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**LINEAR TRANSFORMATIONS THAT PRESERVE THE  
NILPOTENT MATRICES**

PETER BOTTA, STEPHEN J. PIERCE AND WILLIAM E. WATKINS

## LINEAR TRANSFORMATIONS THAT PRESERVE THE NILPOTENT MATRICES

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Let  $\mathfrak{sl}_n$  be the algebra of  $n \times n$  matrices with zero trace and entries in a field with at least  $n$  elements. Let  $\mathcal{N}$  be the set of nilpotent matrices. The main result in this paper is that the group of nonsingular linear transformations  $L$  on  $\mathfrak{sl}_n$  such that  $L(\mathcal{N}) = \mathcal{N}$  is generated by the inner automorphisms:  $X \rightarrow S^{-1}XS$ ; the maps:  $X \rightarrow aX$ , for  $a \neq 0$ ; and the map:  $X \rightarrow X'$  that sends a matrix  $X$  to its transpose.

**Introduction.** Let  $M_n$  be the algebra of  $n \times n$  matrices over a field  $K$  and let  $S$  be an algebraic set in  $M_n$ . There are a number of theorems characterizing the linear maps  $L$  on  $M_n$  that preserve  $S$ , i.e.  $L(S) \subseteq S$ . For example there are results for  $\{X: \det X = 0\}$  by Dieudonné [4],  $\{X: \text{rank } X \leq 1\}$  by Jacob [8] and Marcus and Moyls [10], the orthogonal group by Pierce and Botta [2] and other linear groups by Dixon [5]. In every instance the transformations  $L$  that preserve these various algebraic sets have one of these two forms:

$$(1) \quad L(X) = PXQ, \quad \text{for all } X$$

or

$$(2) \quad L(X) = PX'Q, \quad \text{for all } X$$

where  $P$  and  $Q$  are in  $M_n$ . There are conditions on  $P$  and  $Q$  which depend on the algebraic set  $S$ . For example if  $S = \{X: \det X = 0\}$  and  $L$  is nonsingular then  $P$  and  $Q$  are nonsingular; if  $S$  is the orthogonal group then  $PQ = I$  and  $P$  must be a scalar multiple of a matrix in the orthogonal group over the algebraic closure of  $K$ . For a good survey of further results of this type see Marcus [9].

In this paper we characterize the nonsingular linear transformations  $L$  that preserve the set  $\mathcal{N}$  of nilpotent matrices. Since the linear span of  $\mathcal{N}$  is the space  $\mathfrak{sl}_n$  of matrices with trace zero, we may as well assume that  $L$  is a transformation on  $\mathfrak{sl}_n$ . (In order to see that  $\mathcal{N}$  spans  $\mathfrak{sl}_n$ , let  $E_{ij}$  be the matrix whose only nonzero entry is a 1 in position  $(i, j)$ . The nilpotent matrices  $E_{ij}$  and  $E_{ii} + E_{ij} - E_{ji} - E_{jj}$  for  $i \neq j$  span  $\mathfrak{sl}_n$ .)

Actually we characterize all nonsingular semilinear mappings that preserve nilpotence. The main theorem can be extended to matrices with entries from an integral domain. The extension follows from a modification of a result of Chevalley [3, p. 104, Théorème 3].

**THEOREM.** *Let  $n \geq 3$ ,  $K$  be a field with at least  $n$  elements and suppose that  $L$  is a nonsingular linear transformation on  $\mathfrak{sl}_n$  such that  $L(\mathfrak{N}) \subseteq \mathfrak{N}$ . Then  $L$  either has form (1) or (2), where  $PQ$  is a non-zero scalar matrix.*

Without the assumption that  $L$  is nonsingular the theorem is false. Any map whose image is contained in the algebra  $\mathfrak{U}$  of the strictly upper triangular matrices preserves nilpotence. The proof of the theorem depends on a result of Gerstenhaber about maximal spaces of nilpotent matrices. We also use some elementary algebraic geometry and the fundamental theorem of projective geometry [1, p. 88, Theorem 2.26].

**LEMMA 1** (Gerstenhaber [6]). *Suppose  $K$  has at least  $n$  elements and  $\mathfrak{N}$  is a space of nilpotent matrices. Then  $\dim \mathfrak{N} \leq n(n-1)/2$ . If  $\dim \mathfrak{N} = n(n-1)/2$ , then there exists a non-singular matrix  $S$  such that  $\mathfrak{N} = S^{-1}\mathfrak{U}S$ , where  $\mathfrak{U}$  is the algebra of strictly upper triangular matrices. Moreover, any matrix of nilindex  $n$  is contained in exactly one maximal nilpotent algebra.*

**Tangent Spaces.** Let  $K[X] = K[X_{11}, \dots, X_{nn}]$  be the ring of polynomials in  $n^2$  variables with coefficients in  $K$ . For  $r = 1, 2, \dots, n$ , let  $E_r(X) \in K[X]$  be the  $r$ th elementary symmetric function of the matrix  $X = (X_{ij})$ , i.e.  $E_r(X)$  is the sum of all principal  $r \times r$  subdeterminants of  $X$ . We let  $J$  be the ideal in  $K[X]$  generated by  $E_1(X), \dots, E_n(X)$  and  $\text{rad } J = \{F \in K[X]: F^k \in J \text{ for some positive integer } k\}$ . Clearly we have  $\mathfrak{N} = \{A \in M_n: F(A) = 0 \text{ for all } F \in J\}$ . If  $A \in \mathfrak{N}$  then

$$\tan(J, A) = \left\{ B \in M_n: \left. \frac{dF}{dt}(A + tB) \right|_{t=0} = 0 \text{ for all } F \in J \right\}$$

and

$$\tan(\text{rad } J, A) = \left\{ B \in M_n: \left. \frac{dF}{dt}(A + tB) \right|_{t=0} = 0 \text{ for all } F \in \text{rad } J \right\}.$$

Both of these are vector spaces and the second is the usual tangent space at the point  $A$  of the algebraic set  $\mathfrak{N}$ . Further, the second is a subspace of the first.

If  $A$  and  $B$  belong to  $\mathfrak{N}$  and are similar then their tangent spaces defined above are related by the appropriate similarity. Further note that  $C \in \tan(J, A)$  if and only if  $(d/dt)E_r(A + tC)|_{t=0} = 0$  for all  $r = 1, 2, \dots, n$ . If  $A \in \mathfrak{N}$  is of nilindex  $n$ , then, by taking  $A$  into upper Jordan canonical form, one sees that the equations for  $X \in \tan(J, A)$  are, up to a similarity,

$$0 = \sum_{i=0}^{n-j} X_{j+i, i+1}, \quad j = 1, 2, \dots, n.$$

Therefore  $\dim \tan(J, A) = n^2 - n$ . Since  $J$  is generated by  $n$  polynomials, if  $N$  is of nilindex  $n$  we have [7, p. 28, 37]

$$n^2 - n \leq \dim \mathcal{U} \leq \dim \tan(\text{rad } J, N) \leq \dim \tan(J, N) = n^2 - n.$$

So if  $N$  is of nilindex  $n$  then  $\tan(\text{rad } J, N) = \tan(J, N)$ .

LEMMA 2. *If  $A, B \in \mathcal{U}$  are both of nilindex  $n$  then  $AB = BA$  if and only if  $\tan(\text{rad } J, A) = \tan(\text{rad } J, B)$ .*

*Proof.*  $A$  is of nilindex  $n$  so its minimal and characteristic polynomials are equal. Therefore, if  $AB = BA$ , then  $B$  is a polynomial in  $A$ . By the above remarks, we may assume that

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

so

$$B = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where  $a_i \in K$ . Since  $B$  is of nilindex  $n$ ,  $a_1 \neq 0$ . A direct computation shows that

$$\left. \frac{d}{dt} E_n(B + tX) \right|_{t=0} = a_1^{n-1} X_{n1}.$$

Hence the equation for  $B$  arising from  $E_n$  is  $X_{n1} = 0$ , which is the same as for  $A$ . One has that

$$\left. \frac{d}{dt} E_r(B + tX) \right|_{t=0} = a_1^{r-1} \sum_{i=0}^{n-r} X_{r+ii+1} + \sum_{j=1}^{r-1} A_j \sum_{i=0}^{n-j} X_{j+ii+1}$$

for suitable constants  $A_j$  depending on  $a_1, \dots, a_{n-1}$ . By induction, the equation for  $B$  arising from  $E_r$  is

$$a_1^{r-1} \sum_{i=0}^{n-r} X_{r+i+1} = 0,$$

and since  $a_1 \neq 0$  this is the same as for  $A$ . Since  $\tan(\text{rad } J, A) = \tan(\text{rad } J, A)$  the results follows.

On the other hand, suppose  $\tan(\text{rad } J, A) = \tan(\text{rad } J, B)$ . We may assume  $A$  is as above. Let  $E_{ij}$  be the matrix with 1 in the  $(i, j)$  position and zeros elsewhere. Then

$$E_{ji} \in \tan(\text{rad } J, A), \quad i > j,$$

and

$$E_{ji} - E_{j+1, i+1} \in \tan(\text{rad } J, A), \quad i \leq j.$$

Writing  $B = (b_{ij})$ , we have

$$\begin{aligned} \frac{d}{dt} E_2(B + tE_{ji}) \Big|_{t=0} &= b_{ij}, \quad \text{if } i > j, \\ \frac{d}{dt} E_2(B + t(E_{ji} - E_{j+1, i+1})) \Big|_{t=0} &= \pm (b_{ij} - b_{i+1, j+1}) \quad \text{if } i \leq j. \end{aligned}$$

Therefore  $b_{ij} = 0$  if  $i > j$  and  $b_{ij} = b_{i+1, j+1}$  if  $i \leq j$ , and  $B$  is a polynomial in  $A$ .

**LEMMA 3.** *If  $L: \mathfrak{sl}_n \rightarrow \mathfrak{sl}_n$  is a nonsingular linear transformation with the property that  $L(\mathfrak{U}) = \mathfrak{U}$ , and  $A \in \mathfrak{U}$ , then  $L(\tan(\text{rad } J, A)) = \tan(\text{rad } J, L(A))$ .*

*Proof.* The map  $\tilde{L}: K[X] \rightarrow K[X]$  defined by  $\tilde{L}(f)(A) = f(L(A))$  is a  $K$ -algebra homomorphism. Since  $L$  is nonsingular and  $L(\mathfrak{U}) = \mathfrak{U}$  and  $\text{rad } J = \{f \in K[X]: f(N) = 0, \text{ for all } N \in \mathfrak{U}\}$ , we have  $\tilde{L}(\text{rad } J) = \text{rad } J$ . Thus

$$\begin{aligned} \tan(\text{rad } J, L(A)) &= \left\{ B \in M_n: \frac{df}{dt}(L(A) + tB) \Big|_{t=0} \text{ for all } f \in \text{rad } J \right\} \\ &= \left\{ L(C) \in M_n: \frac{df}{dt}(L(A) + tL(C)) \Big|_{t=0} \text{ for all } f \in \text{rad } J \right\} \\ &= \left\{ L(C) \in M_n: \frac{d\tilde{L}(f)}{dt}(A + tC) \Big|_{t=0} = 0 \text{ for all } f \in \text{rad } J \right\} \\ &= L(\tan(\text{rad } J, A)). \end{aligned}$$

**Proof of theorem.** First we observe that  $L(\mathfrak{N}) = \mathfrak{N}$ . This follows from Lemma 1 of Dixon [5] and the fact that  $L$  is nonsingular.

We now show that  $L$  preserves nilindex  $n$ . If  $A \in \mathfrak{N}$  and  $\text{rank } A \leq n - 2$ , then  $A$  kills two linearly independent vectors  $v, w$ . Let  $\mathfrak{N}_1, \mathfrak{N}_2$  be maximal nilpotent algebras containing  $A$  and killing  $v, w$  respectively. Every maximal nilpotent algebra kills exactly one line, so  $\mathfrak{N}_1 \neq \mathfrak{N}_2$ . By Lemma 1,  $L$  maps maximal nilpotent algebras to maximal nilpotent algebras and again by lemma 1,  $L$  preserves the matrices of nilindex  $n$ .

Now we show that if  $A \in \mathfrak{N}$  has rank one, then so does  $L(A)$ . Let  $U$  be the unit auxiliary matrix  $E_{12} + \cdots + E_{n-1,n}$ .

First note that the only members of  $\mathfrak{U}$  which commute with  $U$  and  $E_{12}$  are multiples of  $E_{1n}$ . Thus the centre of any maximal nilpotent algebra is one-dimensional and is generated by a rank one matrix.

Let  $A \in \mathfrak{N}$  have rank one. Then for some nonsingular  $S$ ,  $S^{-1}AS = E_{1n}$ . Let  $\mathfrak{N} = S\mathfrak{U}S^{-1}$ . Then  $A$  generates the centre of  $\mathfrak{N}$ . Let  $V \in \mathfrak{N}$  have nilindex  $n$ . Then  $V$  and  $A + V$  have nilindex  $n$  and commute. It follows from Lemmas 2 and 3 that  $L(A + V)$  commutes with  $L(V)$ . Hence  $L(A)$  commutes with  $L(V)$ . Since the nilindex  $n$  matrices in  $\mathfrak{N}$  generate  $\mathfrak{N}$ ,  $L(A)$  is in the centre of the maximal nilpotent algebra  $L(\mathfrak{N})$ . Hence  $L(A)$  has rank one.

We next define two bijections on the lines (through the origin) of  $K^n$  and use the fundamental theorem of projective geometry. For each line  $\langle v \rangle \in K^n$ , define two  $n - 1$  dimensional subspaces of  $\mathfrak{N}$  by

$$M(v) = \{X \in \mathfrak{N} \mid \text{Im } X = \langle v \rangle\},$$

$$M'(v) = \{X^t \mid X \in M(v)\}.$$

We will show that  $L(M(v)) = M(w)$  or  $M'(w)$  and  $L(M'(v)) = M(w')$  or  $M'(w')$  for some  $w, w' \in K^n$ . The bijections will be  $\varphi(v) = w$  and  $\theta(v) = w'$ .

We note a few facts about  $M(v)$ . Any nonzero member of  $M(v)$  has rank one. If  $v, w \in K^n$ , and are nonzero, then  $M(v)$  and  $M(w)$  are conjugate, and if  $w = Av$ ,  $A$  nonsingular, then  $M(w) = AM(v)A^{-1}$ . In tensor notation,  $M(v) = v \otimes v^\perp$  and  $M'(v) = v^\perp \otimes v$ . (Here,  $\perp$  means orthogonal complement with respect to the dot product.) It is easily verified that  $M(u) \cap M(v) = M(u) = M(v)$  if  $u$  and  $v$  are linearly dependent and 0 otherwise, and that  $M(u) \cap M'(v) = \langle u \otimes v \rangle$  if  $u \cdot v = 0$  and is 0 otherwise. Finally, observe that any  $n - 1$  dimensional subspace of  $\mathfrak{N}$  with all of its nonzero matrices having rank one must be an  $M(v)$  or an  $M'(v)$ . It follows that for  $v \in K^n$ , there is a  $w \in K^n$  such that  $L(M(v)) = M(w)$  or  $M'(w)$ .

Suppose we have  $v, w \in K^n$  with  $L(M(v)) = M(v')$  and  $L(M(w)) = M'(w')$ . Since  $n \geq 3$ , pick  $u$  orthogonal to  $v$  and  $w$ . Then  $M(v) \cap M'(u)$

and  $M(w) \cap M'(u)$  are one dimensional. If  $L(M'(u)) = M(u')$  then  $M(u') \cap M(v') = L(M'(u) \cap M(v))$  has dimension 1; which is impossible, as  $M(u') \cap M(v')$  has dimension 0 or  $n - 1 \geq 2$ . If  $L(M'(u)) = M'(u')$ , we reach a similar contradiction. A similar argument holds when we examine the images of  $M'(v)$  and  $M'(w)$ . Thus, by replacing  $L$  with the map  $X \rightarrow L(X)'$  if necessary, we may assume that for any nonzero  $v \in K^n$ ,  $L(M(v)) = M(w)$  and  $L(M'(v)) = M'(u)$  for appropriate  $u, w \in K^n$ .

Thus we define two maps  $\varphi, \theta$  induced by  $L$  on the lines of  $K^n$ . We have  $L(M(v)) = M(\varphi(v))$  and  $L(M'(v)) = M'(\theta(v))$  for  $v \in K^n$ .

Since  $L(\mathcal{U}) = \mathcal{U}$ ,  $L^{-1}$  also preserves nilpotence and hence  $\varphi$  and  $\theta$  are bijections on the lines of  $K^n$ .

Now we show that  $\varphi$  and  $\theta$  preserve coplanarity of lines in  $K^n$  and thus satisfy the hypothesis of the fundamental theorem of projective geometry. Let  $\langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle$  be three distinct coplanar lines in  $K^n$ . Then

$$\begin{aligned} 2n - 1 &= \dim(M(u_1) + M(u_2) + M(u_3)) \\ &= \dim L(M(u_1) + M(u_2) + M(u_3)) \\ &= \dim(M(\varphi(u_1)) + M(\varphi(u_2)) + M(\varphi(u_3))). \end{aligned}$$

If  $\varphi(u_1), \varphi(u_2), \varphi(u_3)$  are linearly independent then

$$\dim(M(\varphi(u_1)) + M(\varphi(u_2)) + M(\varphi(u_3))) = 3n - 3$$

and this is impossible since  $n \geq 3$ . Thus  $\varphi(u_1), \varphi(u_2), \varphi(u_3)$  are coplanar and  $\varphi$  satisfies the hypothesis of the fundamental theorem of projective geometry. So does  $\theta$ . Thus there exist semilinear maps  $S$  and  $T$  on  $K^n$  such that  $\varphi(u) = \langle Su \rangle$  and  $\theta(u) = \langle Tu \rangle$ , for all nonzero  $u$  in  $K^n$ .

There are linear maps  $P$  and  $Q$  on  $K^n$  and automorphisms  $\sigma$  and  $\tau$  on  $K$  such that  $Sv = P(\sigma v)$  and  $Tv = Q(\tau v)$ . (The automorphisms act componentwise.) Then

$$L(M(v)) = M(P\sigma v) = PM(\sigma v)P^{-1}$$

and

$$L(M'(v)) = M'(Q\tau v) = Q'^{-1}M'(\tau v)Q'.$$

Suppose  $u \cdot v = 0$ . Then  $\dim(M(u) \cap M'(v)) = 1$  and so

$$\dim(M(P\sigma u) \cap M'(Q\tau v)) = 1$$

and thus  $(P\sigma u) \cdot (Q\tau v) = 0$ , i.e.,

$$u \cdot \sigma^{-1}(P'Q\tau v) = 0.$$

Let  $R$  be the semilinear map defined by

$$Rv = \sigma^{-1}(P'Q\tau v).$$

Then  $u \cdot v = 0$  implies  $u \cdot Rv = 0$ . Thus  $R = dI$  is a scalar map,  $\sigma = \tau$  and  $P'Q = dI$ .

Replace the map  $L$  with the map  $X \rightarrow P^{-1}L(X)P$ . Then  $L(M(v)) = M(\sigma v)$  and  $L(M'(v)) = M'(\sigma v)$ , for all nonzero  $v$  in  $K^n$ . Thus if  $u \cdot v = 0$  then  $L(u \otimes v) = c(u \otimes v)\sigma(u \otimes v)$ , where  $c$  is a scalar valued function. If  $v \in \langle u_1, u_2 \rangle^\perp$ , then by comparing  $L((u_1 + u_2) \otimes v)$  with  $L(u_1 \otimes v) + L(u_2 \otimes v)$  we get  $c(u_1 \otimes v) = c(u_2 \otimes v)$ . Similarly if  $u \in \langle v_1, v_2 \rangle$ , then  $c(u \otimes v_1) = c(u \otimes v_2)$ .

Now we show that  $c$  is a constant function. Suppose that  $u_1 \cdot v_1 = 0$  and  $u_2 \cdot v_2 = 0$ . Pick  $v_3 \in \langle u_1, u_2 \rangle^\perp$ . Then  $c(u_1 \otimes v_1) = c(u_1 \otimes v_3) = c(u_2 \otimes v_3) = c(u_2 \otimes v_2)$ . Thus  $c$  is a constant function say  $c(u \otimes v) = k$ . Then  $L(u \otimes v) = k\sigma(u \otimes v)$ , for all  $u, v$  with  $u \cdot v = 0$ .

Since  $L$  is linear,  $\sigma$  is the identity automorphism. The rank one nilpotent matrices span  $\mathfrak{sl}_n$  and so the theorem is proved.

REMARK. When  $n = 2$ , the same result is obtained by a simple computation.

#### REFERENCES

1. E. Artin, *Geometric Algebra*, Interscience, New York, 1957.
2. E. P. Botta and S. Pierce, *The preservers of any orthogonal group*, Pacific J. Math., **70** (1977), 37–50.
3. C. Chevalley, *Théorie des Groupes de Lie*, Hermann, Paris, 1968.
4. J. Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables*, Arch. Math., **1** (1948), 282–287.
5. J. Dixon, *Rigid embeddings of simple groups in the general linear group*, Canad. J. Math., **29** (1977), 384–391.
6. M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices I*, Amer. J. Math., **80** (1958), 614–622.
7. J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975.
8. H. G. Jacob, *Coherence invariant mappings on Kronecker products*, Amer. J. Math., **77** (1955), 177–189.
9. M. Marcus, *Linear transformations on matrices*, J. Res., Nat. Bur. of Stand., v. 75B, Nos. 3 and 4 (1971), 107–113.
10. M. Marcus and B. Moysl, *Transformations on tensor product spaces*, Pacific J. Math., **9** (1959), 1215–1221.



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