M - 2), \\ \quad \text{if } L \equiv M \equiv 1 \pmod{2}, L \equiv M \pmod{4} \text{ and} \\ \quad L \equiv kM \pmod{q} \text{ for some } k \in \mathcal{L}, \\ \text{or} \\ L \equiv M \equiv 0 \pmod{2} \text{ and } L \equiv -kM \pmod{q} \\ \quad \text{for some } k \in \mathcal{L}. \end{array} \right.$$

We now treat the case $q = 3$. We have from [15: Theorem 1]

$$(3.15) \quad 3^{(p-1)/3} = \begin{cases} (L + 9M)/(L - 9M) \pmod{p}, & \text{if } M \equiv -1 \pmod{3}, \\ (L - 9M)/(L + 9M) \pmod{p}, & \text{if } M \equiv 1 \pmod{3}. \end{cases}$$

As in the proof of Theorem 2 we obtain

THEOREM 3. *Let p be a prime congruent to 1 modulo 6, for which 3 is a cubic nonresidue \pmod{p} . Then*

$$\Phi_3(3) = \left\{ \begin{array}{l} L - 1, \quad \text{if } L \equiv M \equiv 1 \pmod{2}, \quad L \equiv M \pmod{4} \text{ and} \\ \quad M \equiv 1 \pmod{3}, \\ \text{or} \\ L \equiv M \equiv 1 \pmod{2}, \quad L \equiv -M \pmod{4} \text{ and} \\ \quad M \equiv -1 \pmod{3}, \\ \frac{1}{2}(-L - 9M - 2), \\ \text{if } L \equiv M \equiv 1 \pmod{2}, \quad L \equiv M \pmod{4} \text{ and} \\ \quad M \equiv -1 \pmod{3}, \\ \text{or} \\ L \equiv M \equiv 0 \pmod{2} \text{ and } M \equiv 1 \pmod{3}, \\ \frac{1}{2}(-L + 9M - 2), \\ \text{if } L \equiv M \equiv 1 \pmod{2}, \quad L \equiv -M \pmod{4} \text{ and} \\ \quad M \equiv 1 \pmod{3}, \\ \text{or} \\ L \equiv M \equiv 0 \pmod{2} \text{ and } M \equiv -1 \pmod{3}. \end{array} \right.$$

We remark that $\Phi_3(q^2)$, where q is a prime, which is a cubic nonresidue \pmod{p} , is easily determined using Theorems 2 and 3 and the relation

$$(3.16) \quad \Phi_3(1) + \Phi_3(q) + \Phi_3(q^2) = -3,$$

see for example [14: p. 92]. Moreover, as in § 2, we can treat $\Phi_3(D)$ for composite D .

Finally we remark that the ideas of this paper can be used in conjunction with results in [10], [11] and [16] to treat $\Phi_5(D)$ and $\Phi_7(D)$ for certain values of D .

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| | |
|---|-----|
| Thomas E. Armstrong and Karel Libor Prikry , On the semimetric on a Boolean algebra induced by a finitely additive probability measure | 249 |
| Walter Russell Bloom , Strict local inclusion results between spaces of Fourier transforms | 265 |
| Richard Clark Brown , Notes on generalized boundary value problems in Banach spaces. II. Infinite-dimensional extension theory | 271 |
| Sui Sun Cheng , Isoperimetric eigenvalue problem of even order differential equations | 303 |
| Lung O. Chung and Jiang Luh , Derivations of higher order and commutativity of rings | 317 |
| Ali Ahmad Fora , A fixed point theorem for product spaces | 327 |
| Barry J. Gardner , Radical classes of regular rings with Artinian primitive images | 337 |
| John Brady Garnett and Peter Wilcox Jones , BMO from dyadic BMO | 351 |
| Allen E. Hatcher , On the boundary curves of incompressible surfaces | 373 |
| Richard Howard Hudson and Kenneth S. Williams , Resolution of ambiguities in the evaluation of cubic and quartic Jacobsthal sums | 379 |
| Viktor Losert , Counter-examples to some conjectures about doubly stochastic measures | 387 |
| Kenneth Derwood Magill, Jr., P. R. Misra and Udai Bhan Tewari , Structure spaces for sandwich semigroups | 399 |
| Mark Mandelker , Continuity of monotone functions | 413 |
| Kenneth Guy Miller , An index theorem and hypoellipticity on nilpotent Lie groups | 419 |
| Evelyn M. Nelson , Homomorphisms of mono-unary algebras | 427 |
| Marvin E. Ortel , The support of an extremal dilatation | 431 |
| R. S. Pathak and O. P. Singh , Finite Hankel transforms of distributions | 439 |
| Richard Cole Penney , The theory of ad-associative Lie algebras | 459 |
| Linda Ruth Sons , Zero distribution of functions with slow or moderate growth in the unit disc | 473 |
| Russell Bruce Walker , Transversals to laminations | 483 |