A GRAPH AND ITS COMPLEMENT WITH SPECIFIED PROPERTIES. VI. CHROMATIC AND ACHROMATIC NUMBERS

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Dedicated to Ruth Bari

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We characterize the graphs $G$ such that both $G$ and its complement $\bar{G}$ are $n$-colorable, and we specify explicitly all 171 graphs for the case $n = 3$. We then determine the 41 graphs for which both $G$ and $G$ have achromatic number 3.

1. Introduction. We follow the terminology and notation of [1] but we include some basic definitions for completeness. A coloring of a graph $G$ is an assignment of colors to its points so that whenever two points are adjacent they are colored differently. An $n$-coloring of $G$ is a coloring of $G$ which uses $n$ colors. A complete $n$-coloring of $G$ is an $n$-coloring of $G$ such that, for every pair of distinct colors there exists a pair of adjacent points in $G$ which receive the given pair of colors. The chromatic number $\chi = \chi(G)$ of a graph $G$ is the least integer $n$ such that $G$ has an $n$-coloring. We say that $G$ is $n$-colorable if $\chi(G) \leq n$. Alternatively, $\chi(G)$ can be characterized as the least integer $n$ such that $V(G)$ has a partition into $n$ subsets each of which induces a totally disconnected subgraph. Obviously if $n = \chi(G)$ then every $n$-coloring of $G$ is complete. The achromatic number $\psi = \psi(G)$ of a graph $G$ is the greatest integer $m$ such that $G$ has a complete $m$-coloring. Clearly every graph $G$ of order $p$ has a $p$-coloring, but this coloring is only complete if $G$ is $K_p$.

A homomorphism of a graph $G$ onto a graph $G'$ is a function $\phi$ from $V(G)$ onto $V(G')$ such that, whenever $u$ and $v$ are adjacent points of $G$, their images $\phi(u)$ and $\phi(v)$ are adjacent in $G'$. Since no point of a graph is adjacent with itself, two adjacent points of $G$ cannot have the same image under any homomorphism of $G$. If $G'$ is the image of $G$ under a homomorphism $\phi$, we write $G' = \phi(G)$. The order of $\phi$ is $|V(\phi(G))|$. A homomorphism $\phi$ of $G$ is complete of order $n$ if $\phi(G) = K_n$. Thus every graph $G$ has a complete homomorphism of order $\chi(G)$ and also a complete homomorphism of order $\psi(G)$, and $\chi(G)$ and $\psi(G)$ are the smallest and largest orders of the complete homomorphisms of $G$. It was shown by Harary, Hedetniemi and Prins [2] that $G$ also has a complete homomorphism of order $n$ for all intermediate $n$.

It is convenient to write $G > H$ when $H$ is an induced subgraph of $G$. If $X$ is a set of points in a graph $G$ then we use $\langle X \rangle$ to denote the
subgraph $G$ induced by $X$. If necessary to avoid ambiguity we can write $\langle X \rangle_G$ and $\langle X \rangle_H$ if $X$ is a set of points in two different graphs $G$ and $H$. We write $\overline{\chi}(G)$ for $\chi(\overline{G})$ and $\overline{\psi}(G)$ for $\psi(\overline{G})$.

2. The chromatic number. We are concerned in this section with those graphs $G$ for which both $G$ and $\overline{G}$ are $n$-colorable.

**Theorem 1.** Let $G_1, G_2, \ldots, G_k$ be the components of a graph $G$. Then $\overline{\chi}(G) = \Sigma \overline{\chi}(G_i)$.

**Proof.** We first prove the inequality $\chi(G) \leq \Sigma \chi(G_i)$ holds if $G_1, G_2, \ldots, G_k$ are induced subgraphs of $G$ such that $V(G) = \cup V(G_i)$. For each $1 \leq i \leq k$ there exists a family $S_i$ of subsets $V(G_i)$ whose union is $V(G_i)$, with $|S_i| = \chi(G_i)$, and such that each $S \in S_i$ induces in $G_i$ a totally disconnected subgraph. Let $S = \cup S_i$. Then $S$ is a family of subsets of $V(G)$, whose union is $V(G)$, such that each $S \in S$ induces in $G$ a totally disconnected subgraph. Thus $\chi(G) \leq |S| \leq \Sigma |S_i| = \Sigma \chi(G_i)$.

Next we show that $\overline{\chi}(G) \geq \Sigma \overline{\chi}(G_i)$ if $G_1, G_2, \ldots, G_k$ are the components of $G$. There exists a family $S$ of subsets of $V(G)$, whose union is $V(G)$, with $|S| = \overline{\chi}(G)$, such that each $S \in S$ induces in $\overline{G}$ a totally disconnected subgraph. For each $1 \leq i \leq k$, let $S_i = \{S \in S \mid S \cap V(G_i) \neq \emptyset\}$. Points from different components of $G$ are adjacent in $\overline{G}$, so the subfamilies $S_i, 1 \leq i \leq k$, constitute a partition of $S$. Each $S_i$ is such that every $S \in S_i$ induces in $\overline{G_i}$ a totally disconnected subgraph, so $|S_i| \leq \overline{\chi}(G_i)$. Thus $\overline{\chi}(G) = |S| \leq \Sigma |S_i| \geq \Sigma \overline{\chi}(G_i)$.

Since each $\overline{G_i}$ is an induced subgraph of $\overline{G}$, the theorem is an immediate consequence of the discussion above.

The corollaries which follow include a characterization of graphs $G$ such that $G$ and $\overline{G}$ are both $n$-colorable.

**Corollary 1a.** Let $G_1, G_2, \ldots, G_k$ be the components of $G$. Then $G$ and $\overline{G}$ are both $n$-colorable if and only if

(i) $\chi(G_i) \leq n$ for every $1 \leq i \leq k$, and

(ii) $\Sigma \overline{\chi}(G_i) \leq n$.

**Proof.** This follows directly from Theorem 1 and the fact that $\chi(G) = \max \chi(G_i)$.

**Corollary 1b.** If $G$ has $k$ components, then $\overline{\chi}(G) \geq k$. If $k = \overline{\chi}(G)$, then each component of $G$ is complete.

**Proof.** As $G$ has $k$ components $G_i$, $\overline{G}$ must contain $K_k$. If $k = \overline{\chi}(G)$, then $\Sigma \overline{\chi}(G_i) = k$, so for each $i$, $\overline{\chi}(G_i) = 1$, whence $\overline{G_i}$ is totally disconnected and therefore $G_i$ is complete.
For the special case of disconnected graphs $G$ such that $G$ and $\overline{G}$ are both 3-colorable, Theorem 1 leads to a particularly simple characterization.

**Corollary 1c.** If a graph $G$ is disconnected then $G$ and $\overline{G}$ are both 3-colorable if and only if one of the following conditions is satisfied.

(i) $G$ has exactly 3 components each of which is a complete graph of order no greater than 3.

(ii) $G$ has exactly 2 components, $G_1$ and $G_2$, and $G_1$ is a complete graph of order no greater than 3, and $G_2$ is 3-colorable and $\overline{G_2}$ is 2-colorable.

**Proof.** Let $G_1, G_2, \ldots, G_k$ be the components of a disconnected graph $G$.

Suppose first that $G$ and $\overline{G}$ are both 3-colorable. By Corollary 1b we need consider only two possible values of $k$.

**Case 1.** $k = 3$.
In this case $k = \chi(G)$ so Corollary 1b applies and each $G_i$ is complete. Then $\chi(G) \leq 3$ implies that each $G_i$ is of order no greater than 3. In this case $G$ satisfies condition (i).

**Case 2.** $k = 2$.
From Theorem 1 we get $\chi(G_1) + \chi(G_2) = \chi(G) \leq 3$. Without loss of generality we may conclude that $\chi(G_1) = 1$ and $\chi(G_2) \leq 2$. As in Case 1 it follows that $G_1$ is complete of order no greater than 3. Thus $G_2$, being a subgraph of $G$, is 3-colorable, and $\overline{G_2}$ is 2-colorable because $\chi(G_2) \leq 2$. In this case $G$ satisfies condition (ii).

Suppose conversely that $G$ satisfies either (i) or (ii).

**Case 1'.** $G$ satisfies (i).
Let $G_1, G_2$ and $G_3$ be the components of $G$. Then each $G_i$ is complete so $V(G_i)$ induces in $\overline{G}$ a totally disconnected subgraph, thus $\chi(G) \leq 3$. Because each $G_i$ is of order no greater than 3 we can partition $V(G)$ into three subsets $V_1', V_2'$ and $V_3'$ such that $|V_i' \cap V(G_j)| \leq 1$ for $1 \leq j, j \leq 3$. Then each $V_i'$ induces in $G$ a totally disconnected subgraph, so $\chi(G) \leq 3$. In this case $G$ and $\overline{G}$ are both 3-colorable.

**Case 2'.** $G$ satisfies (ii).
In this case Corollary 1a clearly implies that $G$ and $\overline{G}$ are both 3-colorable.

**Theorem 2.** If a graph $G$ is $n$-colorable, then $\chi(G)$ is the least integer $t$ such that $V(G)$ can be partitioned into $t$ subsets $V_1, V_2, \ldots, V_t$ and for each $1 \leq i \leq t$, $|V_i| \leq n$ and $V_i$ induces a complete subgraph.
Proof. By definition $\chi(G)$ is the least integer $t$ such that $V(G)$ can be partitioned into $t$ subsets $V_1, V_2, \ldots, V_t$ each of which induces in $\overline{G}$ a totally disconnected subgraph. Also for any subset $S$ of $V(G)$, $S$ induces in $\overline{G}$ a totally disconnected subgraph if and only if $S$ induces in $G$ a complete subgraph, in which case $|S| \leq \chi(G) \leq n$.

The corollaries which follow include another characterization of graphs $G$ such that $G$ and $\overline{G}$ are both $n$-colorable which can usefully be applied to connected graphs.

**Corollary 2a.** A graph $G$ and its complement are both $n$-colorable if and only if there exist positive integers $s, t \leq n$ such that

(i) For each $1 \leq i \leq s$ there is a positive integer $a_i \leq t$ such that $\bigcup K_{a_i}$ is a spanning subgraph of $G$.

(ii) For each $1 \leq i \leq t$ there is a positive integer $b_i \leq s$ such that $\bigcup K_{b_i}$ is a spanning subgraph of $G$.

Moreover the minimum values of $s$ and $t$ which satisfy these conditions are $\chi(G)$ and $\overline{\chi}(G)$ respectively.

Proof. Suppose first that $G$ and $\overline{G}$ are both $n$-colorable. Let $s = \chi(G)$ and $t = \overline{\chi}(G)$, so $s, t \leq n$. As $G$ is $s$-colorable, by Theorem 2 there is a partition of $V(G)$ into $t = \overline{\chi}(G)$ subsets $V_1, \ldots, V_t$ such that for each $1 \leq i \leq t$, $|V_i| \leq s$ and $V_i$ induces a complete subgraph in $G$. Writing $b_i = |V_i|$, we have $\bigcup K_{b_i} = \bigcup \langle V_i \rangle$ as a spanning subgraph of $G$.

Similarly, since $\overline{G}$ is $t$-colorable and $\overline{\chi}(G) = s$, the same argument applied to $\overline{G}$ yields $\bigcup K_{a_i}$ as a spanning subgraph of $\overline{G}$ for some sequence of positive integers $a_i \leq t$.

Now suppose conversely that $G$ is a graph which satisfies conditions (i) and (ii). By condition (i), there is a partition of $V(G)$ into $s$ subsets $V_1, \ldots, V_s$ such that for each $1 \leq i \leq s$, $V_i$ induces a complete subgraph in $\overline{G}$. Then each $V_i$ induces in $G$ a totally disconnected subgraph. Thus $\chi(G) \leq s \leq n$, so $G$ is $n$-colorable. Also note that the least value of $s$ which can satisfy (i) is $\chi(G)$ since $\chi(G) \leq s$. Similarly by (ii) we deduce $\overline{\chi}(G) \leq t \leq n$, so $\overline{G}$ is $n$-colorable and $\overline{\chi}(G)$ is the minimum possible value for $t$.

**Corollary 2b.** If a graph $G$ and its complement are both $n$-colorable then the order of $G$ is at most $n^2$.

Although this corollary is clearly a consequence of the partition described in Theorem 2, we should also point out that it is also a special case of the well known result of Nordhaus and Gaddum [3] that the order $p$ of a graph satisfies the inequality, $p \leq \chi \overline{\chi}$. It is convenient to include here another useful consequence of the Nordhaus-Gaddum theorem.
COROLLARY 2c. If a graph $G$ and its complement are both $n$-colorable and the order of $G$ exceeds $n(n - 1)$, then $\chi(G) = \bar{\chi}(G) = n$.

Proof. Since $\chi(G) \leq n$ and $\bar{\chi}(G) \leq n$, if either were actually less than $n$ then $\chi(G) \cdot \bar{\chi}(G)$ would be no greater than $n(n - 1)$.

Our final corollary of this theorem deals again with the special case $n = 3$.

COROLLARY 2d. If a graph $G$ of order $p$ and its complement $\bar{G}$ are both 3-colorable, then $p \leq 9$ and

(i) if $p = 9$, then $G$ and $\bar{G}$ each contain $3K_3$ as a subgraph,
(ii) if $p = 8$, then $G$ and $\bar{G}$ each contain $2K_3 \cup K_2$ as a subgraph,
(iii) if $p = 7$, then $G$ and $\bar{G}$ each contain either $K_3 \cup 2K_2$ or $2K_3 \cup K_1$ as a subgraph.

Proof. Suppose that $G$ and $\bar{G}$ are both 3-colorable. Then by Corollary 2b the order $p$ of $G$ is at most 9. If $p \geq 7$ then by Lemma 2c, $\chi(G) = \bar{\chi}(G) = 3$. Thus by Corollary 2a, depending on the value of $p$, $G$ and $\bar{G}$ must contain the subgraphs described above.

We complete this section by cataloguing all graphs $G$ of order 6 or less and all disconnected graphs $G$ of order 7, 8 or 9 for which $G$ and $\bar{G}$ are both 3-colorable. Because there are 171 graphs in this category we will not illustrate them. Rather we describe each such graph by specifying an ordered triple $(p, q, n)$ where $p$ denotes the order and $q$ the size of the graph and $n$ denotes its numerical designation in the Graph Diagrams in Appendix 1 of [1]. Every graph of order 6 or less appears in these diagrams and the triple $(p, q, n)$ completely describes such graphs. The disconnected graphs of order 7, 8, and 9 for which $\chi \leq 3$ and $\bar{\chi} \leq 3$ do not appear in the diagrams, but their components do, and we indicate such graphs by specifying their components. There are pairs $(p, q)$ for which only one graph of order $p$ and size $q$ exists. Such graphs do not have a numerical designation in the Graph Diagrams. We hereby confer the designation 1 on all such graphs. Thus in the lists which follow the triple $(2, 1, 1)$ represents the unique graph of order 2 and size 1, namely $K_2$. Our list of disconnected graphs of order 7 through 9 with $\chi = \bar{\chi} = 3$ are really complete, by the following argument. By Corollary 1c, all such graphs have 3 components each of order 3 or less or 2 components, $G_1$ and $G_2$, with $G_1$ complete of order 3 or less and $\chi(G_1) \leq 3$, $\bar{\chi}(G_2) \leq 2$. By the Nordhaus-Gaddum theorem we conclude that the order of $G_2$ is no greater than 6, so $G_2$ is in List C, our list of all graphs of order 6 or less with $\chi = 3$, $\bar{\chi} = 2$. 
List A. $\chi + \overline{\chi} \leq 4$.
\begin{align*}
\chi &= \overline{\chi} = 1: (1, 0, 1) \text{ which is } K_1.
\chi &= 1 \text{ and } \overline{\chi} = 2: (2, 0, 1) \text{ which is } \overline{K}_1.
\chi &= 2 \text{ and } \overline{\chi} = 1: (2, 1, 1) \text{ which is } K_2.
\chi &= 1 \text{ and } \overline{\chi} = 3: (3, 0, 1) \text{ which is } K_3.
\chi &= \overline{\chi} = 2, \text{ connected: } (3, 2, 1), (4, 3, 2), \text{ and } (4, 4, 2) \text{ which are } P_3, P_4 \text{ and } C_4.
\chi &= \overline{\chi} = 2, \text{ disconnected: } (3, 1, 1) \text{ and } (4, 2, 2) \text{ which are } K_1 \cup K_2 \text{ and } 2K_2.
\end{align*}

List B. $\chi = 2$ and $\overline{\chi} = 3$.
Connected: (4, 3, 3), (5, 4, 4), (5, 4, 6), (5, 5, 3), (5, 6, 5) and $p = 6$ with $(q, n) = (5, 7), (5, 10), (5, 14), (6, 7), (6, 9), (6, 11), (7, 5), (7, 14), (8, 23), (9, 17)$.
Disconnected: (4, 1, 1), (4, 2, 1), (5, 2, 2), (5, 3, 1), (5, 3, 4), (5, 4, 1), (6, 3, 5), and (6, 4, 8).

List C. $\chi = 3$ and $\overline{\chi} = 2$.
Connected: (4, 4, 1), (4, 5, 1), (5, 5, 4), (5, 6, 1), (5, 6, 4), (5, 6, 6), (5, 7, 1), (5, 8, 2), and $p = 6$ with $(q, n) = (7, 23), (8, 5), (8, 14), (9, 7), (9, 9), (9, 11), (10, 7), (10, 10), (10, 14), (11, 8), (12, 5)$.
Disconnected: (4, 3, 1), (5, 4, 5) and (6, 6, 17).

List D. $\chi = \overline{\chi} = 3$, order 6 or less.
Connected: $p = 5$ with $(q, n) = (5, 2), (5, 5), (5, 6), (6, 2), (7, 2); (6, 5, 3); (p, q) = (6, 6)$ with $n = 8, 10, 13, 14, 18, 20$;
$(p, q) = (6, 7)$ with $n = 6, 7, 8, 9, 10, 11, 12, 13, 16, 19, 20, 21, 24$;
$(p, q) = (6, 8)$ with $n = 1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 16, 19, 20, 21, 24$;
$(p, q) = (6, 9)$ with $n = 2, 3, 5, 8, 10, 13, 14, 18, 19, 20; (6, 10, 3), (6, 10, 12), (6, 10, 15)$.
Disconnected: (5, 3, 2), (5, 4, 2), (5, 5, 1);
$p = 6$ with $(q, n) = (4, 6), (5, 12), (5, 15), (6, 2), (6, 3), (6, 5), (6, 19), (7, 1), (7, 2)$.

List E. $\chi = \overline{\chi} = 3$, of order 7, 8, or 9, disconnected $3K_3, 2K_3 \cup K_2, K_3 \cup 2K_2, 2K_3 \cup K_1,$ and $K_3 \cup G$ where $G$ is any connected graph in List C, and $K_2 \cup G$ where $G$ is any connected graph of order 5 or 6 in List C, and $K_1 \cup G$ where $G$ is any connected graph of order 6 in List C.

Of the 171 graphs which appear in these lists, 116 have $\chi = \overline{\chi} = 3$. In addition to these the complements of the 51 graphs in List E are connected graphs of order 7 through 9 with $\chi = \overline{\chi} = 3$. And Corollary 2d implies that there are many other graphs of order 7 through 9 with
\( \chi = \bar{\chi} = 3 \) which are not in our lists, of which one example is \( G = C_7 + e \) where the edge \( e \) joins two points whose distance in \( C_7 \) is 2. In this case clearly both \( G \) and \( \bar{G} \) contain \( K_3 \cup 2K_2 \) as a subgraph so \( \chi(G) = \bar{\chi}(G) = 3 \).

3. The achromatic number. We first characterize graphs \( G \) with \( \psi(G) = 2 \).

**Theorem 3.** A graph \( G \) has achromatic number 2 if and only if each component of \( G \) is complete bipartite.

**Proof.** Obviously the union of complete bipartite graphs has \( \psi = 2 \). For the converse, assume that \( \psi = 2 \), then \( \chi \leq 2 \) since \( \chi \leq \psi \) for any graph. Thus \( G \) must be bipartite. Moreover each component of \( G \) cannot contain \( P_4 \) as an induced subgraph since \( \psi(P_4) = 3 \). Thus each component of \( G \) must be complete bipartite.

**Corollary 3a.** The only graphs with \( \psi = \bar{\psi} = 2 \) are \( C_4, 2K_2, K_{1,2} \) and \( K_2 \cup K_1 \).

We now develop some results in the form of five lemmas for finding all graphs with \( \psi = \bar{\psi} = 3 \). We write \( uAv \) to indicate adjacency and \( u \bar{A}v \) for nonadjacency. The first lemma was proved by exhaustion and we omit the detailed verification.

**Lemma 4a.** Among all graphs of order 6, only the six graphs \( 2K_3, 2K_2 + K_2, C_4 + K_2 \) and their complements \( K_{3,3}, C_4 \cup K_2 \) and \( 3K_2 \) satisfy the property that either \( G \) or \( \bar{G} \) contains two point-disjoint triangles and \( \psi = \bar{\psi} \leq 3 \).

\[
\begin{align*}
2K_3; & \\
2K_2 + K_2; & \\
C_4 + K_2; & \\
K_{3,3}; & \\
C_4 \cup K_2; & \\
3K_2; & \\
\end{align*}
\]

**Figure 1.** The six graphs of order 6 with \( \psi, \bar{\psi} \leq 3 \)
LEMMA 4b. Among all graphs of order 7, only the six graphs $2K_3 \cup K_1$, $2K_3 + \overline{K}_3$, $C_4 + \overline{K}_3$ and their complements satisfy the property that either $G$ or $\overline{G}$ contains two point-disjoint triangles and $\psi, \overline{\psi} \leq 3$.

Proof. Assume that $\psi = \overline{\psi} = 3$ and that $G$ contains two point-disjoint triangles $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_4, v_5, v_6\}$. Then the subgraph $H$ of $G$ induced by these six points in one of the three graphs, $2K_3$, $K_2 + \overline{K}_2$ or $C_4 + \overline{K}_2$, of Lemma 4a; otherwise either $G$ or $\overline{G}$ contains an induced subgraph of order 6 which has achromatic number at least 4 and so $\psi$ or $\overline{\psi}$ would be at least 4, a contradiction to the hypothesis. By $w$ we denote the seventh point in $V(G) - V(H)$, and divide the proof into three cases according to whether $H$ is $2K_3$, $2K_2 + \overline{K}_2$, or $C_4 + \overline{K}_2$.

If $G = H \cup K_1$, it is easily verified that $\psi = \overline{\psi} = 3$. Now we may assume that $G \supset H \cup K_1$ properly. Then there is a point $v_i$ in $G$ which is adjacent to $w$. Without loss of generality we may assume that $wAv_i$. On the other hand, there is at least one point $v_i$, $i = 4, 5$ or 6, which is not adjacent to $w$, say $v_4$ as shown in Figure 3, otherwise all three points $v_i$, $i = 4, 5$, and 6 are adjacent to $w$ and so $\{v_4, v_5, v_6, w\}$ induces $K_4$, a contradiction.
Then it is easy to see that $\psi(G) = 4$ regardless of whether or not $wAv_i$ for $i = 2, 3, 5, 6$, a contradiction.

**Case 2.** $H = 2K_2 + \overline{K}_2$.

As $\psi = \bar{\psi} = 3$, we know that $\chi, \bar{\chi} \leq 3$ so by Lemma 2c, $\chi = \bar{\chi} = 3$. Thus by Corollary 2d, $\overline{G}$ contains a triangle. As $H = 2K_2 + \overline{K}_2 = G - w$, it follows that $G$ contains $C_4 \cup K_2$ as an induced subgraph. Hence there are two possibilities: either $G \supset F_1$ or $\overline{G} \supset F_2$, where $F_1, F_2$ are the graphs illustrated in Figure 4, which we now consider as two subcases.

**Figure 4.** A step in the proof of Case 2

**Case 2a.** $\overline{G} \supset F_1$.

If $\overline{G} \not= F_1$, then $w$ is adjacent to at least one more point of $G$, i.e., to $v_1, v_2, v_4$, or $v_5$. We may assume that $w$ is adjacent to $v_1$ or $v_2$ from the symmetry of $F_1$. In either case, $\bar{\psi} = 4$, a contradiction. On the other hand, if $\overline{G} = F_1$ then $\bar{\psi} = 4$, a contradiction.

**Case 2b.** $\overline{G} \supset F_2$.

If $\overline{G} = F_2$, then $\psi = \bar{\psi} = 3$. If $\overline{G} \not= F_2$, then $w$ is adjacent to one of the points $v_i$, $i = 1, 3, 4$ or 6. From the symmetry of $F_2$, we may assume that $wAv_1$. Then it is easy to see that $\psi = 4$, a contradiction.
Case 3. \( H = C_4 + \overline{K}_2 \).

Since \( \overline{G} \supseteq K_3 \) from Corollary 2d, and \( H = 3K_2 \), it follows that \( \overline{G} \supseteq 2K_2 \cup K_3 \). We may assume without loss of generality that \( \{v_2, v_5, w\} \) induces \( K_3 \) in \( \overline{G} \); see Figure 5. If \( \overline{G} = 2K_2 \cup K_3 \), then \( \psi = \overline{\psi} = 3 \). If \( \overline{G} \neq 2K_2 \cup K_3 \), then \( w \) must be adjacent to at least one of \( v_i \), \( i = 1, 3, 4 \) or 6. Assuming now that \( wAv_1 \), we see that \( \overline{\psi} = 4 \), a contradiction.

![Figure 6. A labelling of \( K_3 \cup 2K_2 \)](image)

**Lemma 4c.** If \( G \) is a graph of order 7 such that neither \( G \) nor \( \overline{G} \) contains two point-disjoint triangles, then \( \psi \) or \( \overline{\psi} \) is at least 4.

**Proof.** Assume that \( \psi = \overline{\psi} = 3 \), then \( \chi, \overline{\chi} \leq 3 \) since \( \chi \leq \psi \). By applying Lemma 2c, \( \chi = \overline{\chi} = 3 \). Thus \( G \supseteq K_3 \cup 2K_2 \) or \( G \supseteq 2K_3 \cup K_2 \) by Corollary 2d. But by the hypothesis, \( G \) cannot contain two point-disjoint triangles and so, \( G, G \supseteq K_3 \cup 2K_2 \). Now we label the points of \( K_3 \cup 2K_2 \) as in Figure 6.
By the symmetry of $G$ and $\bar{G}$, it is sufficient to handle only the case $u_2Aw_2$. By the hypothesis that $G$ cannot contain two point-disjoint triangles, $v_1Aw_2$ and $v_2Au_2$. Then regardless of the presence or absence of other lines, we can easily verify that $\psi = 4$, a contradiction.

**Lemma 4d.** There are no graphs of order at least 8 such that $\psi = \bar{\psi} = 3$.

**Proof.** Assume that $G$ has order 8 and $\psi = \bar{\psi} = 3$. Then $\chi = \bar{\chi} = 3$ by Lemma 2c. Thus both $G$ and $\bar{G}$ contain $2K_3 \cup K_2$ as a spanning subgraph by Corollary 2d. The subgraph of $G$ induced by the set of points of $2K_3$ must be one of the three graphs, $2K_3, 2K_2 + \bar{K}_2$ or $C_4 + \bar{K}_2$ of Lemma 4a. We now divide the proof into three cases:

**Case 1.** $G$ contains $2K_3$ as an induced subgraph.

By Corollary 2d, both $G$ and $\bar{G}$ contain $2K_3 \cup K_2$ hence of course $\bar{G} \supseteq 2K_3$. It is convenient to label $G$ as in Figure 7.

![Figure 7](image)

**Figure 7.** A subgraph of $\bar{G}$

By symmetry, we may assume that both point sets $\{u_3, u_6, v_1\}$ and $\{u_2, u_5, v_2\}$ induce $K_3$ in $G$. Then it is easily verified that $\bar{\psi} = 4$.

**Case 2.** $G$ contains $2K_2 + \bar{K}_2$ as an induced subgraph.

Let $F_1, F_2$ be the graphs illustrated in Figure 8.
Since $\bar{G} \supset 2K_3$ by Corollary 2d, there are two possibilities: either $\bar{G} \supset F_1$ or $\bar{G} \supset F_2$. However in either case, $\psi = 4$.

**Case 3.** $G$ contains $C_4 + \bar{K}_2$ as an induced subgraph.

Since $\bar{G} \supset 2K_3$ by Corollary 2d, we may assume that both $\{v_1, u_2, u_3\}$ and $\{v_2, u_3, u_4\}$ induce $K_3$ in $\bar{G}$, see Figure 9, and thus $\psi = 4$, a contradiction.
Combining the preceding four lemmas, we obtain the following result.

**Lemma 4e.** Let $G$ be a graph of order at least 7, then $G$ has $\psi = \bar{\psi} = 3$ if and only if $G$ is one of the six graphs, $2K_3 \cup K_1$, $K(3, 3, 1)$, $C_4 \cup C_3$, $2K_2 + \bar{K}_3$, $2K_2 \cup K_3$ and $K(3, 2, 2)$.

We are now ready to specify all the graphs with $\psi = \bar{\psi} = 3$.

**Theorem 4.** There are exactly 41 graphs $G$ such that both $G$ and $\bar{G}$ have achromatic number 3: six have order 7, twenty are of order 6, fourteen of order 5 and just one of order 4.

*Proof.* By Lemma 4d, we know that there are no such graphs of order $p \geq 8$. Lemma 4e lists all six graphs with $p = 7$ and Figure 2 shows them. To complete the list of all the graphs with $\psi = \bar{\psi} = 3$, we had to resort to the method of brute force by an exhaustive inspection of Appendix I of [1] for $p = 4, 5, \text{and} 6$.

As the determination of all graphs with $\psi = \bar{\psi} = n \geq 4$ appears to be hopelessly complicated, we can realistically ask only for the construction of additional families of graphs with $\psi = \bar{\psi}$.

**References**


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