AN INDEX THEOREM AND HYPOELLIPTICITY ON NILPOTENT LIE GROUPS

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Extending results of Grushin we determine the index of
\( p(x, D) \) where \( p(x, \xi) \) is a polynomial homogeneous with respect
to some family of dilations on \( \mathbb{R}^{2d} \) and \( p(x, \xi) \neq 0 \) if \( (x, \xi) \neq (0, 0) \).
In general these operators are not elliptic. If \( G \) is a step
two nilpotent Lie group and \( P \) is a left invariant differential
operator on \( G \) which is homogeneous with respect to some
family of dilations, we apply this index theorem to prove
that \( P \) is hypoelliptic if and only if \( P^* \) is hypoelliptic. This
extends a result of Helffer and Nourrigat.

1. An index theorem. A family of dilations on a Lie algebra
\( \mathfrak{g} \) is a one parameter family of automorphisms \( \{ \delta_r : r > 0 \} \) of \( \mathfrak{g} \) of
the form \( \delta_r = \exp ((\log r)A) \), where \( A \) is a diagonalizable automorphism
of \( \mathfrak{g} \) with positive real eigenvalues. There is no loss of generality
in assuming that the smallest eigenvalue is 1. A finite dimensional
normed vector space \( V \) with norm \( | \cdot | \) determines an abelian Lie
algebra. Let \( \{ \delta_r \} \) be a family of dilations on \( V \). For \( w \in V \) define
\( \| w \| \) by \( \| w \| = r \) if \( |\delta_r^{-1}(w)| = 1 \). Then \( w \rightarrow \| w \| \) is continuous on
\( V \) and \( C^\infty \) on \( V - \{ 0 \} \) by the implicit function theorem. Let \( \mathbb{B} = \{ w_1, w_2, \ldots, w_n \} \)
be a basis for \( V \) consisting of eigenvectors of \( A \) with corresponding eigenvalues \( \mu_1, \ldots, \mu_n \). If \( w = a_1 w_1 + \cdots + a_n w_n \),
then
\begin{align}
\delta_r w &= \sum r^{\mu_j} a_j w_j \\
\| w \| &\approx \sum |a_j|^{1/\mu_j} \tag{1.1}
\end{align}

Throughout this section we will be considering a family of
dilations on the abelian Lie algebra \( \mathbb{R}^{2d} = \mathbb{R}^d \oplus \mathbb{R}^d \). We do not
necessarily assume that either \( \mathbb{R}_x^d \) or \( \mathbb{R}_\xi^d \) is invariant under \( \{ \delta_r \} \). Let
\( f \in C^\infty(\mathbb{R}^{2d}) \), \( f(w) = 0 \) for \( \| w \| \leq 1/2 \), and \( f(w) = 1 \) for \( \| w \| \geq 1 \). Define
\( \Phi(w) = 1 + f(w) \| w \| \) and \( \varphi(w) = 1 \) for all \( w = (x, \xi) \in \mathbb{R}^{2d} \). Note that
there is a \( C \) such that if \( |w - w'| \leq \Phi(w) \) then \( \Phi(w') \leq C\Phi(w) \). Thus
(\( \Phi, \varphi \)) is a pair of weight functions on \( \mathbb{R}^d \) as defined in Beals [1].
We will usually not mention \( \varphi \) and will refer to \( \Phi \) as the weight
function for the family of dilations \( \{ \delta_r \} \). Note that \( \Phi \) satisfies the
coercive estimate
\begin{align}
|w| &\leq C\Phi(w) \tag{1.3}
\end{align}
where \( \bar{\mu} = \max \{ \mu_1, \ldots, \mu_{2d} \} \).
For $m \in \mathbb{R}$, let $S^m_\varphi$ denote the set of all smooth functions $p$ on $\mathbb{R}^d$ such that for each $\alpha$ and $\beta \in N^d$

$$
\sup \{ \Phi(x, \xi)^{-m+|\alpha|} | D^\alpha_x D^\beta_\xi p(x, \xi) | : (x, \xi) \in \mathbb{R}^{2d} \} < \infty .
$$

$\mathcal{L}^m_\varphi$ is the set of pseudodifferential operators with symbols in $S^m_\varphi$, $H^s_\varphi$ is the associated (global) Sobolev space as defined in [1] and $\| \ |_{m, \varphi}$ is a norm for the topology on $H^s_\varphi$. We note that in the special case where $m \in N$ and $m_j/\mu_j \in N$ for all $j$ (this is necessarily the case in the context of Theorem 2 below, by Proposition 1.3 of [7]), then $\| \ |_{m, \varphi}$ can be given explicitly as follows: Let $\mathcal{B}$ be a basis for $\mathbb{R}^{2d}$ consisting of eigenvectors for $\{\delta_r\}$ and let $a_j(x, \xi)$ be the $j$th coordinate of $(x, \xi)$ with respect to the basis $\mathcal{B}$. By (1.2) above and 6.17 of [1]

\begin{equation}
(1.4) \quad \| u \|_{m, \varphi} \approx \sum | a_j(x, D)^{m_j/\mu_j} u | + \| u \|
\end{equation}

where $\| \ |$ is the $L^2$ norm.

We shall denote by $\tilde{S}^m_\varphi$ the subset of $S^m_\varphi$ consisting of functions $p$ such that for all $\alpha$ and $\beta$ in $N^d$

$$
\sup \{ \Phi(x, \xi)^{-m+|\alpha|} | D^\alpha_x D^\beta_\xi p(x, \xi) | : (x, \xi) \in \mathbb{R}^{2d} \} < \infty .
$$

We say that $p \in C^\infty(\mathbb{R}^{2d})$ is homogeneous of degree $m$ with respect to $\{\delta_r\}$ for large $w$ if there is a $0 < c < 1$, such that $p(\delta_r w) = r^m p(w)$ for all $r \geq 1$ and all $w$ for which $\|w\| \geq c$. If $p$ is homogeneous of degree $m$ with respect to $\{\delta_r\}$ for large $w$ and if $v$ is an eigenvector for the generator $A$ of $\{\delta_r\}$ with eigenvalue $\mu$, then

$$
r^m D_v p(\delta_r w) = r^m D_v p(w) .
$$

If $\|w\| \geq 1$, let $r = \|w\|$ and $w' = \delta_r^{-1}(w)$. Then $\|w'\| = 1$ and $D_v p(w) = \|w\|^{m-r} D_v p(w')$. Thus there is a $C$ such that

\begin{equation}
(1.5) \quad |D_v p(w)| \leq C \|w\|^{m-r} \leq C \|w\|^{m-1}
\end{equation}

for all $w$, $\|w\| \geq 1$. Consequently if $p$ is homogeneous of degree $m$ with respect to $\{\delta_r\}$ for large $w$, then $p \in \tilde{S}^m_\varphi$. It follows from this remark that $\Phi \in \tilde{S}^m_\varphi$ and hence $\Phi^m \in \tilde{S}^m_\varphi$ for all $m \in R$.

We say that $p \in S^m_\varphi$ is $\Phi$-elliptic if there is a $C$ such that $\Phi(w)^m \leq C |p(w)|$ for $|w| \geq C$. Note that if $p$ is a polynomial and $p$ is homogeneous of degree $m$ with respect to $\{\delta_r\}$, then $p$ is $\Phi$-elliptic and only if $p(w) \neq 0$ for $|w| \neq 0$. Note that in general $\Phi$-ellipticity does not imply ellipticity in the usual sense. For example on $\mathbb{R}^1 \times \mathbb{R}^3$, $p(x, \xi) = \xi_1^4 + x_1^3 + 2x_1\xi_1 + \xi_1^2 + \xi_2 + x_2^2$ is $\Phi$-elliptic and homogeneous of degree two, where the dilations are given in terms of coordinates $a_1 = \xi_1, a_2 = x_1 + \xi_1, a_3 = \xi_2$ and $a_4 = x_2$, with $\mu_1 = 2, \mu_2 = \mu_3 = \mu_4 = 1$.

If $\Gamma$ is an oriented curve and $p$ maps the range of $\Gamma$ into
$C - \{0\}$, let $\Delta r \arg p$ denote the change in the argument of $p$ along $\Gamma$. In the following theorem $\Gamma$ is the curve in $R_x  \oplus R_\xi$ given by $x(\theta) = \cos \theta$, $y(\theta) = \sin \theta$, $0 \leq \theta \leq 2\pi$. In the case where $R^2$ and $R^2_\xi$ are eigenspaces for $A$ with eigenvalues 1 and $1 + \delta$ respectively, $\delta > 0$, this theorem was proved in [2].

**Theorem 1.** Let $\delta_r = \exp((\log r)A)$, $r > 0$, be a family of dilations on $R^{2d}$, $\Phi$ the weight function for $\{\delta_r\}$. Let $p = p_0 + p_1$ where $p_0$ is $\Phi$-elliptic and homogeneous of degree $m$ with respect to $\{\delta_r\}$ for large $w$ and $p_1 \in S^m_{\Phi} \Phi_1$ for some $m_1 < m$. Then $p(x, D): H^m_{\Phi} \rightarrow L^2$ is Fredholm. If $d > 1$, then $\text{ind} p(x, D) = 0$. If $d = 1$, then $2\pi \text{ind} p(x, D) = \Delta \Gamma \arg p_0$. If $d = 1$ and $p_0$ is a polynomial, then $\text{ind} p(x, D)$ is also given by (1.6) below.

**Proof.** By Theorem 7.2 of [1] and (1.3) above, $p(x, D): H^m_{\Phi} \rightarrow L^2$ is Fredholm. By Corollary 6.13 of [1], $p_0(x, D): H^m_{\Phi} \rightarrow L^2$ is compact. Hence $\text{ind} p_0(x, D) = \text{ind} p(x, D)$. Let $f \in C^\infty(R^{2d})$ be real valued, $f(w) = 0$ for $\|w\| \leq 1/2$, $f(w) = 1$ for $\|w\| \geq 1$. Let $a(w) = f(w)/\|w\|^m/2$, $q = p_0a^2$. Then $A = a(x, D) \in \mathcal{L}_{\Phi_1}^{m_1}$, and by the pseudodifferential operator calculus $p_0(x, D)A^*A = q(x, D) + R$ where $R \in \mathcal{L}_{\Phi_1}$. Thus $\text{ind} q(x, D) = \text{ind} p_0(x, D)$. Also $q(\delta, w) = p_0(w) \neq 0$ for all $r \geq 1$ and all $w$, $\|w\| = 1$. If $d > 1$, $\{w \in R^{2d}: \|w\| = 1\}$ is simply connected, so $q$ can be continuously deformed to a nonzero constant through $\Phi$-elliptic symbols which are homogeneous of degree $0$ for large $w$. Hence $\text{ind} q(x, D) = 0$.

Now consider the case $d = 1$. Although $q$ is not elliptic in the classic sense, $q$ is included in the class of symbols for which Hormander proves the index theorem in §7 of [5]. In [5] it is shown that $2\pi \text{ind} q^w(x, D) = \Delta_r \arg q$, where $q^w(x, D)$ is the Weyl pseudodifferential operator with symbol $q$. By (4.10) of [5] $q^w(x, D) = a(x, D)$ where $a = q + r$, $r \in \mathcal{S}_{\Phi_1}$. Thus $\text{ind} q(x, D) = \text{ind} q^w(x, D)$. Clearly $\Delta_r \arg q = \Delta_r \arg p_0$.

If $d = 1$ and $p_0$ is a polynomial, then $\text{ind} p(x, D)$ can also be computed as follows: Let $v_1$ and $v_2$ be eigenvectors for the generator $A$ of $\{\delta_r\}$, chosen so that if $(x_1, \xi_1)$ and $(x_2, \xi_2)$ are the respective $x, \xi$ coordinates of $v_1$ and $v_2$, then $x_1\xi_2 - x_2\xi_1 > 0$. Let $\Gamma_+$ be the line $t \rightarrow v_1 + tv_2$ and $\Gamma_-$ the line $t \rightarrow -v_1 + tv_2$, $t \in R$. Let $m_2 = m/\mu_2$. Let $\nu_+$ be the number of complex roots $z$ of $p_0(v_1 + zv_2)$ with positive imaginary part and $\nu_-$ the number of complex roots of $p_0(-v_1 + zv_2)$ with negative imaginary part. By the homogeneity of $p_0$,

$$\Delta_r \arg p_0 = \Delta_{r+} \arg p_0 - \Delta_{r-} \arg p_0 \quad \text{and} \quad \Delta_{r+} \arg p_0 = -i \int_{-\infty}^{\infty} \frac{d}{dt} |p_0(v_1 + tv_2)| dt = 2\pi(\nu_+ - m_2/2).$$
Thus

\begin{equation}
\arg p_0 = -i \int_{-\infty}^{\infty} \frac{d}{dt} |p_0(tv_2 - \nu_1)| dt = 2\pi (m_2/2 - \nu_-).
\end{equation}

(1.6)

ind \, p(x, D) = \nu_+ + \nu_- - m_2.

2. Hypoellipticity of \( P^* \). Let \( \mathcal{G} \) be a nilpotent Lie algebra of step 2; i.e., \([\mathcal{G}, \mathcal{G}_2] = 0\) where \( \mathcal{G}_2 = [\mathcal{G}, \mathcal{G}] \). Let \( G \) be the corresponding connected, simply connected Lie group. A family of dilations \( \{\delta_r\} \) on \( \mathcal{G} \) induces a family of algebra automorphisms, also denoted \( \{\delta_r\} \), of \( \mathcal{U}(\mathcal{G}) \), the complexified universal enveloping algebra of \( \mathcal{G} \). An element \( P \) of \( \mathcal{U}(\mathcal{G}) \) is said to be homogeneous of degree \( m \) with respect to \( \{\delta_r\} \) if \( \delta_r(P) = r^m P \) for all \( r > 0 \). The set of all \( P \in \mathcal{U}(\mathcal{G}) \) such that \( P \) is homogeneous of degree \( m \) with respect to a given family of dilations \( \{\delta_r\} \) will be denoted \( \mathcal{U}_m(\mathcal{G}, \{\delta_r\}) \) or simply \( \mathcal{U}_m(\mathcal{G}) \) when there is no chance of confusion. We consider the elements of \( \mathcal{U}(\mathcal{G}) \) as left invariant differential operators on \( G \).

**THEOREM 2.** Let \( \mathcal{G} \) be a nilpotent Lie algebra of step two and \( \{\delta_r\} \) a family of dilations on \( \mathcal{G} \). If \( P \in \mathcal{U}_m(\mathcal{G}, \{\delta_r\}) \) is hypoelliptic, then \( P^* \) is hypoelliptic.

When \( \{\delta_r\} \) is the natural family of dilations for a grading \( \mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \) of \( \mathcal{G} \), then this result was proved in Helffer and Nourrigat [4]. For the Heisenberg group such a result was proved in Miller [6]. It follows from this theorem that any hypoelliptic \( P \in \mathcal{U}_m(\mathcal{G}) \) is locally solvable.

The proof is based on the Helffer-Nourrigat-Rockland characterization of the hypoelliptic operators in \( \mathcal{U}_m(\mathcal{G}) \): \( P \in \mathcal{U}_m(\mathcal{G}) \) is hypoelliptic if and only if \( \pi(P) \) is injective in \( \mathcal{U}_m(\mathcal{G}) \) for every nontrivial irreducible unitary representation \( \pi \) of \( G \). (See [3] and [8]. That this result holds for arbitrary dilations is shown in [7].) We shall also need some other preliminary information before beginning the proof of Theorem 2.

By Lemma 1.2 of [7] there is a basis \( \{X_i, \cdots, X_N; \cdots, X_n\} \) of \( \mathcal{G} \) such that each \( X_j \) is an eigenvector for the generator \( A \) of \( \{\delta_r\} \), \( \{X_{N+1}, \cdots, X_n\} \) spans \( \mathcal{G}_2 \), and for each \( k > N \) there are \( i \) and \( j \leq N \) such that \( [X_i, X_j] = X_k \). Let \( \mu_j \) be the eigenvalue of \( A \) corresponding to \( X_j \). If \( \alpha \in \mathbb{N}_n \), let \( \alpha \mu = \sum \alpha_j \mu_j \) and \( X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \). Then \( P \in \mathcal{U}_m(\mathcal{G}) \) if and only if

\begin{equation}
P = \sum_{\alpha = m} a_\alpha X^\alpha
\end{equation}

for some \( a_\alpha \in \mathbb{C} \).

Let \( \mathcal{G}_i \) be the subspace of \( \mathcal{G} \) spanned by \( \{X_i, \cdots, X_N\} \). Letting
\( \mathcal{F}^* \) denote the vector space dual of \( \mathcal{F} \), we define \( \delta_r \) on \( \mathcal{F}^* \) to be the transpose of \( \delta_r \) on \( \mathcal{F} \) for each \( r > 0 \). Since \( \mathcal{F}_1 \) is invariant under \( \{ \delta_r \} \) (on \( \mathcal{F}^* \)) restricts to a family of dilations on the vector space \( \mathcal{F}^* \). For \( \eta \in \mathcal{F}^*_1 \) define \( ||\eta|| \) as in \( \S 1 \). If \( X \in \mathcal{F} \), let \( X = X' + X'' \) where \( X' \in \mathcal{F}_1, X'' \in \mathcal{G}_2 \). For \( \eta \in \mathcal{F}^*_1 \),

\[
(2.2) \quad \pi_\eta(\exp X) = \exp i\langle \eta, X' \rangle
\]
defines a unitary representation of \( G \) on \( C \). It follows from (2.1) that if \( P \in \mathcal{U}_m(\mathcal{F}) \), then

\[
(2.3) \quad \pi_{\delta_r P}(P) = r^m \pi_\eta(P) = \pi_\eta(\delta_r P) ; \quad \eta \in \mathcal{F}^*_1 .
\]

We next recall some facts about the representation theory for \( G \). More details are given in [7]. Let \( \zeta \in \mathcal{F}^*_2 \). Then there is a \( d = d(\zeta) \leq N/2 \) and a basis \( \mathcal{B}(\zeta) = \{ Y_1(\zeta), \ldots, Y_N(\zeta) \} \) for \( \mathcal{F}_1 \) such that \( \mathcal{B}(\zeta) \) is orthogonal with respect to the inner product determined by the basis \( \{ X_1, \ldots, X_N \} \) and such that

\[
(2.4) \quad \langle \zeta, [Y_j(\zeta), Y_{j+d}(\zeta)] \rangle = 1 \quad \text{for } j \leq d ;
\]
\[
\langle \zeta, [Y_j(\zeta), Y_k(\zeta)] \rangle = 0 \quad \text{for all other choices } j < k \leq N .
\]
(In [7] we had \( [Y_j(\zeta), Y_{j+d}(\zeta)] = \lambda_j I \). This was necessary because we wanted the basis to be orthonormal, but that is not needed here.) For any \( \rho \in \mathbb{R}^{N-2d} \) there is an irreducible unitary representation \( \pi_{\rho, \zeta} \) of \( G \) on \( L^2(\mathbb{R}^d) \) such that

\[
\begin{align*}
\pi_{\rho, \zeta}(Y_j(\zeta))u(t) &= \partial u/\partial t_j , \quad j \leq d ; \\
\pi_{\rho, \zeta}(Y_{j+d}(\zeta))u(t) &= i\rho_j u(t) , \quad j \leq d ; \\
\pi_{\rho, \zeta}(Y_{j+2d}(\zeta))u(t) &= i\rho_j u(t) , \quad j \leq N - 2d ; \\
\pi_{\rho, \zeta}(Z)u(t) &= i\langle \zeta, Z \rangle u(t) , \quad Z \in \mathcal{G}_2 .
\end{align*}
\]

Furthermore every irreducible unitary representation of \( G \) is unitarily equivalent to \( \pi_{\rho, \zeta} \) for some \( \zeta \in \mathcal{F}_2^* \) and some \( \rho \in \mathbb{R}^{N-2d(\zeta)} \). Note that if \( \zeta = 0 \) we obtain the representation defined by (2.2).

For \( \zeta \in \mathcal{F}_2^* \), \( t \in \mathbb{R}^d \), \( \tau \in \mathbb{R}^d \) and \( \rho \in \mathbb{R}^{N-2d} \), \( d = d(\zeta) \), let \( \eta(t, \tau; \rho, \zeta) \) be that element \( \eta \) of \( \mathcal{F}_1^* \) such that

\[
\begin{align*}
\langle \eta, Y_j(\zeta) \rangle &= \tau_j , \quad \langle \eta, Y_{j+d}(\zeta) \rangle = t_j , \quad j \leq d ; \\
\langle \eta, Y_{j+2d}(\zeta) \rangle &= \rho_j , \quad j \leq N - 2d .
\end{align*}
\]

Let \( f \in C(\mathbb{R}^N) \) satisfy \( f \equiv 0 \) in a neighborhood of 0 and \( f \equiv 1 \) outside some bounded set. Define

\[
\Phi_{\rho, \zeta}(t, \tau) = 1 + f(t, \tau, \rho) \| \eta(t, \tau; \rho, \zeta) \| .
\]

Let \( \zeta \in \mathcal{F}_2^* \), \( \zeta \neq 0 \), be fixed. If for all \( \rho \in \mathbb{R}^{N-2d} \), \( q_\rho \in C(\mathbb{R}^{2d}) \) and for all multi-indices \( \alpha \) and \( \beta \) there is a \( C_{\alpha \beta} \) such that
for all \((t, \tau, \rho) \in \mathbb{R}^N\) we will write \("q_{\rho} \in S_{\rho, \zeta}^k\) uniformly in \(\rho\). \(\mathcal{L}_{\rho, \zeta}^k\) is the space of pseudodifferential operators with symbols in \(S_{\rho, \zeta}^k\); \(H_{\rho, \zeta}^k\) the corresponding global Sobolev space as defined in [1].

It follows from (2.5), (2.6) and (2.2) that, for \(X \in \mathcal{G}\),

\[
\text{sym} \pi_{\rho, \zeta}(X)(t, \tau) = \pi_{\tau(t, \tau, \rho, \zeta)}(X),
\]

where \(\text{sym} Q\) denotes the symbol of the operator \(Q\). Let \(\zeta \in \mathcal{G}_\ast^\ast\) be fixed and let \(\{X_1, \ldots, X_N\}\) be the basis for \(\mathcal{G}\) described at the beginning of this section. By (2.7) and (1.2),

\[
\pi_{\rho, \zeta}(X_j) \in \mathcal{L}_{\rho, \zeta}^{\mu_j} \quad \text{uniformly in } \rho \text{ if } j \leq N,
\]

\[
\pi_{\rho, \zeta}(X_j) \in \mathcal{L}_{\rho, \zeta}^{\nu_j} \quad \text{uniformly in } \rho \text{ if } j > N.
\]

Thus if \(P \in \mathcal{U}_{m}(\mathcal{G})\), then \(\pi_{\rho, \zeta}(P) \in \mathcal{L}_{\rho, \zeta}^{m} \) uniformly in \(\rho\).

**Lemma.** Let \(P \in \mathcal{U}_{m}(\mathcal{G})\) satisfy \(\pi_{\eta}(P) \neq 0\) for each of the one dimensional unitary representations \(\pi_{\eta}, \eta \in \mathcal{G}_\ast^\ast, \eta \neq 0\). Then for fixed \(\zeta \in \mathcal{G}_\ast^\ast\), \(\zeta \neq 0\), there is a \(c > 0\) and a \(C > 0\) such that

\[
|\text{sym} \pi_{\rho, \zeta}(P)(t, \tau)| \geq c \Phi_{\rho, \zeta}(t, \tau)^m
\]

for all \(\rho \in \mathbb{R}^{N-\delta}\) and all \((t, \tau) \in \mathbb{R}^{\delta}\) such that \(|t| + |\tau| \geq C\).

**Proof.** Let \(S = \{\eta \in \mathcal{G}_\ast^\ast : \|\eta\| = 1\}\) and let \(c_1 = \min \{\pi_{\eta}(P) : \eta \in S\}\). For arbitrary \(\eta \in \mathcal{G}_\ast^\ast, \eta \neq 0\), let \(r = \|\eta\|^{-1}\). Then \(\|\delta_{\rho, \zeta}\| = 1\). (2.3) implies that \(|\pi_{\eta}(P)| \geq c_1 \|\eta\|^m\). Thus letting \(p_{\rho, \zeta}(t, \tau) = \pi_{\tau(t, \tau, \rho, \zeta)}(P)\), we have

\[
|p_{\rho, \zeta}(t, \tau)| \geq c_1 \|\pi_{\tau(t, \tau, \rho, \zeta)}(P)\|^m.
\]

Let \(p_{\rho, \zeta} = \text{sym} \pi_{\rho, \zeta}(P)\). By (2.7), the pseudodifferential operator calculus, (2.9) and the remark following (2.9),

\[
p_{\rho, \zeta} - p'_{\rho, \zeta} \in S_{\rho, \zeta}^{m-1} \quad \text{uniformly in } \rho.
\]

Now there exist \(c_2 > 0\) and \(C_2\) such that if \(|t| + |\tau| \geq C_2\) then \(|\eta(t, \tau; \rho, \zeta)|^m \geq c_2(|t| + |\tau|)\) for all \(\rho\). Thus, by (2.10), there exist \(c_3 > 0\) and \(C_3\) such that if \(|t| + |\tau| \geq C_3\), then \(|p'_{\rho, \zeta}(t, \tau)| \geq c_3 \Phi_{\rho, \zeta}(t, \tau)^m\) for all \(\rho\). Also, by (2.11), it follows that given \(\varepsilon > 0\) there is a \(C_\varepsilon\) such that if \(|t| + |\tau| \geq C_\varepsilon\), then for all \(\rho\)

\[
|p_{\rho, \zeta}(t, \tau) - p'_{\rho, \zeta}(t, \tau)| < 1/2s \Phi_{\rho, \zeta}(t, \tau)^m.
\]

The lemma follows by taking \(C = \max\{C_2, C_\varepsilon\}(c_3)\).
Proof of Theorem 2. By the theorem of Helffer-Nourrigat-Rockland, to prove $P^*$ hypoelliptic it suffices to show that $\ker \pi_{\rho, \zeta}(P^*) = 0$ for all $\zeta \in \mathcal{S}_z^*$ and all $\rho \in R^{N-2d(\zeta)}$, except $\zeta = 0, \rho = 0$. (We consider $\pi_{\rho, \zeta}(P)$ and $\pi_{\rho, \zeta}(P^*)$ as bounded operators from $H^m_{\rho, \zeta}$ to $H^0_{\rho, \zeta}$). If $\zeta = 0$, then

$$\pi_{\rho, \zeta}(P^*) = \pi_{\rho, \zeta}(P) \neq 0$$

for all $\rho \neq 0$. If $\zeta \neq 0$, then by Theorem 7.2 of [1] and the above lemma, $\pi_{\rho, \zeta}(P)$ is Fredholm for all $\rho$. Also by Remark 1.4 of [4] and the Helffer-Nourrigat-Rockland Theorem, $\ker \pi_{\rho, \zeta}(P) = \ker \pi_{\rho, \zeta}(P) \cap \mathcal{S}_z^* = 0$. Hence it suffices to prove that $\text{ind} \pi_{\rho, \zeta}(P) = 0$.

We consider first the case when $d = d(\zeta) < N/2$. Let $q_{\rho, \zeta} = \text{sym} \pi_{\rho, \zeta}(P^*)$. By (2.12) and the above lemma there is a $c > 0$ and a $C$ such that $|q_{\rho, \zeta}(t, \tau)| \geq c\Phi_{\rho, \zeta}(t, \tau)^m$ for all $(t, \tau, \rho) \in R^N$ with $|t| + |\tau| \geq C$. Choose $f \in C^\infty(R^{2d})$ such that $f(t, \tau) \equiv 0$ if $|t| + |\tau| \leq C$, $f(t, \tau) \equiv 1$ if $|t| + |\tau| \geq 2C$. Let $a_{\rho, \zeta} \in S_{\rho, \zeta}^m$ uniformly in $\rho$ and $b_{\rho, \zeta} = 1 - a_{\rho, \zeta} \circ q_{\rho, \zeta} \in S_{\rho, \zeta}^m$ uniformly in $\rho$, where $p \circ q$ denotes the symbol of $p(t, D)q(t, D)$. Let $\psi(\tau) = (1 + |\tau|^2)^{1/2m}$. There is a $C > 0$ (depending on $\zeta$), such that $\psi(\tau) \leq C(1 + |\tau|^2)^{1/2m}$, and, by (2.8), such that $|\rho|^i \leq C\Phi_{\rho, \zeta}(t, \tau)$. Let $a_{\rho, \zeta} \in S_{\rho}^m$ uniformly in $\rho$ and $|\rho|^i b_{\rho, \zeta} \in S_{\rho}^0$ uniformly in $\rho$. By the $L^2$ boundedness theorem for pseudodifferential operators there is a $C_1$ such that $\|a_{\rho, \zeta}(t, D)u\| \leq C_1 \|u\|$ and $|\rho|^i \|b_{\rho, \zeta}(t, D)u\| \leq C_1 \|u\|$, for all $u \in L^2(R^d)$ and all $\rho$. Thus if $|\rho|^i \geq 2C_1$,

$$\|u\| \leq \|a_{\rho, \zeta}(t, D)\pi_{\rho, \zeta}(P^*)u\| + \|b_{\rho, \zeta}(t, D)u\| \leq C_1 \|\pi_{\rho, \zeta}(P^*)u\| + 1/2 \|u\|.$$ 

Hence $\pi_{\rho, \zeta}(P^*)$ is injective and thus $\text{ind} \pi_{\rho, \zeta}(P) = 0$ if $|\rho|^i \geq 2C_1$. Since $\text{ind} \pi_{\rho, \zeta}(P)$ is independent of $\rho$, $\text{ind} \pi_{\rho, \zeta}(P) = 0$ for all $\rho$. If $d = d(\zeta) = N/2$, we write $\pi_{\zeta}$ for $\pi_{0, \zeta}$. Define $\varphi: R_+^d \oplus R_+^d \to \mathcal{S}_z^*$ by $\varphi(t, \tau) = \eta(t, \tau; 0, \zeta)$, as defined before (2.6). Let $\delta_{\tau} = \varphi^{-1} \circ \delta_{\tau} \circ \varphi$. Then $\{\delta_{\tau}\}$ is a family of dilations on $R^{2d}$. Let $p_{\zeta}(t, \tau) = \pi_{\eta(t, \tau; 0, \zeta)}(P)$. It follows from (2.3) that $p_{\tau}$ is homogeneous of degree $m$ with respect to $\{\delta_{\tau}\}$ and by (2.12) $p_{\zeta}$ is $\Phi_{\zeta}$-elliptic. Since $p_{\zeta} = \text{sym} \pi_{\zeta}(P) \in S_{\zeta}^{-1}$ we can apply Theorem 1 to find $\text{ind} \pi_{\zeta}(P)$. If $d > 1$, then $\text{ind} \pi_{\zeta}(P) = 0$.

If $d = 1$ and $B(\zeta) = \{Y_1(\zeta), Y_2(\zeta)\}$, set $Y_1(-\zeta) = Y_2(\zeta), Y_2(-\zeta) = Y_1(\zeta)$. Then $B(-\zeta) = \{Y_1(-\zeta), Y_2(-\zeta)\}$ satisfies (2.4) for $-\zeta$. Also $\eta(t, \tau; -\zeta) = \eta(\tau, t; \zeta)$ and $p_{-\zeta}(t, \tau) = p_{\zeta}(\tau, t)$. By Theorem 1

$$2\pi \text{ind} \pi_{-\zeta}(P) = \Delta_{p_{-\zeta}} \arg p_{-\zeta} = -\Delta_{p_{\zeta}} \arg p_{\zeta} = -2\pi \text{ind} \pi_{\zeta}(P).$$

But $\ker \pi_{\zeta}(P) = \ker \pi_{-\zeta}(P) = 0$ implies $\text{ind} \pi_{\zeta}(P) \geq 0$ and $\text{ind} \pi_{-\zeta}(P) \geq 0$. Thus $\text{ind} \pi_{\zeta}(P) = 0$. 


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