COUNTER-EXAMPLES TO SOME CONJECTURES ABOUT DOUBLY STOCHASTIC MEASURES

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Some new types of doubly stochastic measures are constructed. Using measure preserving transformations, one can construct examples of nontrivial extreme doubly stochastic measures which are absolutely continuous with respect to another extreme doubly stochastic measure (disproving a conjecture by Feldman). By combinatorial arguments, one gets an extreme doubly stochastic measure that is not concentrated on a countable union of function graphs and whose support is the whole unit square.

Let $I$ be the unit interval, $m$ the ordinary Lebesgue measure on $I$. A probability measure $\mu$ on $I \times I$ is called doubly stochastic, if its marginal distributions coincide with $m$ (i.e., $\mu(A \times I) = \mu(I \times A) = m(A)$ for each Borel set $A \subseteq I$). This is thought of as a continuous analogue of the notion of a doubly stochastic matrix (see [8] for a survey of results about doubly stochastic matrices). By a theorem of G. Birkhoff and von Neumann, the extreme points of the set of doubly stochastic matrices are the permutation matrices. The continuous analogue of a permutation matrix would be the graph of a bijective, measure preserving function, but it is easy to construct extreme points that are not of this type (a deep study, showing that for some purposes these special measures may well suffice has been given in [13]). On the other hand, all concrete examples of extreme doubly stochastic measures (e.d.s.m.) that can be found in the literature are concentrated on the graphs (or inverse graphs) of functions, mostly even linear functions and there was some common belief that any double stochastic measure must in some sense be made up from graphs (cf. the beginning of §2 of [2]). A functional analytic characterization of the e.d.s.m. has been given by Douglas [4] and Lindenstrauss [7]. Several authors have tried to generalize properties of permutation matrices to these measures, see e.g., [1], [2], [10]. One of the aims of this paper is to present some new constructions of e.d.s.m. which will also disprove some natural conjectures. The second construction yields a measure that is not concentrated on graphs. While the measures in the first construction are still concentrated on two graphs, it turns out that even in this case the geometric interrelations are much more complicated. In [2] the following conjecture (attributed to J. Feldman) was mentioned: if $\mu$ is an e.d.s.m. and if $\nu$ is a doubly stochastic measure which is
absolutely continuous with respect to \( \mu \), then \( \nu = \mu \). In [2] this conjecture was confirmed by Brown and Shiflett for a class of extreme doubly stochastic measures which is geometrically related to permutation matrices. In Theorem 1 of our paper we will give a functional analytic characterization of measures \( \mu \) that satisfy a slightly stronger property as above (\( \nu \) need not be positive — the result of [2] holds also for this stronger property). It is along the lines of the results by Douglas in [5]. Then we study a special class of doubly stochastic measures, defined in a certain way by a measure preserving transformation \( T \). A similar type of measures (but with a different behavior) has been studied in [12]. In Theorem 2 we give conditions on \( T \) that ensure that the corresponding measure be extremal (resp. satisfies the properties of Theorem 1). Then it is easy to give examples of transformations for which the properties of Theorem 1 do not hold and which disprove also Feldman's conjecture mentioned above. The idea to use ergodicity properties of transformations for the construction of doubly stochastic measures was first used in [13] p. 87.

The second part of the paper concerns the support of an e.d.s.m. It has been proved in [7] that any such measure is singular with respect to the ordinary Lebesgue measure \( m \times m \) on the unit square, i.e., it is concentrated on a set of \( m \times m \) — measure zero. Nevertheless, we will give an example showing that the support of the measure may be the whole unit square, i.e., the measure is not concentrated on a closed set whose \( m \times m \)-measure is less than one. This measure has also the property that any graph (or inverse graph) of some measurable function has measure zero (Theorem 3).

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1. If \( m' \) is an arbitrary probability measure on \( I = [0, 1] \), we write \( E_{m'} \), for the set of all probability measures on \( I \times I \) whose marginal distributions are equal to \( m' \). We write \( \mu' < \mu \), if \( \mu' \) is absolutely continuous with respect to \( \mu \).

**Theorem 1.** If \( \mu \in E_m \), then the following statements are equivalent:

(i) If \( \nu \) is an arbitrary (complex) measure which is absolutely continuous with respect to \( \mu \) and whose marginal distributions are equal to \( m \), then \( \nu = \mu \).

(ii) The space of functions \( F = \{(x, y) \to f(x) + g(y) : f, g \in L^\infty (m)\} \) is weak \( *= \)-dense in \( L^\infty (\mu) \).

(iii) If \( m' \) is a probability measure which is absolutely continuous with respect to \( m \) and \( \mu' \in E_{m'} \) is absolutely continuous with
respect to $\mu$, then $\mu'$ is an extreme point in $E_m$, (i.e., $\mu$ is a sort of "hereditary extreme point").

The main content of the theorem may be rephrased as follows: if $F$ is $w^*$-dense in $L^\infty(\mu)$, $m' < m$, $\mu' < \mu$, and $\mu' \in E_m'$, then $F$ is also $w^*$-dense in $L^\infty(\mu')$.

**Proof.** (ii) $\Rightarrow$ (i) is Theorem 4 of [5].

(i) $\Rightarrow$ (ii) follows also from the methods of [5]: if $u \in L^\infty(\mu)$ annihilates the space $F'$, then $\nu = (1 + u)\mu$ has the same marginal distributions as $\mu$.

(iii) $\Rightarrow$ (ii). Given finitely many elements $u_1, \ldots, u_k \in L^\infty(\mu)$, $\varepsilon > 0$ and $h \in L^\infty(\mu)$, there exists a nonnegative function $u \in L^1(\mu)$ and $c > 0$ such that $u(x, y) = u(y, x)$ for all $x, y \in I$, $\int u(t)d\mu(t) = 1$ and $|u_i| \leq cu$ for $i = 1, \ldots, k$. Put $\mu' = u\mu$ and $m' = (\int u(\cdot, y)d\mu(y))m$; then $m' < m$ and $\mu' \in E_m'$. By (iii) and Theorem 1 of [4], there exists a function $h_0 \in F$ such that $\|h - h_0\|_{L^1(\mu')}$ < $\varepsilon/c$. This implies clearly $\left| \int u_i(t)(h(t) - h_0(t))d\mu(t) \right| < \varepsilon$ for $i = 1, \ldots, k$.

(ii) $\Rightarrow$ (iii). By Theorem 1 of [4] it is sufficient to show that $F$ is dense in $L^\infty(\mu)$ with respect to the norm topology induced by $L^1(\mu')$.

The result follows from a general lemma on convex subsets of $L^\infty(\mu)$.

**Lemma.** If $F$ is a convex subset of $L^\infty(\mu)$ and $\mu' < \mu$, then the $w^*$-closure of $F$ in $L^\infty(\mu)$ is contained in the closure taken with respect to the norm topology induced by $L^1(\mu')$.

**Proof.** Since $\mu' < \mu$, we get $\mu' = u\mu$ for some $u \in L^1(\mu)$. The continuous functionals for the $w^*$-topology are given by elements of $L^\infty(\mu)$. The continuous functionals for the norm topology induced by $L^1(\mu')$ are represented by elements of $\{hu: h \in L^\infty(\mu)\} \subseteq L^1(\mu)$ (if one uses the same duality as above). Now the lemma follows easily from the Hahn Banach theorem ([11], Ch. II, 9.2).

Now let $T: [0, 1/2] \to [0, 1/2]$ be an arbitrary measure preserving transformation. We consider measures $\mu$ supported by the following four sets: $F_1 = \{(x, Tx); 0 \leq x \leq 1/2\}$, $F_2 = \{(x, x - 1/2); 1/2 \leq x \leq 1\}$, $F_3 = \{(x, x + 1/2); 0 \leq x \leq 1/2\}$, $F_4 = \{(x, x); 1/2 \leq x \leq 1\}$ (i.e., the unit square is partitioned into four congruent sub squares. In the first of them we consider the graph of $T$, in the other squares, the diagonal). Let $j: [0, 1/2] \to F_1$ be defined by $j(x) = (x, Tx)$. $j$ induces a bijective correspondence between measures on $[0, 1/2]$ and $F_1$. |
The corresponding measures will be denoted by the same letter. If \( \mu \) is supported by \( F = F_1 \cup F_2 \cup F_3 \cup F_4 \), then \( \mu_1 \) shall denote its restriction to \( F_1 \).

**Theorem 2.** If \( \mu \) is a doubly stochastic measure, supported by \( F \), then the following statements hold:

(i) \( \mu \) is uniquely determined by \( \mu_1 = \mu|_{F_1} \). Conversely, a measure \( \mu_1 \) on \([0, 1/2]\) induces a doubly stochastic measure on \( F \), iff \( \mu_1 \) is \( T \)-invariant and \( 0 \leq \mu_1 \leq m \).

(ii) Write \( \mu_1 = h_1 m(h_1 \in L^\infty(m), 0 \leq h_1 \leq 1) \). \( \mu \) is an extreme doubly stochastic measure iff \( \mu_1 \) (resp. \( h_1 \)) has the following property: if \( A \subseteq [0, 1/2] \) is a \( T \)-invariant, measurable set with \( \mu_1(A) > 0 \), then \( \text{ess sup}_{x \in A} h_1(x) = 1 \).

(iii) \( \mu \) satisfies the conditions of Theorem 1, iff \( \mu_1 \) (resp. \( h_1 \)) has the following property: if \( A \subseteq [0, 1/2] \) is a \( T \)-invariant, measurable set with \( \mu_1(A) > 0 \), then \( m\{x \in A: h_1(x) = 1\} > 0 \). In particular, if \( \{x: h_1(x) = 1\} \) has \( m \)-measure zero, then \( \mu \) does not satisfy the conditions of Theorem 1.

**Proof.** Assume that \( \mu \) is given. Let \( A \) be a measurable subset of \([0, 1/2]\).

Since \( \mu \) is doubly stochastic, we get:

\[
m(A) = \mu(I \times A) = \mu((I \times A) \cap F_1) + \mu((I \times A) \cap F_2)
\]

\[
= \mu(T^{-1}A \times A) + \mu([1/2, 1] \times A)
\]

\[
= \mu_1(T^{-1}A) + \mu((A + 1/2) \times [0, 1/2]). \tag{1}
\]

Similarly

\[
m(A) = \mu((A + 1/2) \times I) = \mu((A + 1/2) \times [0, 1/2]) + \mu((A + 1/2) \times [1/2, 1])
\]

\[
= m(A) - \mu_1(T^{-1}A) + \mu([1/2, 1] \times (A + 1/2)). \tag{a}
\]

This gives:

\[
\mu([1/2, 1] \times (A + 1/2)) = \mu_1(T^{-1}A). \tag{2}
\]

Furthermore

\[
m(A) = \mu(I \times (A + 1/2)) = \mu([0, 1/2] \times (A + 1/2)) + \mu([1/2, 1] \times (A + 1/2))
\]

\[
gives
\]

\[
\mu([0, 1/2] \times (A + 1/2)) = m(A) - \mu_1(T^{-1}A). \tag{3}
\]

And finally
The last equation shows that\( \mu_x(A) = \mu_x(T^{-1}A) \), i.e., \( \mu_x \) is\( T \)-invariant. (1), (2), (3) show that the values of \( \mu \) on \( F_2, F_3, F_4 \) are uniquely
determined by \( \mu_x \). \( \mu_x \leq m \) holds by (3).

Conversely, if one defines the measure \( \mu \) on \( F_2, F_3, F_4 \) by the
formulas (1), (2), (3), it is easily seen that \( \mu \) is a doubly stochastic
measure. This proves (i).

(ii). Assume that \( A \subseteq [0, 1/2] \) is \( T \)-invariant \( \mu_x(A) > 0 \), \( 0 < \alpha \leq 1/2 \) and \( h_1 \leq 1 - \alpha \) a.e. on \( A \). Let \( e_A \) be the characteristic function
of \( A \). If we put \( \mu'_x = (1 + \alpha(1 - \alpha)^{-1}e_A)\mu_x \) and \( \mu''_x = (1 - \alpha(1 - \alpha)^{-1}e_A)\mu_x \),
then it follows easily from (i) that \( \mu'_x \) and \( \mu''_x \) define doubly stochastic
measures \( \mu' \) and \( \mu'' \) such that \( \mu = (\mu' + \mu'')/2 \).

For the converse assume that \( \mu = (\mu' + \mu'')/2 \) and \( \mu' \neq \mu'' \). Then
we get measures \( \mu'_x \) and \( \mu''_x \) such that \( \mu_x = (\mu'_x + \mu''_x)/2 \). Put \( \mu'_x = f'\mu_x \) and
\( \mu''_x = f''\mu_x \). Then \( f' + f'' = 2 \) and \( f' \neq f'' \). Therefore we may
assume that there exists \( \beta > 1 \) such that \( A = \{ x: f'(x) > \beta \} \) satisfies \( \mu_x(A) > 0 \).
Since \( \mu_x \) and \( \mu'_x \) are \( T \)-invariant, the same holds for \( A \). Since \( \mu'_x = f'hm \leq m \) by (i), we have \( f'h \leq 1 \) and therefore \( h \leq \beta^{-1} \)
on \( A \).

(iii). By an argument similar to (i), it can be shown that an
arbitrary (complex) measure \( \mu' \) on \( F_1 \cup F_2 \cup F_3 \cup F_4 \) with marginal
distributions \( m \) is uniquely determined by its restriction \( \mu'_x \) to \( F_1 \).
\( \mu'_x \) has to be \( T \)-invariant furthermore \( \mu' < \mu \) if \( \mu'_x < \mu_x \) and \( m - \mu'_x < m - \mu_x \) Now the same construction as in (ii) gives the result.

**Example.** It is now easy to give examples of extreme doubly
stochastic measures which do not satisfy the properties of Theorem
1. Let \( T_0: I \rightarrow I \) be an arbitrary ergodic transformation (with respect
to \( m \)), e.g., \( T_0(x) = x + \alpha(\text{mod} 1), \alpha \) irrational. We consider the mapping
\( g: I \rightarrow [0, 1/2] \) defined by \( g(x) = x^{1/2}/2 \). Then \( T = g \circ T_0 \circ g^{-1} \) is an ergodic
transformation of \( [0, 1/2] \) with invariant measure \( g(m) = 8ym \). Now
define \( h_1(y) = 2y \) for \( y \in [0, 1/2] \). By Theorem 2, \( \mu_1 = h_1m \) defines a
doubly stochastic measure \( \mu_1 \), which is extremal (by (ii) — observe
that \( A = [0, 1/2] \) is the only invariant set) but does not satisfy the
properties of Theorem 1 (by (iii)). It is easily seen that there exists
even a probability measure \( \mu' \neq \mu \) which is doubly stochastic and
absolutely continuous with respect to \( \mu \) (e.g., the diagonal measure
on \( F_2 \cup F_3 \)).

2. In the next section we present another method to construct
extreme doubly stochastic measures. It uses approximations by
measures on finite subalgebras. Let $\Sigma$ be the $\sigma$-algebra of Borel sets on $I$. We will consider increasing sequences $(\Sigma_n)$ resp. $(\Sigma'_n)$ of finite subalgebras of $I$, where $\Sigma_n(\Sigma'_n)$ is generated by a partition $P_n(Q_n)$ of $I$ into finitely many subintervals. We will define a measure $\mu$ successively on the subalgebras $\Sigma_n$. To assure that the limit measure is extremal, we will use an idea similar to [2] Thm. 1 and [3] Thm. 2. There is a connection between extremality and the existence of "loops". If $\mu$ is not extremal, one will also get such loops in one of the approximating algebras $\Sigma_n \otimes \Sigma'_n$. The aim of the following construction will be, to cut off each such loop at a later step.

**Theorem 3.** There exists an extreme doubly stochastic measure $\mu$ whose support is the whole space. In addition, $\mu$ has the following property: if $\mu = \int \mu_{1,x} \, dm(x) = \int \mu_{2,x} \, dm(x)$ are the disintegrations of $\mu$ with respect to the two coordinate projections ($\mu_{1,x}$ is concentrated on $[x] \times I$, $\mu_{2,x}$ on $I \times [x]$), then the measures $\mu_{1,x}$ and $\mu_{2,x}$ are continuous a.e. (i.e., each point has measure zero). In particular any graph or inverse graph of some measurable function $f : I \to I$ has $\mu$-measure zero.

**Proof.** By induction we will define partitions $P_n = \{I^a_n\}_{a \in A(n)}$ and $Q_n = \{J^i_n\}_{i \in A(n)}$ of $I$. For $n = 0$ we put $A^{(0)} = \{1\}$, $I^{(0)}_1 = I$, $\mu(I \times I) = 1$. Now assume that $P_n, Q_n$ have already been defined, and $\mu(I^a_n \times J^i_n) > 0$ for all $a, i \in A(n)$. We consider two different cases:

(a) if $n$ is even, we choose $k > \max\{(n + 1)\mu(I^a_n \times J^i_n)(m(I^a_n))^{-1} + m(J^i_n)^{-1}) : a, i \in A(n)\}$. We put $A^{(n+1)} = A^{(n)} \times \{1, \ldots, k\}$. Each interval $I^a_n$ is partitioned into $k$-subintervals $I^a_{n,1}, \ldots, I^a_{n,k}$ of equal length. We put $P_{n+1} = \{I^a_{n,j} : a \in A(n), j = 1, \ldots, k\}$. $Q_{n+1}$ is constructed in the same way from $Q_n$. Finally $\mu(I^a_{n+1} \times J^i_{n+1}) = k^{-1} \mu(I^a_n \times J^i_n)$. Then it is easily seen that

$$\mu(J^a_{n+1} \times J^i_{n+1}) < (n + 1)^{-1} \min(m(I^a_{n+1}), m(J^i_{n+1}))$$

for $a, i \in A^{(n+1)}$, $n + 1$ odd.

(b) if $n$ is odd, let $r = r(n)$ be the cardinality of $A^{(n)} \times A^{(n)}$ (i.e., the number of atoms in $\Sigma_n \otimes \Sigma'_n$). We choose $k > 2^r r$ and put $A^{(n+1)} = A^{(n)} \times \{1, \ldots, k\}$. Let $\varphi_n : A^{(n)} \times A^{(n)} \to \{1, \ldots, r\}$ be a fixed bijection. Finally put $c_n = \min\{\mu(I^a_n \times J^i_n) : a, i \in A(n)\}$. To get $P_{n+1}$ (resp. $Q_{n+1}$) we partition each interval $I^a_n$ (resp. $J^i_n$) into $k$-subintervals $I^{a_{i_1} \ldots i_r}_n$ (resp. $J^{a_{i_1} \ldots i_r}_n$) where $1 \leq i_1 \leq k$ for $1 = 1, \ldots, r$, whose length will be decided later on. We put
The text contains a detailed mathematical argument involving doubly stochastic measures. The key points are:

1. The expression for the measure $\mu(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta_1,\ldots,\beta_r}^{(n)} + 1)$ is given as:

   \[
   \mu(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta_1,\ldots,\beta_r}^{(n)} + 1) = (\mu(I_{\alpha}^{(n)}) \times J_{\beta}^{(n)} - c_\alpha k^{-2}r^{-2})k^{1-r}(k-1)^{-1} + c_\alpha k^{-4}r^{-2}
   \]

   if $s = \phi_n(a, \beta)$.

2. For all other choices of indices, the measure is:

   \[
   \mu(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta_1,\ldots,\beta_r}^{(n)} + 1) = c_\alpha k^{-4}r^{-2}
   \]

3. Summation over $\beta$ gives:

   \[
   m(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta_1,\ldots,\beta_r}^{(n)} + 1) - k^{1-r}(k-1)^{-1}m(I_{\alpha}^{(n)}) = -c_\alpha k^{-5}r^{-3/2}(k-1)^{-1}
   \]

   Since by (4) $m(I_{\alpha}^{(n)}) \geq n \mu(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta}^{(n)})$ and $c_\alpha \leq \mu(I_{\alpha}^{(n)} \times J_{\beta}^{(n)})$, we get:

   \[
   m(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta_1,\ldots,\beta_r}^{(n)} + 1) \leq m(I_{\alpha}^{(n)} \times J_{\beta}^{(n)})(n-1)k^{1-r}(k-1)^{-1}.
   \]

4. Combined with (7), we find:

   \[
   m(I_{\alpha_1,\ldots,\alpha_r}^{(n)} \times J_{\beta_1,\ldots,\beta_r}^{(n)} + 1) \leq (n-1)^{-1}m(I_{\alpha}^{(n+1)})
   \]

   for all $\beta \in A^{(n+1)}$, if $I_{\alpha_1,\ldots,\alpha_r}^{(n+1)}$ is not contained in $I_{\alpha}^{(n)}$. If $\alpha \in A^{(n)}$ is fixed, then $I_{\alpha}^{(n)}$ is a union of $(k-1)^r$ intervals for which (8) holds. Therefore:

   \[
   m(I_{\alpha}^{(n)} \times J_{\beta}^{(n+1)}) \leq (n-1)^{-1}m(I_{\alpha}^{(n+1)})
   \]

   (by our choice of $k$ and since clearly $r \geq 2^n$). Summing over $\alpha \in A^{(n)}$, we get:

   \[
   m(I^{(n)}) \leq 2^{1-n}.
   \]

Having defined all partitions $P_n, Q_n$, we want to extend $\mu$ to a measure on $\Sigma \otimes \Sigma$. This is done as follows: Let $\mu^{(n)}$ be the pro-
bability measure on $I \times I$, whose restriction to each rectangle $I_{a}^{(n)} \times J_{\beta}^{(n)}(\alpha, \beta \in A^{(n)})$ is a multiple of the ordinary Lebesgue measure and such that $\mu^{(n)}(I_{a}^{(n)} \times J_{\beta}^{(n)}) = \mu(I_{a}^{(n)} \times J_{\beta}^{(n)})$. Let $\mu'$ be a cluster point of the sequence $(\mu^{(n)})$ in the weak topology of the set of all Radon measures on $I \times I$ with respect to continuous functions. Since each $\mu^{(n)}$ is doubly stochastic, the same is true for $\mu'$. In particular, the boundary of each rectangle $I_{a}^{(n)} \times J_{\beta}^{(n)}$ has $\mu'$-measure zero. If $(\mu^{(n)})$ is a subsequence of $(\mu^{(n)})$ converging towards $\mu'$, it follows from [9] Ch. II, Thm. 6.1 that $\mu^{(n)}(I_{a}^{(n)} \times J_{\beta}^{(n)})$ converges to $\mu(I_{a}^{(n)} \times J_{\beta}^{(n)})$. This shows that $\mu'(I_{a}^{(n)} \times J_{\beta}^{(n)}) = \mu(I_{a}^{(n)} \times J_{\beta}^{(n)})$, i.e., $\mu'$ is an extension of $\mu$. (From now on we will again write $\mu$ for $\mu'$.) We claim that $\mu$ has the properties stated in Theorem 3.

We have already remarked that $\mu$ is doubly stochastic. Let $\mu = \int \mu_{1,x} \, dm(x)$ be the disintegration of $\mu$ with respect to the first coordinate projection ($\mu_{1,x}$ is concentrated on $\{x\} \times I$). Similarly $\mu^{(n)} = \int \mu^{(n)}_{1,x} \, dm(x)$. If $x \in I_{a}^{(n)}(\alpha \in A^{(n)})$ and $M$ is a Borel subset of $I$, then $\mu^{(n)}_{1,x}(M) = \mu^{(n)}(I_{a}^{(n)} \times M)/m(I_{a}^{(n)})$. Put $I^\circ = \bigcap_{n} \bigcup_{n \geq m} I^{(n)}$. Then $m(I^\circ) = 0$ by (10). If $x \in I^\circ$, then by (4) and (9) $\mu^{(n)}_{1,x}(J_{a}^{(m)}) = \mu^{(m)}_{1,x}(J_{a}^{(m)}) < (m - 2)^{-1}$ for all $\alpha \in A^{(m)}, m \geq n \circ(x), n \geq m$. It follows that any cluster point $\mu'_{1,x}$ of the sequence $(\mu^{(n)}_{1,x})$ is a continuous measure and by the same argument as used for $(\mu^{(n)})$, the sequence $(\mu^{(n)}_{1,x})$ converges in the weak topology towards $\mu'_{1,x}$. By dominated convergence, we get $\mu = \int \mu'_{1,x} \, dm(x)$ and therefore $\mu_{1,x} = \mu'_{1,x}$ a.e. The same argument works for the second coordinate.

The interiors of the rectangles $I_{a}^{(n)} \times J_{\beta}^{(n)}(\alpha, \beta \in A^{(n)}, n \geq 0)$ form clearly a basis for the topology of $I \times I$. By our construction $\mu(I_{a}^{(n)} \times J_{\beta}^{(n)}) = \mu^{(n)}(I_{a}^{(n)} \times J_{\beta}^{(n)}) > 0$ and therefore $\text{supp} \, \mu = I \times I$.

The last thing that we have to show is the extremality of $\mu$. Equivalently: if $f \in L^\infty(\mu)$ and the conditional expectations $E_{1}f$ and $E_{2}f$ with respect to the two coordinate projections are zero, then $f$ equals zero. Assume the contrary, i.e., $\|f\|_{\infty} = 1$ and $E_{1}f = E_{2}f = 0$. By the martingale convergence theorem, there exists an odd integer $n$ and $\alpha', \beta' \in A^{(n)}$ such that

$$\left| \int_{I_{a}^{(n)} \times J_{\beta}^{(n)}} f \, d\mu \right| \geq 2^{-1} \mu(I_{a}^{(n)} \times J_{\beta}^{(n)}) .$$

Put

$$B^{(n+1)} = \{(\alpha i_{1} \cdots i_{r}, \beta i_{1} \cdots i_{s} + 1 \cdots i_{r}) \mid \alpha, \beta \in A^{(n)}, \varphi_{n}(\alpha, \beta) = s \},$$

$$1 \leq i_{s} < k, 1 \leq i_{l} \leq k \quad \text{for} \quad l = 1, \cdots, r, l \neq s \subseteq A^{(n+1)} \times A^{(n+1)}$$

and put
By (6) we have \( \mu \left( \bigcup \left\{ I^T_{\alpha} \times J^T_{\beta} \mid (\alpha, \beta) \in B^{(n+1)} \right\} \right) \leq c_n k^{-2r} r^{-1} \). Since \( \|f\|_{\infty} = 1 \) and \( E_{i-f} = E_{i-f} = 0 \) we conclude that:

\[
\begin{align*}
\sum_{\beta \in A^{(n+1)}} \left| \sum_{\alpha \in A^{(n+1)}} a_{\alpha \beta} \right| (a, \beta) \in B^{(n+1)} \right\} & < c_n k^{-2r} r^{-1} \\
\sum_{\alpha \in A^{(n+1)}} \left| \sum_{\beta \in A^{(n+1)}} a_{\alpha \beta} \right| (a, \beta) \in B^{(n+1)} \right\} & < c_n k^{-2r} g^{-1}.
\end{align*}
\]

Since \( 2^{-1} \mu (I^T_{\alpha} \times J^T_{\beta}) - c_n k^{-2r} r^{-1} \geq 4^{-1} c_n \), (11) can be reformulated as follows:

\[
\sum_{\alpha \in A^{(n+1)}} \left| \sum_{\beta \in A^{(n+1)}} a_{\alpha \beta} : (a, \beta) \in B^{(n+1)} \right\} \geq 4^{-1} c_n
\]

Now choose numbers \( (a_{\alpha \beta})_{(\alpha, \beta) \in B^{(n+1)}} \) such that (12), (13), (14) are fulfilled and the cardinality of \( C = \{ (\alpha, \beta) : a_{\alpha \beta} \neq 0 \} \) becomes minimal. By (14), there exists \( (\alpha, \beta) \in C \) such that \( |a_{\alpha \beta}| \geq 4^{-1} c_n k^{-2r} \). Put \( D_0 = \{ (\alpha, \beta) \} \) and then inductively:

\[
D_{2i+1} = \{ (\delta_1, \delta_2) \in C \mid D_{2i} : \exists \delta_3 \in A^{(n+1)} : (\delta_1, \delta_3) \notin D_{2i} \}
\]

\[
D_{2i} = \{ (\delta_1, \delta_2) \in C \mid D_{2i-1} : \exists \delta_3 \in A^{(n+1)} : (\delta_3, \delta_2) \notin D_{2i-1} \}
\]

(e.g., \( D_{2i} \) stands for those rectangles that belong to a row determined by some rectangle from \( D_{2i-1} \), but do not belong to \( D_{2i-1} \) itself).

Now assume that \( D_i \cap D_j \neq \emptyset \) for some \( i < j \) and choose \( i \) minimal. Assume that \( i \neq 0 \). Take \( (\delta_1, \delta_2) \in D_i \cap D_j \). If \( i \) is odd, there exists \( \delta_3 \in A^{(n+1)} \) such that \( (\delta_1, \delta_3) \notin D_{i-1} \). If \( j \) is also odd, then \( (\delta_1, \delta_3) \in D_j \cup D_{j-1} \) (since \( (\delta_1, \delta_2) \in D_j \)), but both possibilities contradict the minimality of \( i \). Similarly if \( j \) is even, then \( (\delta_1, \delta_3) \in D_j \cup D_{j+1} \) which is also impossible. An analogous argument works in the case that \( i \) is even. This shows that either \( (\alpha, \beta) \in D_j \) for some \( j \geq 2 \) or all \( D_j \) are disjoint. To exclude the second possibility, observe that \( \bigcup_{j \geq 0} D_j \) contains with each element also the whole column corresponding to that element. Therefore, by (12):

\[
\left| \sum_{\gamma, \delta} a_{\gamma \delta} : (\gamma, \delta) \in \bigcup_{j \geq 0} D_j \right| \leq 8^{-1} c_n k^{-2r}.
\]

Similarly \( \bigcup_{j \geq 1} D_j \) contains only complete rows, therefore by (13):

\[
\left| \sum_{\gamma, \delta} a_{\gamma \delta} : (\gamma, \delta) \in \bigcup_{j \geq 1} D_j \right| \leq 8^{-1} c_n k^{-2r}.
\]

But if the sets \( D_j \) are disjoint, the two sums differ exactly by \( a_{\alpha \beta} \).
and this contradicts $|a_{a_\beta}| \geq 4^{-1}c_0L^{-2r}$.

This shows the existence of a "loop", i.e., there exist pairwise different elements $\alpha_1, \ldots, \alpha_m \in A^{(n+1)}$, such that $(\alpha_{2i-1}, \alpha_{2i}), (\alpha_{2i+1}, \alpha_{2i}) \in C$ for all $i$ (we put $\alpha_{m+1} = \alpha_1$); $m$ is clearly even. Define $a = a_{a_\alpha a_{\alpha_2}}$ and $b_{a_{a_\alpha a_{\alpha_2}}}, a_{a_{a_\alpha a_{\alpha_2}}}, b_{a_{a_\alpha a_{\alpha_2}}} = a$, $b_{a_{a_\alpha a_{\alpha_2}}} = -a$ for all $i$ and $b_{a_\alpha} = 0$ in the other cases. The sums in (12) and (13) are zero for $\langle b_{a_\alpha} \rangle$. If the sum in (14) would also be zero (for all choices of $\alpha', \beta' \in A^{(n)}$), then $\langle a_{a_\alpha} - b_{a_\alpha} \rangle$ would have the properties (12), (13), (14), contradicting our minimal choice of $\langle a_{a_\alpha} \rangle$ (since $a_{a_\alpha} - b_{a_\alpha} = 0$). Therefore $\langle b_{a_\alpha} \rangle$ has the properties (12), (13), (14) (possibly with a smaller constant in (14)) and we will assume from now on that $a_{a_\alpha} = b_{a_\alpha}$ for all $\alpha, \beta$.

Now recall the definition of $A^{(n+1)}$: we have $\alpha_i = (\rho(\alpha_i), \rho_1(\alpha_i), \ldots, \rho_r(\alpha_i))$ with $\rho(\alpha_i) \in A^{(n)}$, $1 \leq \rho_i(\alpha_i) \leq k(l = 1, \ldots, r)$, $I_{a_{a_\alpha}} \subseteq I_{\rho(\alpha_i)}$, $J_{a_{a_\alpha}} \subseteq J_{\rho(\alpha_i)}$. Put $\bar{\varphi}(2i - 1) = \varphi_\alpha(\alpha_{2i-1}, \alpha_{2i})$ and $\bar{\varphi}(2i) = \varphi_\alpha(\alpha_{2i+1}, \alpha_{2i})$. We may assume that $\alpha' = \rho(\alpha_i), \beta' = \rho(\alpha_i)$ in (14), i.e.,

$$\sum_{a_{a_\alpha} \in I_{a_{a_\alpha}}^{(n+1)}} J_{a_{a_\alpha}}^{(n+1)} \subseteq I_{a_{a_\alpha}}^{(n)} \times J_{a_{a_\alpha}}^{(n)} \neq \emptyset.$$

Put $t = \varphi^-(1) = \varphi_\alpha(\alpha_1, \alpha_2)$. It follows from the definition of $B^{(n+1)}$ that for any $\alpha \in A^{(n+1)}$, $\gamma, \delta \in A^{(n)}$ there exists at most one $\beta \in A^{(n+1)}$ such that $(\alpha, \beta) \in B^{(n+1)}$ and $I_{a_{a_\alpha}}^{(n+1)} \times J_{a_{a_\alpha}}^{(n+1)} \subseteq I_{a_{a_\alpha}}^{(n)} \times J_{a_{a_\alpha}}^{(n)}$, similarly for $\alpha, \beta$ interchanged. This shows that $\varphi^-(l) \neq \varphi^-(l + 1)$ for all $l$. Now put $N = \{l: \varphi^-(l) = t\} = \{l_1, l_2, l_3, \ldots\}$ with $1 < l_1 < l_2 < l_3 < \cdots$. Since $\varphi^-(1) = t$, we have $\rho_i(\alpha_i) + 1 = \rho_i(\alpha_2)$. If $\varphi^-(l) \neq t$, then $\rho_i(\alpha_{i+1}) = \rho_i(\alpha_i)$. On the other hand, if $\varphi^-(l) = t$ and $l$ is even, then $\rho_i(\alpha_{i+1}) + 1 = \rho_i(\alpha_i)$ and if $l$ is odd, then $\rho_i(\alpha_i) + 1 = \rho_i(\alpha_{i+1})$. This means that if we put $s(l) = |\{j \geq 1: l_j < l, l_j \text{ is odd}\}| - |\{j \geq 1: l_j < l, l_j \text{ is even}\}|$ (here $| \cdot |$ stands for the cardinality of the set), then $\rho_i(\alpha_i) = \rho_i(\alpha_i) + \sigma(l)$. In particular, for $l = m + 1$, we have $\alpha_{m+1} = \alpha_1$ and therefore $\sigma(m + 1) = 0$. This means that $|\{j \geq 1: l_j < m + 1, l_j \text{ is odd}\}| = |\{j \geq 1: l_j < m + 1, l_j \text{ is even}\}|$. But these numbers $l_j$ determine those indices $(\alpha, \beta)$ for which $a_{a_\beta} \neq 0$ and $I_{a_{a_\beta}}^{(n+1)} \times J_{a_{a_\beta}}^{(n+1)} \subseteq I_{a_{a_\beta}}^{(n)} \times J_{a_{a_\beta}}^{(n)}$. If $l_j$ is odd, then $a_{a_\beta} = +a$ and if it is even then $a_{a_\beta} = -a$. By summing up we would get: $\sum_{a_{a_\beta} \in I_{a_{a_\beta}}^{(n+1)} \times J_{a_{a_\beta}}^{(n+1)}} \subseteq I_{a_{a_\beta}}^{(n)} \times J_{a_{a_\beta}}^{(n)} = 0$, contrary to (14). This contradiction proves that $\mu$ is extremal.

Finally since all the measures in the two disintegrations of $\mu$ are continuous, it follows that any graph or inverse graph has $\mu$-measure zero.

**REFERENCES**


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