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BUNDLES OVER CONFIGURATION SPACES

Frederick Ronald Cohen, Ralph Cohen, Nicholas J. Kuhn and Joseph Alvin Neisendorfer
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F. R. COHEN, R. L. COHEN, N. J. KUHN AND J. L. NEISENDORFER

Let \( F(R^n, k) \) be the configuration space of ordered sets of \( k \) distinct points in \( R^n \). \( F(R^n, k) \) is acted upon freely by the symmetric group on \( k \) letters, \( \Sigma_k \). In this paper we calculate the order of the vector bundles

\[
\xi_{n,k} : F(R^n, k) \times \Sigma_k R^k \to F(R^n, k)/\Sigma_k.
\]

Applications to the study of iterated loop spaces of spheres are also discussed.

1. The study of the stable homotopy type of the spaces \( \Omega^n S^{n+r} \) has received much attention in recent years [2, 8, 13]. The starting point for this study was Snaith's stable decomosition [18]:

\[
\Omega^n S^{n+r} \simeq_s \bigvee_{k \geq 0} F(R^n, k)^+ \wedge_{\Sigma_k} S^{r(k)},
\]

where \( F(R^n, k)^+ \) is the configuration space of \( k \) ordered distinct points in \( R^n \) together with a disjoint basepoint, \( S^{r(k)} \) is the \( k \)-fold smash product of \( S^r \) with itself, \( \Sigma_k \) is the symmetric group of \( k \) letters, and where \( " \simeq_s " \) denotes stable homotopy equivalence.

The space \( F(R^n, k)^+ \wedge_{\Sigma_k} S^{r(k)} \) is clearly the Thom complex of the \( r \)-fold Whitney sum of the vector bundle

\[
\xi_{n,k} : F(R^n, k) \times \Sigma_k R^k \to F(R^n, k)/\Sigma_k.
\]

If \( M(\xi_{n,k}) \) is the associated Thom spectrum, then Snaith's theorem gives an equivalence of spectra

\[
\Sigma^{\infty} \Omega^n S^{n+r} \simeq \bigvee_{k \geq 0} \Sigma^r \Phi M(\xi_{n,k}),
\]

where \( \Sigma^{\infty} \) is the stabilization functor which assigns to a space its associated suspension spectrum.

If \( \phi_{n,k} \) is the stable order of \( \xi_{n,k} \) (i.e., \( \phi_{n,k} \) is the smallest integer such that \( \phi_{n,k} \xi_{n,k} \) is stably trivial) then we have the obvious periodicity

\[
M((r + \phi_{n,k})\xi_{n,k}) \simeq \Sigma^{k \phi_{n,k}} M(\xi_{n,k}).
\]

This, together with Snaith's theorem gives clear interrelationships amongst the stable homotopy types of the spaces \( \Omega^n S^{n+r} \) as \( r \) varies.

The case \( n = 2 \) is well understood by the work of F. Cohen, M. Mahowald, and R. J. Milgram [5], who proved that \( \phi_{2,k} = 2 \) for all \( k \). The resulting periodicity in the homotopy type of the associated Thom
spectra was used by M. Mahowald [13] and R. Cohen [8] to construct new infinite families in the stable homotopy ring $\pi_*^S$.

It is the purpose of this paper to compute the orders $\phi_{n,k}$ for general $n$ and $k$. Our main result can be stated as follows. Let

$$a_{n,k} = 2^{\rho(n-1)} \prod_{3 \leq p \leq k} p^{[(n-1)/2]}$$

where $p$ denotes an odd prime, and where $\rho(m)$ is Adam's vector field number: $\rho(m) = \text{the number of positive integers } \leq m \text{ that are congruent to } 0, 1, 2, \text{ or } 4 \text{ mod } 8.$

**Theorem 1.1.** If $n \equiv 0 \mod 4$, then $\phi_{n,k} = a_{n,k}$. Furthermore, if $n \equiv 0 \mod 4$, then $a_{n,k} | \phi_{n,k}$ and $\phi_{n,k} | 2a_{n,k}$.

**Remarks.** 1. The bundle $\xi_{n,2}$ is easily seen to be stably isomorphic to the canonical line bundle over $\mathbb{R}P^{n-1}$, so the fact that $\phi_{n,2} = 2^{\rho(n-1)}$ is the classical result of Adams [1].

2. Using the Atiyah-Hirzebruch spectral sequence converging to the KO-theory of $F(\mathbb{R}^n, p)/\Sigma p$, S. W. Yang computed the order of $\xi_{n,p}$, and proved that $a_{n,k} | \phi_{n,k}$ [20].

3. The conjecture that $\phi_{n,k} = a_{n,k}$ was made by Yang, Mahowald, and F. Cohen.

The essential idea in the proof of 1.1 is to notice that the classifying map

$$f_{n,k} : F(\mathbb{R}^n, k)/\Sigma_k \to BO$$

of $\xi_{n,k}$ factors as a composition of maps, one of which is the natural inclusion

$$i_n : \Omega^n S^n \to Q_0 S^0,$$

where $QX = \lim_{m \to \infty} \Omega^m \Sigma^m X$, and where $\Omega^n S^n$ denotes the component of $\Omega^n S^n$ containing maps of degree $k$. We then study the order of $i_n$ localized at a prime $p$, using the results of F. Cohen, J. Moore, and J. Neisendorfer [6, 7, 15] and of Toda [19].

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2. Proof of Theorem 1.1. Our first object is to identify the classifying maps of the bundles $\xi_{n,k}$. This is done easily by recalling first that $F(\mathbb{R}^\infty, k) = \lim_{n \to \infty} F(\mathbb{R}^n, k)$ is a contractible space, acted upon freely by $\Sigma_k$, and therefore $F(\mathbb{R}^\infty, k)/\Sigma_k = B\Sigma_k$. For a proof of this, see for instance [14].

Thus the bundle

$$\xi_{\infty,k} : F(\mathbb{R}^\infty, k) \times_{\Sigma_k} \mathbb{R}^k \to F(\mathbb{R}^\infty, k)/\Sigma_k = B\Sigma_k$$

is classified by the map

$$f_k : B\Sigma_k \to BO(k)$$

induced by the regular representation of $\Sigma_k$ in $O(k)$. Moreover, since the bundle $\xi_{n,k}$ is the pull-back of $\xi_{\infty,k}$ under the inclusion $F(\mathbb{R}^n, k)/\Sigma_k \subseteq F(\mathbb{R}^\infty, k)/\Sigma_k$, $\xi_{n,k}$ is classified by the map

$$f_{n,k} : F(\mathbb{R}^n, k)/\Sigma_k \subseteq F(\mathbb{R}^\infty, k)/\Sigma_k = B\Sigma_k \to BO(k).$$

The stable order $\phi_{n,k}$ of $\xi_{n,k}$ is the order of the class determined by $f_{n,k}$ in the abelian group $[F(\mathbb{R}^n, k)/\Sigma_k, BO]$. In order to determine $\phi_{n,k}$ we first recall some of May's iterated loop space machinery [14].

Recall first the “approximations”

$$\alpha_n : C_n X \to \Omega^n \Sigma^n X$$

of [14]. $C_n X$ is a filtered space which approximates $\Omega^n \Sigma^n X$ in the sense that $\alpha_n$ is a weak homotopy equivalence if $X$ is connected. For $X = S^0$,

$$C_n(S^0) \simeq \coprod_k F(\mathbb{R}^n, k)/\Sigma_k$$

and the map $\alpha_n : \coprod_k F(\mathbb{R}^n, k)/\Sigma_k \to \Omega^n S^n$ takes $F(\mathbb{R}^n, k)/\Sigma_k$ to $\Omega^n_k S^n$.

Now consider the map

$$\beta : \coprod_k BO(k) \to BO \times \mathbb{Z}$$

which includes $BO(k)$ into $BO \times \{k\}$ in the obvious manner. Let $\eta : QS^0 \to BO \times \mathbb{Z}$ be the infinite loop map induced by the map $S^0 \to BO \times \mathbb{Z}$ which sends 0 to the basepoint in $BO \times \{0\}$ and 1 to the basepoint in $BO \times \{1\}$. We then have

**Lemma 2.1.** The following diagram homotopy commutes for all positive integers $n$ and $k$. 

\[ F(\mathbb{R}^n, k)/\Sigma_k \subset \bigoplus_j F(\mathbb{R}^n, j)/\Sigma_j \longrightarrow \bigoplus_j F(\mathbb{R}^\infty, j)/\Sigma_j \xrightarrow{\cup j} \bigoplus BO(j) \]
\[ \downarrow \alpha_n \quad \downarrow \alpha_\infty \quad \downarrow \beta \]
\[ \Omega^nS^n \quad \rightarrow \quad Q\Sigma^0 \quad \rightarrow \quad BO \times \mathbb{Z} \]
\[ \downarrow *[{-k}] \quad \downarrow *[{-k}] \quad \downarrow *[{-k}] \]
\[ \Omega^nS^n \quad \rightarrow \quad Q\Sigma^0 \quad \rightarrow \quad BO \times \mathbb{Z} \]

where \([{-k}]\) translates components by \(-k\).

**Proof.** This follows directly from May's iterated loop space machinery, and an explicit proof is found in [4].

Note that the classifying map \(f_{n,k}: F(\mathbb{R}^n, k)/\Sigma_k \to BO = BO \times \{0\} \subset BO \times \mathbb{Z}\) of \(\xi_{n,k}\) is the composition obtained by going along the top and then down the right-hand side of the diagram in Lemma 2.1. Now since \(\eta\) is a map of infinite loop spaces, and therefore like \(i_n\) is an \(H\)-map, Lemma 2.1 implies that the power of \(p\) in the prime factorization of \(\phi_{n,k}\) is bounded by the order of the localization at \(p\) of \(i_n \in [\Omega_0^nS^n, Q_0S^0]\).

**Proposition 2.2.** For a prime \(p\), let \(i_{n,p}: \Omega_0^nS^n(p) \to Q_0S^0(p)\) be the localization of \(i_n\). Then in \([\Omega_0^nS^n(p), Q_0S^0(p)]\) the order of \(i_{n,p}\) divides \(p^q\), where
\[
q = \begin{cases} 
\left\lfloor \frac{n - 1}{2} \right\rfloor & \text{if } p \text{ is odd} \\
p(n - 1) & \text{if } p = 2 \text{ and } n \equiv 0 \text{ mod } 4 \\
p(n - 1) + 1 & \text{if } p = 2 \text{ and } n \equiv 0 \text{ mod } 4.
\end{cases}
\]

Notice that Theorem 1.1 is a corollary of Proposition 2.2 in view of Yang’s results [20] (see the second remark following the statement of Theorem 1.1), and the fact that if \(k < p\), \(F(\mathbb{R}^\infty, k)/\Sigma_k = B\Sigma_k\) is homology \(p\)-equivalent to a point.

**Proof of 2.2.** We prove Proposition 2.2 in several cases.

**Case 1.** \(p\) odd and \(n\) odd (say \(n = 2m + 1\)).
Recent results of Selick [17], Cohen, Moore and Neisendorfer [6, 7], and Neisendorfer [15] imply that the identity element
\[
1 \in \left[ \Omega_0^{2m+1}s_1^{2m+1}, \Omega_0^{2m+1}s_{(p)}^{2m+1} \right]
\]
has order \(p^m\). Since \(i_n\) is an \(H\)-map, the result follows in this case.
Case 2. \( p = 2, n \) odd.

To verify this case we shall use the Kahn-Priddy theorem [10]:

**Theorem 2.3.** There exist maps \( s: \mathbb{Q}R\mathbb{P}^\infty \to \mathbb{Q}_0S^0 \) and \( j: \mathbb{Q}_0S^0 \to \mathbb{Q}R\mathbb{P}^\infty \) such that when localized at the prime 2, \( s \circ j \) is a homotopy equivalence.

In [16], Segal gave a proof of this theorem in which he showed that when restricted to \( \Omega^nS^n \subseteq \mathbb{Q}_0S^0 \), \( j \) factors through a map \( j_n: \Omega^nS^n \to \mathbb{Q}R\mathbb{P}^{n-1} \). In [3], Caruso, Cohen, May, and Taylor also gave a proof of the Kahn-Priddy theorem, obtaining Segal's factorization, and in which explicit formulae for the maps \( j_n, j, \) and \( s \) are given.

In any case, using the proof and the formulae in [3] of this theorem, N. Kuhn verified that the maps \( j_n, j \) and \( s \) are one-fold loop maps [12]. The fact that \( j \) is an \( H \)-map actually follows immediately from Kahn's work in [11]. Using these results, we shall consider the following homotopy commutative diagram of spaces localized at 2.

\[
\begin{array}{ccc}
\Omega^nS^n & \xrightarrow{j_n} & Q_0S^0 \\
\downarrow & & \downarrow (s \circ j)^{-1} \\
\mathbb{Q}R\mathbb{P}^{n-1} & \xrightarrow{i_n} & \mathbb{Q}_0S^0 \\
\end{array}
\]

where \( (s \circ j)^{-1} \) is a homotopy inverse to \( s \circ j \). Since \( s \) is an infinite loop map, and \( j \) deloops once, \( s \circ j \) and therefore \( (s \circ j)^{-1} \) are maps of loop spaces. Thus the order of \( i_n \) (localized at 2) divides the order of the identity of \( \mathbb{Q}R\mathbb{P}^{n-1} \), which Toda showed to be \( 2^{n(n-1)} \) when \( n \) is odd [19]. This proves the proposition in this case.

Case 3. \( n = 2m \).

Consider the following fibration of James [9].

\[
S^{2m-1} \xrightarrow{e} \Omega S^{2m} \xrightarrow{h} \Omega S^{4m-1}
\]

This fibration yields the classical EHP sequence in homotopy groups. Apply \( \Omega^{2m-1} \) to this fibration and consider the following diagram.
where $T$ is twice the identity map, and $[i, i]' = \Omega^{2m}[i, i]$, where $[i, i]$: $S^{4m-1} \to S^{2m}$ is the Whitehead product of the identity with itself.

**Lemma 2.4.** In the above diagram we have
(a) both squares commute,
(b) the lower triangle commutes, and
(c) $i_{2m} \circ [i, i]'$ is null homotopic.

**Proof.** The commutativity of the two squares is obvious, and the commutativity of the lower triangle follows from the standard fact that the Hopf invariant of $[i, i]$ is 2. Similarly, the fact that $i_{2m} \circ [i, i]' = 0$ follows from the standard fact that the Whitehead product $[i, i]$ stabilizes to zero.

**Corollary 2.5.** There exists a map $g$: $\Omega^{2m} S^{2m} \to \Omega^{2m-1} S^{2m-1}$ so that $T = [i, i]' \circ h + e \circ g$.

**Proof.** By the lemma, $h \circ (T - [i, i]' \circ h)$ is null homotopic, and therefore $T - [i, i]' \circ h$ lifts to a map $g$: $\Omega^{2m} S^{2m} \to S^{2m-1}$ satisfying the required property.

We are now ready to prove the proposition in this final case. Localizing at 2, we have that

$$2^{p(2m-2)+1} i_{2m} = 2^{p(2m-2)} (i_{2m} \circ T)$$
$$= 2^{p(2m-2)} (i_{2m} \circ [i, i]' \circ h + i_{2m} \circ e \circ g)$$

by 2.5, and which equals $2^{p(2m-2)} (i_{2m-1} \circ g)$ by 2.4 part c and the fact that $i_{2m-1} = i_{2m} \circ e$. But $2^{p(2m-2)} i_{2m-1}$ is null homotopic by the result in case 2. We may therefore conclude that

$$2^{p(2m-2)+1} i_{2m} = 0.$$  

Similarly, localized at $p$ odd and using the result of case 1, we obtain that $2^{p((n-1)/2)} i_{2m}$ is null homotopic, and therefore so is $p^{((n-1)/2)} i_{2m}$. 
Thus we have proved the proposition when $p$ is odd, and summarizing the results in $p = 2$, we have:

\[
\begin{align*}
2^p(n-1)i_n &= 0 \quad \text{if } n \text{ is odd}, \\
2^p(n-2)i_n &= 0 \quad \text{if } n \text{ is even}, \\
\text{and} \quad 2^p(n)i_n &= 0 \quad \text{if } n \text{ is even}.
\end{align*}
\]

The last equation follows from the first since $i_{2m}$ factors through $i_{2m+1}$.

Notice that if $n \equiv 2 \mod 8$, $\rho(n - 1) = \rho(n - 2) + 1$ and therefore $2^\rho(n-1)i_n = 0$. If $n \equiv 6 \mod 8$, $\rho(n - 1) = \rho(n)$ so $2^\rho(n-1)i_n = 0$. Thus if $n \not\equiv 0 \mod 4$, $2^\rho(n-1)i_n$ is null homotopic. If $n \equiv 0 \mod 4$, $\rho(n - 1) = \rho(n - 2)$ so $2^\rho(n-1)i_n = 0$.

This completes the proof of Proposition 2.2, and therefore of Theorem 1.1.

REFERENCES


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