Pacific Journal of Mathematics

BUNDLES OVER CONFIGURATION SPACES

FREDERICK RONALD COHEN, RALPH COHEN,
NICHOLAS J. KUHN AND JOSEPH ALVIN NEISENDORFER

Vol. 104, No. 1 May 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 1, 1983

BUNDLES OVER CONFIGURATION SPACES

F. R. COHEN, R. L. COHEN, N. J. KUHN AND J. L. NEISENDORFER

Let $F(R^n, k)$ be the configuration space of ordered sets of k distinct points in R^n. $F(R^n, k)$ is acted upon freely by the symmetric group on k letters, Σ_k. In this paper we calculate the order of the vector bundles

$$\xi_{n,k}\colon F(R^n, k) \times_{\Sigma_k} R^k \to F(R^n, k)/\Sigma_k.$$

Applications to the study of iterated loop spaces of spheres are also discussed.

1. The study of the stable homotopy type of the spaces $\Omega^n S^{n+r}$ has received much attention in recent years [**2, 8, 13**]. The starting point for this study was Snaith's stable descomposition [**18**]:

$$\Omega^n S^{n+r} \simeq_s \bigvee_{k\geq 0} F(\mathbf{R}^n, k)^+ \wedge_{\Sigma_k} S^{r^{(k)}},$$

where $F(\mathbf{R}^n, k)^+$ is the configuration space of k ordered distinct points in $\mathbf{R}^n$ together with a disjoint basepoint, $S^{r^{(k)}}$ is the k-fold smash product of S^r with itself, Σ_k is the symmetric group of k letters, and where "$\simeq_s$" denotes stable homotopy equivalence.

The space $F(\mathbf{R}^n, k)^+ \wedge_{\Sigma_k} S^{r^{(k)}}$ is clearly the Thom complex of the r-fold Whitney sum of the vector bundle

$$\xi_{n,k}\colon F(\mathbf{R}^n, k) \times_{\Sigma_k} \mathbf{R}^k \to F(\mathbf{R}^n, k)/\Sigma_k.$$

If $M(\xi_{n,k})$ is the associated Thom spectrum, then Snaith's theorem gives an equivalence of spectra

$$\Sigma^\infty \Omega^n S^{n+r} \simeq \bigvee_{k\geq 0} \Sigma^{rk} M(r\xi_{n,k}),$$

where Σ^∞ is the stabilization functor which assigns to a space its associated suspension spectrum.

If $\phi_{n,k}$ is the stable order of $\xi_{n,k}$ (i.e., $\phi_{n,k}$ is the smallest integer such that $\phi_{n,k}\xi_{n,k}$ is stably trivial) then we have the obvious periodicity

$$M((r + \phi_{n,k})\xi_{n,k}) \simeq \Sigma^{k\phi_{n,k}} M(r\xi_{n,k}).$$

This, together with Snaith's theorem gives clear interrelationships amongst the stable homotopy types of the spaces $\Omega^n S^{n+r}$ as r varies.

The case $n = 2$ is well understood by the work of F. Cohen, M. Mahowald, and R. J. Milgram [**5**], who proved that $\phi_{2,k} = 2$ for all k. The resulting periodicity in the homotopy type of the associated Thom spectra was used by M. Mahowald [13] and R. Cohen [8] to construct new infinite families in the stable homotopy ring π_*^s.

It is the purpose of this paper to compute the orders $\phi_{n,k}$ for general n and k. Our main result can be stated as follows. Let

$$a_{n,k} = 2^{\rho(n-1)} \prod_{3 \leq p \leq k} p^{[(n-1)/2]}$$

where p denotes an odd prime, and where $\rho(m)$ is Adam's vector field number: $\rho(m) =$ the number of positive integers $\leq m$ that are congruent to 0, 1, 2, or 4 mod 8.

THEOREM 1.1. *If* $n \not\equiv 0 \bmod 4$, *then* $\phi_{n,k} = a_{n,k}$. *Furthermore, if* $n \equiv 0 \bmod 4$, *then* $a_{n,k} \mid \phi_{n,k}$ *and* $\phi_{n,k} \mid 2a_{n,k}$.

REMARKS. 1. The bundle $\xi_{n,2}$ is easily seen to be stably isomorphic to the canonical line bundle over $\mathbf{R}P^{n-1}$, so the fact that $\phi_{n,2} = 2^{\rho(n-1)}$ is the classical result of Adams [1].

2. Using the Atiyah-Hirzebruch spectral sequence converging to the KO-theory of $F(\mathbf{R}^n, p)/\Sigma_p$, S. W. Yang computed the order of $\xi_{n,p}$, and proved that $a_{n,k} \mid \phi_{n,k}$ [20].

3. The conjecture that $\phi_{n,k} = a_{n,k}$ was made by Yang, Mahowald, and F. Cohen.

The essential idea in the proof of 1.1 is to notice that the classifying map

$$f_{n,k}\colon F(\mathbf{R}^n, k)/\Sigma_k \to BO$$

of $\xi_{n,k}$ factors as a composition of maps, one of which is the natural inclusion

$$i_n\colon \Omega_0^n S^n \hookrightarrow Q_0 S^0,$$

where $QX = \lim_{m\to\infty} \Omega^m \Sigma^m X$, and where $\Omega_k^n S^n$ denotes the component of $\Omega^n S^n$ containing maps of degree k. We then study the order of i_n localized at a prime p, using the results of F. Cohen, J. Moore, and J. Neisendorfer [6, 7, 15] and of Toda [19].

ACKNOWLEDGEMENTS. The authors are grateful to M. Mahowald, J. P. May, and L. Traylor for helpful conversations concerning the material in this paper.

2. Proof of Theorem 1.1. Our first object is to identify the classifying maps of the bundles $\xi_{n,k}$. This is done easily by recalling first that $F(\mathbf{R}^\infty, k) = \lim_{n\to\infty} F(\mathbf{R}^n, k)$ is a contractible space, acted upon freely by Σ_k, and therefore $F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k$. For a proof of this, see for instance [**14**].

Thus the bundle

$$\xi_{\infty,k}\colon F(\mathbf{R}^\infty, k) \times_{\Sigma_k} \mathbf{R}^k \to F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k$$

is classified by the map

$$f_k\colon B\Sigma_k \to BO(k)$$

induced by the regular representation of Σ_k in $O(k)$. Moreover, since the bundle $\xi_{n,k}$ is the pull-back of $\xi_{\infty,k}$ under the inclusion $F(\mathbf{R}^n, k)/\Sigma_k \subset F(\mathbf{R}^\infty, k)/\Sigma_k$, $\xi_{n,k}$ is classified by the map

$$f_{n,k}\colon F(\mathbf{R}^n, k)/\Sigma_k \subset F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k \underset{f_k}{\to} BO(k).$$

The stable order $\phi_{n,k}$ of $\xi_{n,k}$ is the order of the class determined by $f_{n,k}$ in the abelian group $[F(\mathbf{R}^n, k)/\Sigma_k, BO]$. In order to determine $\phi_{n,k}$ we first recall some of May's iterated loop space machinery [**14**].

Recall first the "approximations"

$$\alpha_n\colon C_n X \to \Omega^n \Sigma^n X$$

of [**14**]. $C_n X$ is a filtered space which approximates $\Omega^n\Sigma^n X$ in the sense that α_n is a weak homotopy equivalence if X is connected. For $X = S^0$,

$$C_n(S^0) \simeq \coprod_k F(\mathbf{R}^n, k)/\Sigma_k$$

and the map $\alpha_n\colon \coprod_k F(\mathbf{R}^n, k)/\Sigma_k \to \Omega^n S^n$ takes $F(\mathbf{R}^n, k)/\Sigma_k$ to $\Omega^n_k S^n$.

Now consider the map

$$\beta\colon \coprod_k BO(k) \to BO \times \mathbf{Z}$$

which includes $BO(k)$ into $BO \times \{k\}$ in the obvious manner. Let $\eta\colon QS^0 \to BO \times \mathbf{Z}$ be the infinite loop map induced by the map $S^0 \to BO \times \mathbf{Z}$ which sends 0 to the basepoint in $BO \times \{0\}$ and 1 to the basepoint in $BO \times \{1\}$. We then have

LEMMA 2.1. *The following diagram homotopy commutes for all positive integers n and k.*

$$\begin{array}{ccccc}
F(\mathbf{R}^n, k)/\Sigma_k \subset \coprod_j F(\mathbf{R}^n, j)/\Sigma_j & \longrightarrow & \coprod_j F(\mathbf{R}^\infty, j)/\Sigma_j & \xrightarrow{\coprod f_j} & \coprod_j BO(j) \\
\downarrow \alpha_n & & \downarrow \alpha_\infty & & \downarrow \beta \\
\Omega^n S^n & \xrightarrow[\iota_n]{} & QS^0 & \xrightarrow[\eta]{} & BO \times \mathbf{Z} \\
\downarrow *[-k] & & \downarrow *[-k] & & \downarrow *[-k] \\
\Omega^n S^n & \xrightarrow[i_n]{} & QS^0 & \xrightarrow[\eta]{} & BO \times \mathbf{Z}
\end{array}$$

where $*[-k]$ *translates components by* $-k$.

Proof. This follows directly from May's iterated loop space machinery, and an explicit proof is found in [**4**].

Note that the classifying map $f_{n,k}$: $F(\mathbf{R}^n, k)/\Sigma_k \to BO = BO \times \{0\} \subset BO \times \mathbf{Z}$ of $\xi_{n,k}$ is the composition obtained by going along the top and then down the right-hand side of the diagram in Lemma 2.1. Now since η is a map of infinite loop spaces, and therefore like i_n is an H-map, Lemma 2.1 implies that the power of p in the prime factorization of $\phi_{n,k}$ is bounded by the order of the localization at p of $i_n \in [\Omega_0^n S^n, Q_0 S^0]$.

PROPOSITION 2.2. *For a prime* p, *let* $i_{n,p}$: $\Omega_0^n S^n_{(p)} \to Q_0 S^0_{(p)}$ *be the localization of* i_n. *Then in* $[\Omega_0^n S^n_{(p)}, Q_0 S^0_{(p)}]$ *the order of* $i_{n,p}$ *divides* p^q, *where*

$$q = \begin{cases} \left[\dfrac{n-1}{2}\right] & \text{if } p \text{ is odd} \\ \rho(n-1) & \text{if } p = 2 \text{ and } n \not\equiv 0 \bmod 4 \\ \rho(n-1) + 1 & \text{if } p = 2 \text{ and } n \equiv 0 \bmod 4. \end{cases}$$

Notice that Theorem 1.1 is a corollary of Proposition 2.2 in view of Yang's results [**20**] (see the second remark following the statement of Theorem 1.1), and the fact that if $k < p$, $F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k$ is homology p-equivalent to a point.

Proof of 2.2. We prove Proposition 2.2 in several cases.

Case 1. p odd and n odd (say $n = 2m + 1$).

Recent results of Selick [**17**], Cohen, Moore and Neisendorfer [**6, 7**], and Neisendorfer [**15**] imply that the identity element

$$1 \in \left[\Omega_0^{2m+1} S^{2m+1}_{(p)}, \Omega_0^{2m+1} S^{2m+1}_{(p)}\right]$$

has order p^m. Since i_n is an H-map, the result follows in this case.

Case 2. $p = 2$, n odd.

To verify this case we shall use the Kahn-Priddy theorem [**10**]:

THEOREM 2.3. *There exist maps* s: $Q\mathbf{R}P^\infty \to Q_0S^0$ *and* j: $Q_0S^0 \to Q\mathbf{R}P^\infty$ *such that when localized at the prime* 2, $s \circ j$ *is a homotopy equivalence.*

In [**16**], Segal gave a proof of this theorem in which he showed that when restricted to $\Omega_0^nS^n \subset Q_0S^0$, j factors through a map j_n: $\Omega_0^nS^n \to Q\mathbf{R}P^{n-1}$. In [3], Caruso, Cohen, May, and Taylor also gave a proof of the Kahn-Priddy theorem, obtaining Segal's factorization, and in which explicit formulae for the maps j_n, j, and s are given.

In any case, using the proof and the formulae in [3] of this theorem, N. Kuhn verified that the maps j_n and j are one-fold loop maps [**12**]. The fact that j is an H-map actually follows immediately from Kahn's work in [**11**]. Using these results, we shall consider the following homotopy commutative diagram of spaces localized at 2.

$$\begin{array}{ccc}
\Omega_0^nS^n & \xrightarrow{\quad i_n \quad} & Q_0S^0 \\
\downarrow{\scriptstyle j_n} & \swarrow{\scriptstyle j} & \uparrow{\scriptstyle (s\circ j)^{-1}} \\
Q\mathbf{R}P^{n-1} \subset Q\mathbf{R}P^\infty & \xrightarrow[\quad s \quad]{} & Q_0S^0
\end{array}$$

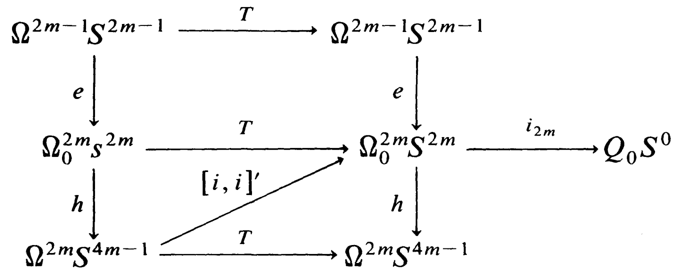
where $(s \circ j)^{-1}$ is a homotopy inverse to $s \circ j$. Since s is an infinite loop map, and j deloops once, $s \circ j$ and therefore $(s \circ j)^{-1}$ are maps of loop spaces. Thus the order of i_n (localized at 2) divides the order of the identity of $Q\mathbf{R}P^{n-1}$, which Toda showed to be $2^{\rho(n-1)}$ when n is odd [**19**]. This proves the proposition in this case.

Case 3. $n = 2m$.

Consider the following fibration of James [**9**].

$$S^{2m-1} \xhookrightarrow{e} \Omega S^{2m} \xrightarrow{h} \Omega S^{4m-1}$$

This fibration yields the classical EHP sequence in homotopy groups. Apply Ω^{2m-1} to this fibration and consider the following diagram.



where T is twice the identity map, and $[i, i]' = \Omega^{2m}[i, i]$, where $[i, i]$: $S^{4m-1} \to S^{2m}$ is the Whitehead product of the identity with itself.

LEMMA 2.4. *In the above diagram we have*
(a) *both squares commute,*
(b) *the lower triangle commutes, and*
(c) $i_{2m} \circ [i, i]'$ *is null homotopic.*

Proof. The commutativity of the two squares is obvious, and the commutativity of the lower triangle follows from the standard fact that the Hopf invariant of $[i, i]$ is 2. Similarly, the fact that $i_{2m} \circ [i, i]' = 0$ follows from the standard fact that the Whitehead product $[i, i]$ stabilizes to zero.

COROLLARY 2.5. *There exists a map* g: $\Omega_0^{2m}S^{2m} \to \Omega_0^{2m-1}S^{2m-1}$ *so that* $T \simeq [i, i]' \circ h + e \circ g$.

Proof. By the lemma, $h \circ (T - [i, i]' \circ h)$ is null homotopic, and therefore $T - [i, i]' \circ h$ lifts to a map g: $\Omega_0^{2m}S^{2m} \to S^{2m-1}$ satisfying the required property.

We are now ready to prove the proposition in this final case. Localizing at 2, we have that

$$2^{\rho(2m-2)+1}i_{2m} = 2^{\rho(2m-2)}(i_{2m} \circ T)$$
$$= 2^{\rho(2m-2)}\left(i_{2m} \circ [i, i]' \circ h + i_{2m} \circ e \circ g\right)$$

by 2.5, and which equals $2^{\rho(2m-2)}(i_{2m-1} \circ g)$ by 2.4 part c and the fact that $i_{2m-1} = i_{2m} \circ e$. But $2^{\rho(2m-2)}i_{2m-1}$ is null homotopic by the result in case 2. We may therefore conclude that

$$2^{\rho(2m-2)+1}i_{2m} = 0.$$

Similarly, localized at p odd and using the result of case 1, we obtain that $2p^{[(n-1)/2]}i_{2m}$ is null homotopic, and therefore so is $p^{[(n-1)/2]}i_{2m}$.

Thus we have proved the proposition when p is odd, and summarizing the results in $p = 2$, we have:

$$2^{\rho(n-1)} i_n = 0 \quad \text{if } n \text{ is odd,}$$
$$2^{\rho(n-2)+1} i_n = 0 \quad \text{if } n \text{ is even,}$$
$$\text{and} \quad 2^{\rho(n)} i_n = 0 \quad \text{if } n \text{ is even.}$$

The last equation follows from the first since i_{2m} factors through i_{2m+1}.

Notice that if $n \equiv 2 \bmod 8$, $\rho(n-1) = \rho(n-2) + 1$ and therefore $2^{\rho(n-1)} i_n = 0$. If $n \equiv 6 \bmod 8$, $\rho(n-1) = \rho(n)$ so $2^{\rho(n-1)} i_n = 0$. Thus if $n \not\equiv 0 \bmod 4$, $2^{\rho(n-1)} i_n$ is null homotopic. If $n \equiv 0 \bmod 4$, $\rho(n-1) = \rho(n-2)$ so $2^{\rho(n-1)+1} i_n = 0$.

This completes the proof of Proposition 2.2, and therefore of Theorem 1.1.

REFERENCES

1. J. F. Adams, *Vector fields on spheres*, Annals of Math., (2) **75** (1962), 603–632.
2. E. H. Brown, Jr. and F. P. Peterson, *On the stable decomposition of* $\Omega^2 S^{r+2}$, Trans. Amer. Math. Soc., **243** (1978), 287–298.
3. J. Caruso, F. Cohen, J. P. May, and L. Taylor, *James maps, Segal maps, and the Kahn-Priddy theorem*, to appear.
4. F. R. Cohen, *Braid orientations and bundles with flat connections*, Invent. Math., **46** (1978), 99–110.
5. F. R. Cohen, M. Mahowald, and R. J. Milgram, *The stable decomposition of the double loop space of a sphere*, Proc. Symp. Pure Math., Vol. 32 part 2 (1978), 225–228.
6. F. R. Cohen, J. Moore, and J. Neisendorfer, *Torsion in homotopy groups*, Annals of Math. **109**, (1979), 121–168.
7. ———, *The double suspension and exponents of the homotopy groups of spheres*, to appear in Annals of Math.
8. R. L. Cohen, *Odd primary infinite families in stable homotopy theory*, Memoirs of the A.M.S., Vol. **30** No. 242 (1981).
9. I. M. James, *On the suspension sequence*, Annals of Math., (2) **65** (1957), 74–107.
10. D. Kahn and S. Priddy, *The transfer and stable homotopy theory*, Math. Proc. Camb. Phil. Soc., **83** (1978), 103–12.
11. D. S. Kahn, *Homology of the Barratt-Eccles decomposition maps*, Proc. Conf. on Homotopy Theory, Northwestern Univ. Notas de Mat. y. Simposia No. 1, Soc. Mat. Mexicana (1975), 60–77.
12. N. Kuhn, Ph.D. Dissertation, University of Chicago, 1980.
13. M. Mahowald, *A new infinite family in* ${}_2\pi_*^s$, Topology, **16** (1977), 249–254.
14. J. P. May, *The Geometry of Iterated Loop Spaces*, Springer Lecture Notes in Math., **271** (1972).
15. J. Neisendorfer, 3-*primary exponents*, to appear.
16. G. Segal, *Operations in stable homotopy theory*, *New Developments in Topology*, J. London Math. Soc. Lecture Notes, Vol. 11 (1974), 105–110.
17. P. Selick, *Odd primary torsion in* $\pi_k(S^3)$, Topology, **17** (1978), 407–412.
18. V. P. Snaith, *A stable decomposition for* $\Omega^n S^n X$, J. London Math. Soc., **7** (1974), 577–583.

19. H. Toda, *Order of the identity class of a suspension space*, Annals of Math., (2) **78** (1963), 300–325.
20. S. W. Yang, Ph.D. Dissertation, Brandeis University, 1978.

Received July 7, 1980. The authors were partially supported by the National Science Foundation. The first author was partially supported by the Alfred Sloan Foundation.

UNIVERSITY OF KENTUCKY
LEXINGTON, KY 40506

STANFORD UNIVERSITY
STANFORD, CA 94305

UNIVERSITY OF WASHINGTON
SEATTLE, WA 98195

AND

OHIO STATE UNIVERSITY
COLUMBUS, OH 43210

Pacific Journal of Mathematics

Vol. 104, No. 1 May, 1983