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THE SUPPORT OF AN EXTREMAL DILATATION

MARVIN E. ORTEL

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We introduce a density condition which applies to subsets, E , of a bounded region Ω in the complex plane. If E satisfies this condition, then it is possible to construct a quasiconformal mapping F , of Ω , subject to the following conditions: F is extremal for its boundary values; F is conformal throughout $\Omega - E$; F is not conformal on E . The construction makes essential use of the Hamilton-Reich-Strebel characterization of extremal quasiconformal maps.

O. Introduction. In all that follows, Ω denotes a bounded domain in the complex plane. Let κ denote an element of $\mathcal{L}^\infty(\Omega)$. We say that κ is an *extremal dilatation* (on Ω) if $\|\kappa\|_\infty \neq 0$ and κ is the complex dilatation of a quasiconformal mapping of Ω which is extremal for its boundary values.

R. Hamilton, E. Reich and K. Strebel have given an incisive characterization of extremal dilatations. Their result follows ([1], [3], [4], [5]):

Let $B(\Omega)$ denote the space of functions, f , analytic on Ω , for which

$$\|f\| = \int |f(z)| dA(z) < \infty \quad (\text{area measure}).$$

Then κ is an extremal dilatation if and only if $(0 < \|\kappa\|_\infty < 1)$ and

$$(*) \quad \sup_{\substack{\|f\| = 1 \\ f \in B(\Omega)}} \left| \int_{\Omega} f(z) \kappa(z) dA(z) \right| = \|\kappa\|_\infty.$$

It is well known that a bounded measurable function κ may be supported on a small subset of Ω and still satisfy condition (*). In this paper we attempt to quantify this feature. We show that subsets of Ω which satisfy a certain density condition will always support extremal dilatations.

Density conditions which are *necessarily* satisfied by the support of an extremal dilatation are known in the case that Ω is the unit disk or the upper half plane. Some of these are discussed in [2]. They have features in common with the present sufficient condition, but in no case is there a complete characterization.

1. A sufficient condition. If E is a subset of Ω , χ_E denotes the indicator function of E :

$$\chi_E(z) = \begin{cases} 1, & z \in E \\ 0, & z \notin E. \end{cases}$$

Let E denote a subset of Ω . We say that E is analytically thick in Ω if there is a bounded analytic function, h , defined on Ω for which $\|h\|_\infty = 1$ and

$$(1.1) \quad \int_{H(x)} \chi_E(z) |h(z)| dA(z) = (1 + o(1)) \int_{H(x)} |h(z)| dA(z),$$

as $x \rightarrow 1$; in (1.1), $H(x) = \{z \in \Omega: |h(z)| > x\}$ for $0 \leq x < 1$; also, $dA(z)$ denotes Lebesgue planar measure.

THEOREM 1. *Suppose E is analytically thick in Ω . Then there is an extremal dilatation, κ , defined on Ω for which*

$$\{z \in \Omega: \kappa(z) \neq 0\} \subset E.$$

Proof. The proof of Theorem 1 depends on Lemma 2 of § 4 and on the theorem of Hamilton-Reich-Strebel.

Let h be given as in the definition. By Lemma 2, condition (1.1) implies

$$(1.2) \quad \int_{\Omega} \chi_E(z) |h(z)|^n dA(z) = (1 + o(1)) \int_{\Omega} |h(z)|^n dA(z), \quad n \rightarrow \infty.$$

Let $N = \{1, 2, 3, \dots\}$ and let $\|\cdot\|_1$ denote the norm in $\mathcal{L}^1(\Omega)$. For each $n \in N$, set $k_n(z) = h^n(z) / \|h^n\|_1$. So, $\|k_n\|_1 = 1$ and, by (1.2)

$$(1.3) \quad \lim_{n \rightarrow \infty} \int_{\Omega-E} |k_n(z)| dA(z) = 0.$$

It can be shown that $\{k_n: n \in N\}$ is a normal family; there are two possibilities:

(1) at least one subsequence of $\langle k_n \rangle_{n \in N}$ converges, uniformly on compact subsets of Ω , to a function $K(z)$ which is analytic and not identically zero on Ω .

(2) $\langle k_n \rangle_{n \in N}$ converges to zero uniformly on compact subsets of Ω .

In Case 1, apply Fatou's theorem to the given subsequence: we see, by (1.3)

$$\int_{\Omega-E} |K(z)| dA(z) \leq \overline{\lim}_{n \rightarrow \infty} \int_{\Omega-E} |k_n(z)| dA(z) = 0.$$

Therefore measure $(\Omega - E) = 0$ since K is analytic and not identically zero.

In Case 2, we construct a sequence, $\langle A_n \rangle$, of mutually disjoint

compact subsets of Ω , and a subsequence, $\langle K_n \rangle$, of $\langle k_n \rangle$, such that

$$(1.4) \quad \int_{\Omega - A_n} |K_n(z)| dA(z) \leq \frac{1}{n}, \quad n \in N.$$

First, A_1 and K_1 are chosen arbitrarily. Suppose K_1, K_2, \dots, K_n and A_1, A_2, \dots, A_n have been chosen; we take K_{n+1} , from the vanishing sequence $\langle k_n \rangle$, so that

$$\int_{\bigcup_{j=1}^n A_j} |K_{n+1}(z)| dA(z) \leq \frac{1}{2(n+1)}.$$

Since $\|K_{n+1}\|_1 = 1$, we may choose A_{n+1} , disjoint from A_1, A_2, \dots, A_n , so that

$$\int_{A_{n+1}} |K_{n+1}(z)| dA(z) \geq 1 - \frac{1}{n+1};$$

this is the same as (1.4).

Now we set

$$\kappa(z) = \begin{cases} \overline{K_n(z)}/|K_n(z)|, & z \in E \cap A_n, \quad n \in N \\ 0 & , \text{ otherwise.} \end{cases}$$

Take $n \in N$; (1.4) implies

$$\begin{aligned} \left| \int_{\Omega} K_n(z) \kappa(z) dA(z) \right| &\geq \left| \int_{A_n} K_n(z) \kappa(z) dA(z) \right| - 1/n \\ &= \int_{E \cap A_n} |K_n(z)| dA(z) - 1/n \geq \int_E |K_n(z)| dA(z) - 2/n. \end{aligned}$$

Combine this with (1.3); since $\langle K_n \rangle$ is a subsequence of $\langle k_n \rangle$, we have

$$\lim_{n \rightarrow \infty} \left| \int_{\Omega} K_n(z) \kappa(z) dA(z) \right| = \|\kappa\|_{\infty}.$$

It now follows, from the theorem of Hamilton, Reich and Strebel, that $\kappa/2$ is an *extremal dilatation*. As κ is supported within E , we are through.

2. An example. Set $\Omega: |z-1| < 1$ and $h(z) = e^{-1} \exp \{-(2i/\pi) \log z\}$. Then $|h(re^{i\theta})| = e^{-1} \exp \{(2/\pi)\theta\}$ and, if $e^{-2} < x < 1$ and $\theta(x) = \pi/2(1 + \log x)$, we have

$$H(x) = \{re^{i\theta}: \theta(x) < \theta < \pi/2 \quad \text{and} \quad 0 < r < 2 \cos \theta\}.$$

For $-\pi/2 < \theta < \pi/2$, we set $l(\theta) = \{re^{i\theta}: 0 < r < 2 \cos \theta\}$.

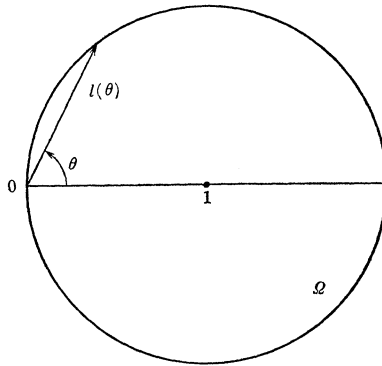


FIGURE 1

Now, let $E \subset \Omega$. We assume that the linear density of E on $l(\theta)$ approaches one as θ approaches $\pi/2$: that is, we assume

$$(2.1) \quad \int_0^{2 \cos \theta} \chi_E(re^{i\theta}) dr = (1 + o(1))2 \cos \theta, \quad \theta \longrightarrow \pi/2 .$$

It is a consequence of (2.1) that

$$(2.2) \quad \int_0^{2 \cos \theta} \chi_E(re^{i\theta}) r dr = (1 + o(1)) \int_0^{2 \cos \theta} r dr, \quad \theta \longrightarrow \pi/2 .$$

This can be seen in a few lines; we integrate $\int_0^{2 \cos \theta} \chi_E(re^{i\theta}) r dr$ by parts, then use (2.1) and the estimate

$$\int_0^{2 \cos \theta} \int_0^t \chi_E(re^{i\theta}) dr dt \leq \int_0^{2 \cos \theta} t dt .$$

In turn, from (2.2), we see

$$\begin{aligned} & \int_{\theta(x)}^{\pi/2} e^{-1} \exp \left\{ \frac{2\theta}{\pi} \right\} \int_0^{2 \cos \theta} \chi_E(re^{i\theta}) r dr d\theta \\ &= (1 + o(1)) \int_{\theta/(x)}^{\pi/2} e^{-1} \exp \left\{ \frac{2\theta}{\pi} \right\} \int_0^{2 \cos \theta} r dr d\theta, \quad \text{as } x \longrightarrow 1 ; \end{aligned}$$

and this is the same as

$$\int_{H(x)} \chi_E(z) |h(z)| dA(z) = (1 + o(1)) \int_{H(x)} |h(z)| dA(z), \quad x \longrightarrow 1 .$$

By Theorem 1, E is an extremal support. So, if E satisfies condition (2.1), there is an extremal quasiconformal mapping of Ω which is conformal outside of E but not conformal throughout Ω .

3. LEMMA 1. Let f and g denote integrable functions defined on $(0, 1)$. We assume: $0 \leq f(r) \leq g(r)$ for all $r, 0 < r < 1; \int_x^1 g(r) dr >$

0 for all x , $0 \leq x < 1$; and

$$(3.1) \quad \int_x^1 r f(r) dr = (1 + o(1)) \int_x^1 r g(r) dr, \quad x \longrightarrow 1.$$

Then

$$(3.2) \quad \int_0^1 r^n f(r) dr = (1 + o(1)) \int_0^1 r^n g(r) dr, \quad n \longrightarrow \infty.$$

Proof. Let $\varepsilon > 0$ be fixed. By condition (3.1), we may choose $x(\varepsilon)$, in $(0, 1)$, so that,

$$(3.3) \quad \int_x^1 r(g(r) - f(r)) dr \leq \varepsilon/2 \int_x^1 r g(r) dr$$

if $x(\varepsilon) \leq x < 1$. This implies that

$$\int_y^1 \int_x^1 r(g(r) - f(r)) dr dx \leq \varepsilon/2 \int_y^1 \int_x^1 r g(r) dr dx$$

holds as long as $x(\varepsilon) \leq y < 1$. We interchange the order of integration and obtain

$$\int_{r=y}^1 r(g(r) - f(r))(r - y) dr \leq \varepsilon/2 \int_{r=y}^1 r g(r)(r - y) dr;$$

then, by (3.3), we see that

$$\int_{r=y}^1 r^2(g(r) - f(r)) dr \leq \varepsilon/2 \int_{r=y}^1 r^2 g(r) dr$$

for any y , $x(\varepsilon) \leq y < 1$.

Repeat this argument with the same $x(\varepsilon)$. We see that (3.3) is valid with r replaced by r^n . Thus,

$$(3.4) \quad \int_{x(\varepsilon)}^1 r^n(g(r) - f(r)) dr \leq \varepsilon/2 \int_{x(\varepsilon)}^1 r^n g(r) dr$$

holds for all $n \in N$.

Set $M = \int_0^1 g(t) - f(t) dt$. Then, by (3.4), if $n \in N$, we have

$$(3.5) \quad \int_0^1 r^n(g(r) - f(r)) dr \leq Mx(\varepsilon)^n + \varepsilon/2 \int_{x(\varepsilon)}^1 r^n g(r) dr.$$

Now, set $x_1(\varepsilon) = (x(\varepsilon) + 1)/2$. Since $\int_{x_1(\varepsilon)}^1 g(t) dt > 0$, we may choose $N(\varepsilon, f, g) \in N$ so that, if $n \geq N(\varepsilon, f, g)$, we have

$$Mx(\varepsilon)^n \leq \varepsilon/2(x_1(\varepsilon))^n \int_{x_1(\varepsilon)}^1 g(r) dr \leq \varepsilon/2 \int_0^1 r^n g(r) dr$$

(just note that $x(\varepsilon) < x_1(\varepsilon)$). Combine this with (3.5); if $n \geq N(\varepsilon, f, g)$, we have

$$\int_0^1 r^n (g(r) - f(r)) dr \leq \varepsilon \int_0^1 r^n g(r) dr .$$

We proved that

$$\int_0^1 r^n (g(r) - f(r)) dr = o(1) \int_0^1 r^n g(r) dr , \quad n \longrightarrow \infty ,$$

and (3.2) now follows.

4. LEMMA 2. The technique here is to perform an iterated integration over the level curves of $|h|$. For the sake of completeness, we establish the existence of an appropriate induced measure on these curves. So, the proof is a little longer than is perhaps necessary.

LEMMA 2. Let h denote a bounded analytic function on Ω with $\|h\|_\infty = 1$. For $0 \leq x < 1$, we set $H(x) = \{z \in \Omega: |h(z)| > x\}$. Then, if $E \subset \Omega$ and

$$(4.1) \quad \int_{H(x)} X_E(z) |h(z)| dA(z) = (1 + o(1)) \int_{H(x)} |h(z)| dA(z)$$

as $x \rightarrow 1$, it follows that

$$(4.2) \quad \int_{\Omega} \chi_E(z) |h(z)|^n dA(z) = (1 + o(1)) \int_{\Omega} |h(z)|^n dA(z) ,$$

as $n \rightarrow \infty$.

Proof. Set $\Omega' = \{z \in \Omega: |h(z)| \neq 0 \text{ and } |h'(z)| \neq 0\}$. The lemma is trivial when h is a constant function. If h is not constant (as we assume from now on), the set $\Omega - \Omega'$ is negligible with regard to integration.

We construct an open cover of Ω' . For each $z \in \Omega'$, $U(z)$ will denote an open subset of Ω' which contains z ; moreover, we assume h is one-to-one in each $U(z)$.

Now, let $\{P_n: n \in N\}$ be a C^∞ partition of unity, on Ω' , subordinate to the cover $\{U(z): z \in \Omega'\}$. So, for each $n \in N$, there is a set $U(n) \in \{U(z): z \in \Omega'\}$ which contains the support of P_n . Set $h[U(n)] = S(n)$ and let $S(n) \xrightarrow{z_n} U(n)$ ($w \rightarrow z_n(w)$) denote the inverse of h defined in $S(n)$. For $0 < r < 1$, $n \in N$ we set

$$\Theta_n(r) = \{\theta: 0 \leq \theta < 2\pi, re^{i\theta} \in S(n)\}$$

and we define

$$f_n(r) = \int_{\Theta_n(r)} P_n(z_n(re^{i\theta})) \chi_E(z_n(re^{i\theta})) |z'_n(re^{i\theta})|^2 r d\theta$$

$$g_n(r) = \int_{\Theta_n(r)} P_n(z_n(re^{i\theta})) |z'_n(re^{i\theta})|^2 r d\theta$$

and

$$f(r) = \sum_{n \in N} f_n(r), \quad g(r) = \sum_{n \in N} g_n(r).$$

It is clear that $0 \leq f(r) \leq g(r)$, $0 < r < 1$.

If $n \in N$ is arbitrary and $0 \leq x < 1$ and $0 \leq \theta < 2\pi$, note that

$$\chi_{H(x)}(z_n(re^{i\theta})) \equiv \Phi_x(r) = \begin{cases} 1, & x < r < 1 \\ 0, & 0 < r \leq x. \end{cases}$$

Take $N \in N$ and suppose $0 \leq x < 1$: then, with $w = re^{i\theta}$,

$$\begin{aligned} \int_{H(x)} \chi_E(z) |h(z)|^N dA(z) &= \sum_{n \in N} \int_{\Omega'} P_n(z) \chi_{H(x)}(z) \chi_E(z) |h(z)|^N dA(z) \\ &= \sum_{n \in N} \int_{S(n)} P_n(z_n(w)) \chi_{H(x)}(z_n(w)) \chi_E(z_n(w)) |w|^N |z'_n(w)|^2 dA(w) \\ &= \sum_{n \in N} \int_{r=0}^1 \int_{\Theta_n(r)} P_n(z_n(\cdot)) \chi_{H(x)}(z_n(\cdot)) \chi_E(z_n(\cdot)) r^N |z'_n(\cdot)|^2 r d\theta dr \\ &= \sum_{n \in N} \int_{r=0}^1 \Phi_x(r) f_n(r) r^N dr = \int_0^1 \Phi_x(r) f(r) r^N dr. \end{aligned}$$

We conclude from the Monotone Convergence Theorem that f is integrable on $(0, 1)$. In summary, if $0 \leq x < 1$ and $N \in N$ we have

$$(4.3) \quad \int_{H(x)} |h(z)|^N \chi_E(z) dA(z) = \int_x^1 f(r) r^N dr$$

and, by the same reasoning,

$$(4.4) \quad \int_{H(x)} |h(z)|^N dA(z) = \int_x^1 g(r) r^N dr.$$

By (4.4), $\int_x^1 g(r) dr > 0$ if $0 < x < 1$. By hypothesis (4.1) and equations (4.3) and (4.4), in the case $N = 1$, we see

$$\int_x^1 f(r) r dr = (1 + o(1)) \int_x^1 g(r) r dr, \quad x \longrightarrow 1.$$

Thus, by Lemma 1,

$$\int_0^1 f(r) r^N dr = (1 + o(1)) \int_0^1 g(r) r^N dr, \quad N \longrightarrow \infty.$$

So, by (4.3) and (4.4), in the case $x = 0$,

$$\int_{H(o)} |h(z)|^N \chi_E(z) dA(z) = (1 + o(1)) \int_{H(o)} |h(z)|^N dA(z),$$

as $N \rightarrow \infty$. Since $\Omega - H(o)$ is countable, we are through.

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