EXTENSIONS OF THEOREMS OF CUNNINGHAM-AIGNER
AND HASSE-EVANS

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If \( k \) is a positive integer and \( p \) is a prime with \( p \equiv 1 \pmod{2^k} \), then \( 2^{(p-1)/2^k} \) is a \( 2^k \)th root of unity modulo \( p \). We consider the problem of determining \( 2^{(p-1)/2^k} \) modulo \( p \). This has been done for \( k = 1, 2, 3 \) and the present paper treats \( k = 4 \) and \( 5 \), extending the work of Cunningham, Aigner, Hasse, and Evans.

1. Introduction. When \( k = 1 \), we have the familiar result

\[
2^{(p-1)/2} \equiv \begin{cases} 
+1 \pmod{p}, & \text{if } p \equiv 1, 7 \pmod{8}, \\
-1 \pmod{p}, & \text{if } p \equiv 3, 5 \pmod{8}.
\end{cases}
\]

When \( k = 2 \) and \( p \equiv 1 \pmod{4} \), there are integers \( a \equiv 1 \pmod{4} \) and \( b \equiv 0 \pmod{2} \) such that \( p = a^2 + b^2 \), with \( a \) and \( |b| \) unique. If \( b \equiv 0 \pmod{4} \) (so that \( p \equiv 1 \pmod{8} \)), Gauss [8: p. 89] (see also [4], [16]) has shown that

\[
2^{(p-1)/4} \equiv \begin{cases} 
+1 \pmod{p}, & \text{if } b \equiv 0 \pmod{8}, \\
-1 \pmod{p}, & \text{if } b \equiv 4 \pmod{8}.
\end{cases}
\]

If \( b \equiv 2 \pmod{4} \) (so that \( p \equiv 5 \pmod{8} \)), we can choose \( b \equiv -2 \pmod{8} \), by changing the sign of \( b \), if necessary, and Gauss [8: p. 89] (see also [4], [11: p. 66], [16]) has shown that

\[
2^{(p-1)/4} \equiv -b/a \pmod{p}.
\]

We note that \((-b/a)^2 \equiv -1 \pmod{p}\).

When \( k = 3 \) and \( p \equiv 1 \pmod{8} \), there are integers \( a \equiv 1 \pmod{4} \) and \( b \equiv 0 \pmod{4} \) such that \( p = a^2 + b^2 \), with \( a \) and \( |b| \) unique. Now \( 2^{(p-1)/8} = 2^{(p-1)/2} \equiv 1 \pmod{p} \), as \( p \equiv 1 \pmod{8} \), so \( 2^{(p-1)/8} \) is a 4th root of unity modulo \( p \). If \( b \equiv 0 \pmod{8} \), Reuschle [14] conjectured and Western [15] (see also [16]) proved that

\[
2^{(p-1)/8} \equiv \begin{cases} 
(-1)^{(p-1)/8} \pmod{p}, & \text{if } b \equiv 0 \pmod{16}, \\
(-1)^{(p+7)/8} \pmod{p}, & \text{if } b \equiv 8 \pmod{16}.
\end{cases}
\]

If \( b \equiv 4 \pmod{8} \), we can choose \( b \equiv 4(-1)^{(p+7)/8} \pmod{16} \), by changing the sign of \( b \), if necessary, and Lehmer [11: p. 70] has shown that

\[
2^{(p-1)/8} \equiv -b/a \pmod{p}.
\]
It is the purpose of this paper to treat the cases \( k = 4 \) and \( 5 \). For \( k = 4 \) and \( p \equiv 1 \pmod{16} \), there are integers \( a \equiv 1 \pmod{4} \), \( b \equiv 0 \pmod{4} \), \( c \equiv 1 \pmod{4} \), \( d \equiv 0 \pmod{2} \), such that \( p = a^2 + b^2 = c^2 + 2d^2 \), with \( a, |b|, c, |d| \) unique. Now \( \{2^{(p-1)/16}\}^8 = 2^{(p-1)/2} \equiv 1 \pmod{p} \), so \( 2^{(p-1)/16} \) is an 8th root of unity modulo \( p \). Since

\[
\left\{ \frac{-(a + b)d}{ac} \right\}^2 \equiv -b/a \pmod{p},
\]

the 8th roots of unity \( \pmod{p} \) are given by \( \{-(a + b)d/ac\}^n \), \( n = 0, 1, \ldots, 7 \). Making use of a congruence due to Hasse [9: p. 232] (see also [5: Theorem 3], [17: p. 411]), we prove in §2 the following extension of the criterion for 2 to be a 16th power \( \pmod{p} \), which was conjectured by Cunningham [3: p. 88] and first proved by Aigner [1] (see also [16: p. 373]).

**Theorem 1.** Let \( p \equiv 1 \pmod{16} \) be a prime. Let \( a \equiv 1 \pmod{4} \), \( b \equiv 0 \pmod{4} \), \( c \equiv 1 \pmod{4} \), \( d \equiv 0 \pmod{2} \) be integers such that \( p = a^2 + b^2 = c^2 + 2d^2 \). It is well known that \( b \equiv 0 \pmod{8} \Leftrightarrow d \equiv 0 \pmod{4} \) (see for example [2: p. 68]). Then the values of \( 2^{(p-1)/16} \pmod{p} \) are given in Table 1.

The case \( b \equiv 0 \pmod{16} \) constitutes the criterion of Cunningham-Aigner.

For \( k = 5 \) and \( p \equiv 1 \pmod{32} \), there are integers \( a \equiv 1 \pmod{4}, \ b \equiv 0 \pmod{4}, \ c \equiv 1 \pmod{4}, \ d \equiv 0 \pmod{2}, \ x \equiv -1 \pmod{8}, \ u \equiv v \equiv w \equiv 0 \pmod{2}, \) such that \( p = a^2 + b^2 = c^2 + 2d^2 \) and

\[
\begin{align*}
p &= x^2 + 2u^2 + 2v^2 + 2w^2, \\
xwv &= u^2 - 2uw - w^2,
\end{align*}
\]

with \( a, |b|, c, |d|, x \) unique. If \( (x, u, v, w) \) is a solution of (1.7), then all solutions are given by \( \pm(x, u, v, w), \pm(x, -u, v, -w), \pm(x, w, -v, -u), \pm(x, -w, -v, u) \) (see for example [12: p. 366]). Now \( \{2^{(p-1)/32}\}^{16} = 2^{(p-1)/2} \equiv +1 \pmod{p} \), so \( 2^{(p-1)/32} \) is a 16th root of unity modulo \( p \). Since

\[
\left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^2 \equiv -\frac{(a + b)d}{ac} \pmod{p},
\]

the 16th roots of unity \( \pmod{p} \) are given by

\[
\left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^n, \quad n = 0, 1, \ldots, 15.
\]
Making use of another congruence due to Hasse [9: p. 233] (see also [7: eqn. (2)]), we prove in §3 the following extension of the criterion for 2 to be a 32nd power (mod \( p \)) due to Hasse [9: p. 232–238] and Evans [6: Theorem 7].

**Theorem 2.** Let \( p \equiv 1 \pmod{32} \) be a prime. Let \( a \equiv 1 \pmod{4} \), \( b \equiv 0 \pmod{4} \), \( c \equiv 1 \pmod{4} \), \( d \equiv 0 \pmod{2} \), \( x \equiv -1 \pmod{8} \), \( u \equiv v \equiv w \equiv 0 \pmod{2} \), be integers such that \( p = a^2 + b^2 = c^2 + 2d^2 \) and \( p = x^2 + 2u^2 + 2v^2 + 2w^2 \), \( 2xv = u^2 - 2uw - w^2 \). Then the values \( 2^{(p-1)/32} \pmod{p} \) are given in Table 2.

Justification of the choices in the left-hand column of Table 2 is made in the proof of Theorem 2, which appears in §3. The cases \( 2^{(p-1)/32} \equiv \pm 1 \pmod{p} \) constitute the criterion of Hasse-Evans.

2. **Evaluation of** \( 2^{(p-1)/16} \pmod{p} \). Let \( p \) be a prime satisfying

\[
(2.1) \quad p \equiv 1 \pmod{16}.
\]

Set

\[
(2.2) \quad p = 8f + 1,
\]

so that

\[
(2.3) \quad f \equiv 0 \pmod{2}.
\]

Let

\[
(2.4) \quad \omega = \exp(2\pi i/8) = (1 + i)/\sqrt{2}.
\]

We note that the ring of integers of \( Q(\omega) = Q(i, \sqrt{2}) \) is a unique factorization domain (see for example [13]). In this ring \( p \) factors as a product of four primes. Denoting one of these by \( \pi \), these four primes are \( \pi_j = \sigma_j(\pi) \), \( j = 1, 3, 5, 7 \), where \( \sigma_j \) denotes the automorphism which maps \( \omega \) to \( \omega^j \).

Let \( g \) be a primitive root (mod \( p \)). Then \( g^{(p-1)/2} \equiv -1 \pmod{p} \), and so

\[
(g^f - \omega)(g^f - \omega^3)(g^f - \omega^5)(g^f - \omega^7) \equiv 0 \pmod{\pi_1\pi_3\pi_5\pi_7}.
\]

Hence

\[
g^f - \omega^j \equiv 0 \pmod{\pi_j},
\]

for some \( j, j = 1, 3, 5, 7 \), and by relabelling the \( \pi \)'s we may assume without loss of generality that

\[
(2.5) \quad g^f \equiv \omega \pmod{\pi}.
\]

Given \( g, \pi \) (apart from units) is uniquely determined by (2.5). Next we define a character \( \chi \pmod{p} \) (depending upon \( g \)) of order 8 by setting

\[
(2.6) \quad \chi(g) = \omega.
\]
For \( r, s = 0, 1, 2, \ldots, 7 \) the Jacobi sum \( J(r, s) \) is defined by
\[
J(r, s) = \sum_{n \equiv r (\text{mod } p)} \chi'(n)\chi'(1-n).
\]

It is known that (see for example [7: §1])
\[
J(2, 2) = -a + bi,
\]
where
\[
p = a^2 + b^2, \quad a \equiv 1 \pmod{4},
\]
and that
\[
J(1, 3) = -c + di\sqrt{2},
\]
where
\[
p = c^2 + 2d^2, \quad c \equiv 1 \pmod{4}.
\]

It is easy to check that replacing the primitive root \( g \) by the primitive root \( g^{8s+t} \), where \( t = 1, 3, 5, 7 \) and \( (8s + t, f) = 1 \), has the effect in (2.8) of replacing \( b \) by \((-1/t)b\) and in (2.10) of replacing \( d \) by \((-2/t)d\).

Our proof depends upon the following important congruence due to Hasse [9: p. 232]
\[
b \equiv 4d + 2m \pmod{32},
\]
where \( m \) is the least positive integer such that
\[
g^m \equiv 2 \pmod{p},
\]
and \( b \) and \( d \) are given by (2.8) and (2.10) respectively. From (2.12) and (2.13) we obtain
\[
2^{(p-1)/16} = 2^{f/2} \equiv g^{mf/2} \equiv g^{(b^2 - d)} \pmod{p}.
\]

It follows from (2.5) and (2.6) that
\[
\chi(n) \equiv n^f \pmod{\pi},
\]
for any integer \( n \) not divisible by \( p \). Hence, for non-negative integers \( r \) and \( s \) satisfying \( 0 \leq r + s < 8 \), we have
\[
J(r, s) \equiv \sum_{n=0}^{p-1} n^f(1-n)^s \pmod{\pi}
\]
\[
\equiv \sum_{n=0}^{p-1} n^f \sum_{j=0}^{s} \binom{s}{j} (-1)^j n^j \pmod{\pi}
\]
\[
\equiv \sum_{j=0}^{s} \binom{s}{j} (-1)^j \sum_{n=0}^{p-1} n^{rf+j} \pmod{\pi},
\]
that is

(2.16) \[ J(r, s) \equiv 0 \pmod{\pi}, \]
as

(2.17) \[ \sum_{n=0}^{p-1} n^k \equiv 0 \pmod{p}, \text{ for } k = 0, 1, \ldots, p - 2. \]

Taking \((r, s) = (2, 2)\) and \((1, 3)\) in (2.16), we have, by (2.8) and (2.10),

(2.18) \[ i \equiv a/b \pmod{\pi}, \sqrt{i} \equiv c/d \pmod{\pi}, \]

so that

(2.19) \[ \sqrt{2} \equiv -ac/bd \pmod{\pi}. \]

Hence we have, appealing to (2.5), (2.18) and (2.19),

\[ g^f \equiv \omega = \frac{1 + i}{\sqrt{2}} \equiv -\frac{(a + b)d}{ac} \pmod{\pi}, \]

and, since \(g^f\) and \(-{(a + b)d/\pi}\) are integers \((\pmod{p})\), we have

(2.20) \[ g^f \equiv -\frac{(a + b)d}{ac} \pmod{p}. \]

Appealing to (2.14) we get

(2.21) \[ 2^{(p-1)/16} \equiv \left\{ -\frac{(a + b)d}{ac} \right\}^{(b/4)-d} \pmod{p}. \]

We consider three cases:

(i) \(2^{(p-1)/4} \equiv -1 \pmod{p}\),
(ii) \(2^{(p-1)/4} \equiv 1, 2^{(p-1)/8} \equiv -1 \pmod{p}\),
(iii) \(2^{(p-1)/8} \equiv 1 \pmod{p}\).

Case (i). From (1.2) we have \(b \equiv 4 \pmod{8}\). Then, from \(p = a^2 + b^2\),

we obtain \(a \equiv 1 \pmod{8}\) and \(p \equiv 2a + 15 \pmod{32}\). The cyclotomic number \((0, 7)_8\) is given by (see for example [10: p. 116])

\[ 64(0, 7)_8 = p - 7 + 2a + 4c, \]

so \(c \equiv 5 \pmod{8}\). Then, from \(p = c^2 + 2d^2\), we get \(d \equiv 2 \pmod{4}\).

Replacing \(g\) by an appropriate primitive root

\[ g^{8s+t} (t = 1, 3, 5, 7; (8s + t, f) = 1) \]
we may take \( b \equiv -4 \equiv 12 \pmod{16} \) and \( d \equiv 2 \pmod{8} \). Then, from (2.21), we obtain

\[
2^{(p-1)/16} \equiv \begin{cases} 
\frac{(a+b)d}{ac} \pmod{p}, & \text{if } b \equiv 12 \pmod{32}, \\
\frac{(a+b)d}{ac} \pmod{p}, & \text{if } b \equiv 28 \pmod{32}.
\end{cases}
\]

**Case (ii).** From (1.2) and (1.4) we have \( b \equiv 8 \pmod{16} \). Then, from \( p = a^2 + b^2 \), we obtain \( a \equiv 1 \pmod{8} \) and \( p \equiv 2a - 1 \pmod{32} \). The cyclotomic number \((1,2)_8\) is given by (see for example [10: p. 116])

\[
64 (1,2)_8 = p + 1 + 2a - 4c,
\]

so \( c \equiv 1 \pmod{8} \). Then, from \( p = c^2 + 2d^2 \), we get \( d \equiv 0 \pmod{4} \). Replacing \( g \) by an appropriate primitive root

\[
g^{8s+t} \quad (t = 1, 3; \ (8s + t, f) = 1)
\]

we may take \( b \equiv 8 \pmod{32} \). Then as

\[
\left\{ \frac{-(a+b)d}{ac} \right\}^2 \equiv \frac{-b}{a} \pmod{p},
\]

we have from (2.21)

\[
2^{(p-1)/16} \equiv \begin{cases} 
-b/a \pmod{p}, & \text{if } d \equiv 0 \pmod{8}, \\
+b/a \pmod{p}, & \text{if } d \equiv 4 \pmod{8}.
\end{cases}
\]

**Case (iii).** From (1.4) we have \( b \equiv 0 \pmod{16} \). Exactly as in Case (ii) we have \( d \equiv 0 \pmod{4} \). Considering four cases according as \( b \equiv 0, 16 \pmod{32} \) and \( d \equiv 0, 4 \pmod{8} \) we obtain from (2.21)

\[
2^{(p-1)/16} \equiv \begin{cases} 
+1 \pmod{p}, & \text{if } b \equiv 0 \pmod{32}, \quad d \equiv 0 \pmod{8} \text{ or } b \equiv 16 \pmod{32}, \quad d \equiv 4 \pmod{8}, \\
-1 \pmod{p}, & \text{if } b \equiv 0 \pmod{32}, \quad d \equiv 4 \pmod{8} \text{ or } b \equiv 16 \pmod{32}, \quad d \equiv 0 \pmod{8}.
\end{cases}
\]

This completes the proof of Theorem 1.

3. **Evaluation of** \( 2^{(p-1)/32} \pmod{p} \). Let \( p \) be a prime satisfying

\[
p \equiv 1 \pmod{32}.
\]
Set
\[(3.2) \quad p = 16f + 1,\]
so that
\[(3.3) \quad f \equiv 0 \pmod{2}.\]
Let
\[(3.4) \quad \theta = \exp(2\pi i/16) = \frac{1}{2} \left\{ \sqrt{2 + \sqrt{2}} + i\sqrt{2 - \sqrt{2}} \right\}.\]

Again, the ring of integers of \( \mathbb{Q}(\theta) \) is a unique factorization domain (see for example [13]). In this ring \( p \) factors as a product of eight primes. Denoting one of these by \( \pi \), these eight primes are given by \( \pi_i = \sigma_i(\pi) \), \( i = 1, 3, 5, 7, 9, 11, 13, 15 \), where \( \sigma_i \) denotes the automorphism which maps \( \theta \) to \( \theta^i \).

Let \( g \) be a primitive root \((\mod p)\). Then
\[(g^f - \theta)(g^f - \theta^3) \cdots (g^f - \theta^{15}) \equiv 0 \pmod{\pi_1\pi_3 \cdots \pi_{15}},\]
and, as before, we can choose \( \pi_1 = \pi \) (unique apart from units) so that
\[(3.5) \quad g^f \equiv \theta \pmod{\pi}.
\]
We define a character \( \Psi \pmod{\pi} \) of order 16 by setting
\[(3.6) \quad \Psi(g) = \theta,\]
and for \( r, s = 0, 1, 2, \ldots, 15 \) we define the Jacobi sum \( J(r, s) \) by
\[(3.7) \quad J(r, s) = \sum_{n \pmod{p}} \psi(r)\psi^*(1 - n).\]

It is known that (see for example [7; §1])
\[(3.8) \quad J(4, 4) = -a + bi, \quad \text{where} \quad p = a^2 + b^2, \quad a \equiv 1 \pmod{4},\]
\[(3.9) \quad J(2, 6) = -c + di\sqrt{2}, \quad \text{where} \quad p = c^2 + 2d^2, \quad c \equiv 1 \pmod{4},\]
and
\[(3.10) \quad J(1, 7) = x + ui\sqrt{2 - \sqrt{2}} + v\sqrt{2} + wi\sqrt{2 + \sqrt{2}}
\quad = x + u(\theta + \theta^7) + v(\theta^2 - \theta^6) + w(\theta^3 + \theta^5),\]
where (see for example [5; eqn. (8)])
\[(3.11) \quad \begin{cases} p = x^2 + 2u^2 + 2v^2 + 2w^2, \quad x \equiv -1 \pmod{8}, \\ 2xu = u^2 - 2uw - w^2. \end{cases}\]
It is easy to check that \( u, \nu \) and \( w \) are all even. Applying the mapping \( \theta \to \theta^3 \) to (3.10), we obtain

\[
(3.12) \quad J(3, 5) = x - wi\sqrt{2 - \sqrt{2}} - \nu\sqrt{2} + ui\sqrt{2 + \sqrt{2}}
\]

\[
= x - w(\theta + \theta^7) - \nu(\theta^2 - \theta^6) + u(\theta^3 + \theta^5).
\]

Further, it is known (see [12: p. 366] and [6: eqn. (48)]) that \( a, b, c, d, x, u, \nu, w \) are related by

\[
(3.13) \quad bd(x^2 - 2\nu^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p}.
\]

The effect on (3.8), (3.9), (3.10) of replacing the primitive root \( g \) by the primitive root \( g^{16s+t} \), where \( t = 1, 3, 5, \ldots, 15 \) and \((16s + t, f) = 1\), is summarized below:

\[
\begin{array}{cccccccc}
g & a & b & c & d & x & u & v & w \\
g^{16s+3} & a & -b & c & d & x & u & -v & -w \\
g^{16s+5} & a & b & c & -d & x & w & -v & -u \\
g^{16s+7} & a & -b & c & -d & x & u & v & w \\
g^{16s+9} & a & b & c & d & x & -u & v & -w \\
g^{16s+11} & a & -b & c & d & x & -w & v & u \\
g^{16s+13} & a & b & c & -d & x & -w & -v & u \\
g^{16s+15} & a & -b & c & -d & x & -u & v & -w
\end{array}
\]

The following important congruence relating \( b, d, u \) and \( w \) has been proved by Hasse [9: p. 233]

\[
(3.15) \quad b + 4d - 8(u + w) \equiv 2m \pmod{64},
\]

where \( m \) satisfies (2.13). From (2.13) and (3.15), we obtain

\[
(3.16) \quad 2^{(p-1)/32} = 2^{f/2} \equiv g^{m/2} \equiv g^{f((b/4) + d - 2(u + w))} \pmod{p}.
\]

As in §2, if \( r \) and \( s \) are non-negative integers satisfying \( 0 \leq r + s < 16 \), we have

\[
(3.17) \quad J(r, s) \equiv 0 \pmod{\pi}.
\]

Thus, in particular, taking \( (r, s) = (4, 4), (2, 6), (1, 7), \) and \( (3, 5) \), in (3.17), we obtain

\[
(3.18) \quad -a + bi \equiv 0 \pmod{\pi},
\]

\[
(3.19) \quad -c + di\sqrt{2} \equiv 0 \pmod{\pi},
\]

\[
(3.20) \quad x + ui\sqrt{2 - \sqrt{2}} + \nu\sqrt{2} + wi\sqrt{2 + \sqrt{2}} \equiv 0 \pmod{\pi},
\]

\[
(3.21) \quad x - wi\sqrt{2 - \sqrt{2}} - \nu\sqrt{2} + ui\sqrt{2 + \sqrt{2}} \equiv 0 \pmod{\pi}.
\]
From (3.18) and (3.19) we get

\[(3.22) \quad i \equiv a/b \pmod{\pi}, \quad i\sqrt{2} \equiv c/d \pmod{\pi},\]

\[\sqrt{2} \equiv -\frac{ac}{bd} \pmod{\pi}.\]

Solving (3.20) and (3.21) simultaneously for \(\sqrt{2} + \sqrt{2}\) and \(\sqrt{2} - \sqrt{2}\) (mod \(\pi\)), and making use of (3.22), we obtain

\[(3.23) \quad \sqrt{2} \pm \sqrt{2} \equiv \frac{x(u \pm w)ad + v(u \mp w)bc}{bd(u^2 + w^2)} \pmod{\pi}.\]

Then, from (3.4), (3.5), (3.22) and (3.23), we have

\[(3.24) \quad g' \equiv \theta \equiv \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \pmod{\pi}.\]

Since both sides of (3.24) are integers (mod \(p\)), we deduce that

\[(3.25) \quad g' \equiv \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \pmod{p}.\]

Appealing to (3.16) we get

\[(3.26) \quad 2^{(p-1)/32} \equiv \left[\frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)}\right]^{(b/4) + d - 2(u + w)} \pmod{p}.\]

We consider four cases:

(i) \(2^{(p-1)/4} \equiv -1 \pmod{p}\),
(ii) \(2^{(p-1)/4} \equiv +1, 2^{(p-1)/8} \equiv -1 \pmod{p}\),
(iii) \(2^{(p-1)/8} \equiv +1, 2^{(p-1)/16} \equiv -1 \pmod{p}\),
(iv) \(2^{(p-1)/16} \equiv +1 \pmod{p}\).

Case (i). From Case (i) of §2 we have \(b \equiv 4 \pmod{8}\) and \(d \equiv 2 \pmod{4}\). Next, from (2.12) and (3.15), we obtain

\[u + w \equiv d \equiv 2 \pmod{4},\]

so that

\[(u, w) \equiv (0, 2) \quad \text{or} \quad (2, 0) \pmod{4}.\]

Replacing \(g\) by an appropriate primitive root \(g^{16s + t}\) (where \(t = 1, 3, 5, \ldots, 15\) and \((16s + t, f) = 1\), we can suppose that

\[(3.27) \quad b \equiv -4 \pmod{16}, \quad u \equiv 0 \pmod{4}, \quad w \equiv 2 \pmod{8}.\]
Exactly one 5-tuple \((b, d, u, v, w)\) satisfies (3.13) and (3.27). Then, from
\[2xv = u^2 - 2uw - w^2,\]
we obtain (recalling \(x \equiv -1 \pmod{8}\))
\[(3.28) \quad v \equiv 2 \pmod{8}.\]
From the work of Evans and Hill [7: Table 2a], we have
\[(3.29) \quad 256\{(2, 4)^{16} - (4, 10)^{16}\} = 32(v - d),\]
so that, by (3.28),
\[(3.30) \quad d \equiv v \equiv 2 \pmod{8}.\]
The choice (3.27) makes the exponent \((b/4) + d - 2(u + w)\) in (3.26) congruent to 1 \(\pmod{4}\). We now consider cases according as
\[b \equiv 12, 28, 44, 60 \pmod{64}; \quad d \equiv 2, 10 \pmod{16}; \quad u \equiv 0, 4 \pmod{8}.
\] For example, if \(b \equiv 12 \pmod{64}, d \equiv 2 \pmod{16}, u \equiv 0 \pmod{8},\) then \((b/4) + d - 2(u + w) \equiv 1 \pmod{16},\) so that (3.26) gives
\[(3.31) \quad 2^{(p-1)/32} \equiv \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \pmod{p},\]
in this case. The other cases can be treated similarly, see Table 2 (VII).

Case (ii). From Case (ii) of §2, we have \(b \equiv 8 \pmod{16}\) and \(d \equiv 0 \pmod{4}\). Appealing to the work of Evans [5: Theorem 4 and its proof], we have
\[(3.32) \quad u \equiv 2 \pmod{4}, \quad v \equiv 4 \pmod{8}, \quad w \equiv 2 \pmod{4},\]
if \(d \equiv 0 \pmod{8},\)
and
\[(3.33) \quad u \equiv 0 \pmod{4}, \quad v \equiv 0 \pmod{8}, \quad w \equiv 0 \pmod{4},\]
if \(d \equiv 4 \pmod{8}.\)

If \(d \equiv 0 \pmod{8},\) replacing \(g\) by \(g^{16s+t}\) (where \(t = 1, 7, 9, 15\) and \((16s + t, f) = 1\), as necessary, we can suppose that
\[(3.34) \quad b \equiv 8 \pmod{32}, \quad w \equiv 2 \pmod{8}.
\]
There are exactly two 5-tuples \((b, d, u, v, w),\) which satisfy (3.13) and (3.34). These are
\[(b, d, u, v, w) \quad \text{and} \quad (b, -d, -w, -v, u), \quad \text{if} \ u \equiv 2 \pmod{8},\]
and
\[(b, d, u, v, w) \quad \text{and} \quad (b, -d, w, -v, -u), \quad \text{if} \ u \equiv 6 \pmod{8}.\]
We note that the 16th root of unity modulo $p$,
\[
\left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^{b/4 + d - 2(u + w)}
\]
is independent of which 5-tuple is used, since
\[
\left\{ \frac{((-d)x + c(-v))(a(\mp w \pm u) - b(\mp w \mp u))}{2b(-d)((\mp w)^2 + (\pm u)^2)} \right\}^{(b/4) - d - 2(\mp w \pm u)}
\]
\[
= \left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^A,
\]
where
\[
A = \begin{cases} 
13 \left( \frac{b}{4} - d - 2u + 2w \right), & \text{if } u \equiv 2 \pmod{8}, \\
5 \left( \frac{b}{4} - d + 2u - 2w \right), & \text{if } u \equiv 6 \pmod{8};
\end{cases}
\]
moreover,
\[
13 \left( \frac{b}{4} - d - 2u + 2w \right) - \left( \frac{b}{4} + d - 2u - 2w \right)
\]
\[
= 3b - 14d - 24u + 28w \equiv 0 \pmod{16},
\]
\[
5 \left( \frac{b}{4} - d + 2u - 2w \right) - \left( \frac{b}{4} + d - 2u - 2w \right)
\]
\[
= b - 6d + 12u - 8w \equiv 0 \pmod{16},
\]
so that
\[
A \equiv \frac{b}{4} + d - 2(u + w) \pmod{16}.
\]

The choice (3.34) makes the exponent $(b/4) + d - 2(u + w)$ in (3.26) congruent to 2 (mod 8). We now consider cases according as $b \equiv 8, 40 \pmod{64}; d \equiv 0, 8 \pmod{16}; u \equiv 2, 6 \pmod{8}$. For example if $b \equiv 8 \pmod{64}, d \equiv 0 \pmod{16}, u \equiv 6 \pmod{8}$, then $(b/4) + d - 2(u + w) \equiv 2 \pmod{16}$, so (3.26) gives
\[
2^{(p-1)/32} \equiv \left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^2 \pmod{p}
\]
\[
\equiv -\frac{(a + b)d}{ac} \pmod{p},
\]
see Table 2(VI). We remark that in applying Theorem 2 in this case, \(d\) must be chosen to satisfy the congruence (3.13). We can do this as
\[
-x^2 - 2v^2 \equiv 0 \pmod{p},
\]
since
\[
-p = -x^2 - 2u^2 - 2v^2 - 2w^2 < x^2 - 2v^2 \leq x^2 < p.
\]

If \(d \equiv 4 \pmod{8}\), replacing \(g\) by \(g^{16s+t}\) (where \(t = 1, 3, 5\) or \(7\) and \((16s + t, f) = 1\)), as necessary, we can suppose that
\[
(3.36) \quad b \equiv -8 \equiv 24 \pmod{32}, \quad d \equiv 4 \pmod{16}.
\]

There are precisely two 5-tuples \((b, d, u, v, w)\), which satisfy (3.13) and (3.36). These are
\[
(b, d, u, v, w) \quad \text{and} \quad (b, d, -u, v, -w).
\]

We note that the 16th root of unity modulo \(p\),
\[
\left\{ \frac{(dx + cw)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^{(b/4) + d - 2(u + w)}
\]
is independent of which 5-tuple is chosen, since
\[
\left\{ \frac{(dx + cw)(a(-u - w) - b(-u + w))}{2bd((-u)^2 + (-w)^2)} \right\}^{(b/4) + d - 2(-u - w)} = \left\{ \frac{(dx + cw)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^B,
\]
where
\[
B = 9\left( \frac{b}{4} + d + 2u + 2w \right) \equiv \frac{b}{4} + d - 2(u + w) \pmod{16}.
\]

The choice (3.36) makes the component \((b/4) + d - 2(u + w)\) in (3.26) congruent to 2 \((\text{mod } 8)\). We now consider cases according as \(b \equiv 24, 56 \pmod{64}; u + w \equiv 0, 4 \pmod{8}\). For example, if \(b \equiv 56 \pmod{64}\), \(u + w \equiv 4 \pmod{8}\), then \((b/4) + d - 2(u + w) \equiv 10 \pmod{16}\), so (3.26) gives
\[
(3.37) \quad 2^{(p-1)/32} \equiv \left\{ \frac{(dx + cw)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^{10} \pmod{p}
\]
\[
\equiv \left\{ \frac{-(a + b)d}{ac} \right\}^5 \pmod{p},
\]
\[
\equiv \left\{ \frac{(a + b)d}{ac} \right\} \pmod{p},
\]
see Table 2(V). However, when applying Theorem 2 in this case, it is not necessary to use the congruence \( bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \) (mod \( p \)) to distinguish the solutions \((x, \pm u, v, \pm w)\) from the solutions \((x, \pm w, \mp v, \mp u)\). Since \( \pm w \mp u \equiv \pm (u + w) \) (mod 8), as \( u \equiv w \equiv 0 \) (mod 4).

**Case (iii)** From Case (iii) of §2 we have

\[
(3.38) \quad b \equiv 0 \pmod{32}, \quad d \equiv 4 \pmod{8},
\]

or

\[
(3.39) \quad b \equiv 16 \pmod{32}, \quad d \equiv 0 \pmod{8}.
\]

If \( b \equiv 0 \pmod{32}, \ d \equiv 4 \pmod{8} \), from the work of Evans [5: Theorem 4 and its proof], we have

\[
(3.40) \quad u \equiv 2 \pmod{4}, \quad v \equiv 4 \pmod{8}, \ w \equiv 2 \pmod{4}.
\]

Replacing \( g \) by \( g^{16s+t} \), where \( t = 1, 7, 9 \) or 15 and \((16s + t, f) = 1\), as necessary, we can suppose that

\[
(3.41) \quad d \equiv 4 \pmod{16}, \ w \equiv 2 \pmod{8}.
\]

There are exactly two 5-tuples \((b, d, u, v, w)\) which satisfy (3.13) and (3.41). These are

\[
(b, d, u, v, w) \quad \text{and} \quad (-b, d, -w, -v, u), \quad \text{if} \ u \equiv 2 \pmod{8},
\]

and

\[
(b, d, u, v, w) \quad \text{and} \quad (-b, d, w, -v, -u), \quad \text{if} \ u \equiv 6 \pmod{8}.
\]

We note that the 16th root of unity modulo \( p \),

\[
\left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^{(b/4) + d - 2(u + w)}
\]

is independent of which 5-tuple is used, since

\[
\left\{ \frac{(dx + c(-v))(a(\mp w \mp u) + b(\mp w \mp u))}{2(-b)d((\mp w)^2 + (\mp u)^2)} \right\}^{(-b/4) + d - 2(\mp w \mp u)}
\]

\[= \left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^c,
\]
where
\[
C = \begin{cases} 
11 \left( \frac{-b}{4} + d - 2u + 2w \right), & \text{if } u \equiv 2 \pmod{8}, \\
3 \left( \frac{-b}{4} + d + 2u - 2w \right), & \text{if } u \equiv 6 \pmod{8},
\end{cases}
\]
and it is easily checked that
\[
C \equiv \frac{b}{4} + d - 2(u + w) \pmod{16}.
\]

Clearly, from (3.38) and (3.40), we have \((b/4) + d - 2(u + w) \equiv 4 \pmod{8}\), and we determine \((b/4) + d - 2(u + w) \pmod{16}\) by considering the cases \(b \equiv 0, 32 \pmod{64}\) and \(u \equiv 2, 6 \pmod{8}\). For example, if \(b \equiv 0 \pmod{64}\) and \(u \equiv 6 \pmod{8}\), we have \((b/4) + d - 2(u + w) \equiv 4 \pmod{16}\), so by (3.26), (1.6) and (1.8),

\[
2^{(p-1)/32} \equiv \left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^4 
\]
\[
\equiv -\frac{b}{a} \pmod{p},
\]
see Table 2 (III). In applying Theorem 2 in this case we must use the congruence \(bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p}\) to distinguish the solutions \((x, \pm u, v, \pm w)\) from the solutions \((x, \mp w, \mp v, \pm u)\).

If \(b \equiv 16 \pmod{32}, d \equiv 0 \pmod{8}\), from the work of Evans [5: Theorem 4 and its proof], we have

\[
u \equiv 0 \pmod{4}, \quad v \equiv 0 \pmod{8}, \quad w \equiv 0 \pmod{4}.
\]

Replacing \(g\) by \(g^{16s+t}\), where \(t = 1\) or \(7\) and \((16s + t, f) = 1\), as necessary, we may suppose that

\[
b \equiv 16 \pmod{64}.
\]

There are exactly four 5-tuples \((b, d, u, v, w)\), which satisfy (3.13) and (3.44). These are

\[(b, d, \pm u, v, \pm w), \quad (b, -d, \pm w, -v, \mp u).\]

We note as before that the 16th root of unity modulo \(p\),

\[
\left\{ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right\}^{(b/4) + d - 2(u + w)}
\]

is independent of which 5-tuple is used.
Clearly, from (3.39) and (3.43), we have \((b/4) + d - 2(u + w) \equiv 4 \pmod 8\), and we determine \((b/4) + d - 2(u + w) \pmod 16\) by considering the cases \(d \equiv 0, 8 \pmod {16}\) and \(u + w \equiv 0, 4 \pmod 8\). For example, if \(d \equiv 0 \pmod {16}\) and \(u + w \equiv 4 \pmod 8\), then \((b/4) + d - 2(u + w) \equiv 12 \pmod {16}\), so by (3.26), (1.6) and (1.8),

\[
2^{(p-1)/32} \equiv \left[ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right]^{(b/4) + d - 2(u + w)} \equiv \frac{b}{a} \pmod p,
\]

see Table 2(IV). When applying Theorem 2 in this case, we can use any one of the four solutions \((x, \pm u, v, \pm w)\), \((x, \pm w, -v, \mp u)\), as \(\pm w \mp u \equiv \pm (u + w) \pmod 8\).

Case (iv). As \(2^{(p-1)/16} \equiv 1 \pmod p\), from Table 1, we have

\[
b \equiv 0 \pmod {32}, \quad d \equiv 0 \pmod 8, \quad (3.46)
\]
or

\[
b \equiv 16 \pmod {32}, \quad d \equiv 4 \pmod 8, \quad (3.47)
\]

If \(b \equiv 0 \pmod {32}\), \(d \equiv 0 \pmod 8\), appealing to the work of Evans [5: Theorem 4 and its proof], we have

\[
u \equiv 0 \pmod 4, \quad v \equiv 0 \pmod 8, \quad w \equiv 0 \pmod 4, \quad (3.48)
\]

There are exactly eight 5-tuples which satisfy (3.13) and (3.48), namely,

\[
(b, d, \pm u, v, \pm w), \quad (b, -d, \pm w, -v, \mp u),
\]

\[
(-b, d, \pm w, -v, \mp u), \quad (-b, -d, \pm u, v, \pm w).
\]

It is straightforward to check that

\[
2^{(p-1)/32} \equiv \left[ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right]^{(b/4) + d - 2(u + w)} \equiv -1 \pmod p,
\]

is the same for all of these. The exponent \((b/4) + d - 2(u + w)\) is congruent to 0 \pmod 8. It is easily determined modulo 16 by considering the cases \(b \equiv 0, 32 \pmod {64}\), \(d \equiv 0, 8 \pmod {16}\), and \(u + w \equiv 0, 4 \pmod 8\). For example, if \(b \equiv 0 \pmod {64}\), \(d \equiv 0 \pmod {16}\), \(u + w \equiv 4 \pmod 8\), we have \(b/4 + d - 2(u + w) \equiv 8 \pmod {16}\) so that, by (1.6), (1.8) and (3.26),

\[
2^{(p-1)/32} \equiv \left[ \frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \right]^8 \equiv -1 \pmod p,
\]
see Table 2 (I). As noted by Evans [6: Comments following Theorem 7], it is unnecessary to use the congruence \( bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p} \) when applying Theorem 2 in this case.

Finally if \( b \equiv 16 \pmod{32}, \ d \equiv 4 \pmod{8} \), appealing to the work of Evans [5: Theorem 4 and its proof], we have

\[
u \equiv 2 \pmod{4}, \quad v \equiv 4 \pmod{8}, \quad w \equiv 2 \pmod{4}.
\]

Replacing \( g \) by \( g^{16s+t} \), where \( t = 1, 3, 5 \) or 7 and \( (16s + t, f) = 1 \), as appropriate, we can choose

\[
(3.49) \quad b \equiv 16 \pmod{64}, \quad d \equiv 4 \pmod{16}.
\]

There are two 5-tuples \((b, d, u, v, w)\) satisfying (3.13) and (3.49), namely,

\[(b, d, \pm u, v, \pm w),\]

and again it is easy to check that

\[
\frac{(dx + cv)(a(u + w) - b(u - w))}{2bd(u^2 + w^2)} \equiv (b/4)^{d-2(u+w)} \pmod{4}.
\]

Now

\[
\frac{b}{4} + d - 2(u + w) \equiv 8 - 2(u + w) \pmod{16}
\]

so, by (3.26), we have

\[
2^{(p-1)/32} \equiv \begin{cases} +1, & \text{if } u + w \equiv 4 \pmod{8}, \\ -1, & \text{if } u + w \equiv 0 \pmod{8}, \end{cases}
\]

see Table 2 (II). In applying Theorem 2 in this case, as noted by Evans [6: Comments following Theorem 7], it is necessary to use the congruence \( bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p} \). This completes the proof of Theorem 2.

4. Numerical examples. (a) \( p = 2113 \) (see Table 2 (I)). We have

\[
(a, b) = (33, \pm 32); \quad a \equiv 1 \pmod{4};
\]

\[
(c, d) = (-31, \pm 24); \quad c \equiv 1 \pmod{4};
\]

\[
(x, u, v, w) = (-17, \pm 28, -8, \pm 8) \quad \text{or} \quad (-17, \pm 8, +8, \mp 28); \quad x \equiv -1 \pmod{8}.
\]
For each choice we have
\[ b \equiv 32 \pmod{64}, \quad d \equiv 8 \pmod{16}, \quad u + w \equiv 4 \pmod{8}, \]
so by Theorem 2(1), we have
\[ 2^{(p-1)/32} = 2^{66} \equiv -1 \pmod{2113}. \]

(b) \( p = 257 \) (see Table 2 (II)). We have
\[ (a, b) = (1, 16); \quad a \equiv 1 \pmod{4}, \quad b \equiv 16 \pmod{64}; \]
\[ (c, d) = (-15, 4); \quad c \equiv 1 \pmod{4}, \quad d \equiv 4 \pmod{16}; \]
\[ (x, u, v, w) = (-9, \pm 6, -4, \mp 6) \text{ or } (-9, \pm 6, +4, \pm 6); \]
\[ x \equiv -1 \pmod{8}. \]

The congruence \( bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p} \) is satisfied by \( (x, u, v, w) = (-9, \pm 6, -4, \mp 6) \). As \( u + w \equiv 0 \pmod{8} \), by Theorem 2(II), we have
\[ 2^{(p-1)/32} = 2^8 \equiv -1 \pmod{257}. \]

(c) \( p = 1249 \) (see Table 2(III)). We have
\[ (a, b) = (-15, 32) \text{ or } (-15, -32); \]
\[ a \equiv 1 \pmod{4}, \quad b \equiv 0 \pmod{32}; \]
\[ (c, d) = (-31, -12); \quad c \equiv 1 \pmod{4}, \quad d \equiv 4 \pmod{16}; \]
\[ (x, u, v, w) = (7, 10, 4, -22) \text{ or } (7, 22, -4, 10); \]
\[ x \equiv -1 \pmod{8}, \quad w \equiv 2 \pmod{8}. \]

The congruence \( bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p} \) is satisfied by \( (a, b) = (-15, 32) \) and \( (x, u, v, w) = (7, 22, -4, 10) \) or by \( (a, b) = (-15, -32) \) and \( (x, u, v, w) = (7, 10, 4, -22) \). Hence, by Theorem 2, taking \( b = 32, u = 22 \equiv 6 \pmod{8} \), we have
\[ 2^{(p-1)/32} = 2^{39} \equiv b/a \equiv 32/15 \equiv 664 \pmod{1249}; \]

taking \( b = -32, u = 10 \equiv 2 \pmod{8} \), we have
\[ 2^{(p-1)/32} = 2^{39} \equiv -b/a \equiv 32/15 \equiv 664 \pmod{1249}. \]

(d) \( p = 1217 \) (see Table 2 (IV)). We have
\[ (a, b) = (-31, 16); \quad a \equiv 1 \pmod{4}, \quad b \equiv 16 \pmod{64}; \]
\[ (c, d) = (33, +8) \text{ or } (33, -8); \quad c \equiv 1 \pmod{4}; \]
\[ (x, u, v, w) = (-17, \pm 12, -8, \mp 16), \quad (-17, \pm 16, +8, \pm 12), \]
\[ x \equiv -1 \pmod{8}. \]
As \( d \equiv 8 \pmod{16} \) and \( u + w \equiv 4 \pmod{8} \) (for each possibility), we have, by Theorem 2,

\[
2^{(p-1)/32} = 2^{38} \equiv -b/a \equiv 16/31 \equiv 1139 \pmod{1217}.
\]

(e) \( p = 577 \) (see Table 2 (V)). We have

\[
(a, b) = (1, 24); \quad a \equiv 1 \pmod{4}, \quad b \equiv 24 \pmod{32};
\]

\[
(c, d) = (17, -12); \quad c \equiv 1 \pmod{4}, \quad d \equiv 4 \pmod{16};
\]

\[
(x, u, v, w) = (-1, \pm 4, -16, \mp 4) \quad \text{or} \quad (-1, \pm 4, +16, \pm 4).
\]

As \( b \equiv 24 \pmod{64}, u + w \equiv 0 \pmod{8} \), by Theorem 2(V), we have

\[
2^{(p-1)/32} = 2^{18} \equiv \frac{(a + b)d}{ac} \equiv \frac{-300}{17} \equiv 186 \pmod{577}.
\]

(f) \( p = 353 \) (see Table 2 (VI)). We have

\[
(a, b) = (17, 8); \quad a \equiv 1 \pmod{4}, \quad b \equiv 8 \pmod{32};
\]

\[
(c, d) = (-15, 8) \quad \text{or} \quad (-15, -8); \quad c \equiv 1 \pmod{4};
\]

\[
(x, u, v, w) = (7, -10, -4, -6) \quad \text{or} \quad (7, -6, 4, 10);
\]

\[
x \equiv -1 \pmod{8}, \quad w \equiv 2 \pmod{8}.
\]

The congruence \( bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2) \pmod{p} \) is satisfied by \((c, d) = (-15, 8)\) and \((x, u, v, w) = (7, -10, -4, -6)\), or by \((c, d) = (-15, -8)\) and \((x, u, v, w) = (7, -6, 4, 10)\). Hence, by Theorem 2, taking the first possibility, we have \( b \equiv 8 \pmod{64}, d \equiv 8 \pmod{16}, u = -10 \equiv 6 \pmod{8}\), so

\[
2^{(p-1)/32} = 2^{11} \equiv \frac{(a + b)d}{ac} \equiv \frac{40}{-51} \equiv 283 \pmod{353}.
\]

(g) \( p = 97 \) (see Table 2 (VIII)). We have

\[
(a, b) = (9, -4); \quad a \equiv 1 \pmod{4}, \quad b \equiv 12 \pmod{16};
\]

\[
(c, d) = (5, -6); \quad c \equiv 1 \pmod{4}, \quad d \equiv 2 \pmod{8};
\]

\[
(x, u, v, w) = (7, -4, 2, 2); \quad x \equiv -1 \pmod{8}, \quad w \equiv 2 \pmod{8}.
\]

As \( b \equiv 60 \pmod{64}, d \equiv 10 \pmod{16}, u \equiv 4 \pmod{8}\), by Theorem 2(VII), we have

\[
2^{(p-1)/32} = 2^3 \equiv \frac{(-32)(-46)}{(48)(20)} \equiv \frac{23}{15} \equiv 8 \pmod{97}.
\]

5. Acknowledgement. We wish to thank Mr. Lee-Jeff Bell for doing some calculations for us in connection with the preparation of this paper.
\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{$b$} & \textbf{$d$} & \textbf{Cases} & \textbf{$2^{(p-1)/16}$ (mod $p$)} & \textbf{Examples} \\
\hline
$b \equiv 0 \pmod{16}$ & $d \equiv 0 \pmod{4}$ & $b \equiv 0 \pmod{32}, d \equiv 0 \pmod{8}$ & +1 & $p = 2113$ \\
& & or & & $p = 257$ \\
& & $b \equiv 16 \pmod{32}, d \equiv 4 \pmod{8}$ & & \\
\hline
$b \equiv 8 \pmod{16}$ & $d \equiv 0 \pmod{8}$ & & $-b/a$ & $p = 353$ \\
$b$ chosen $\equiv 8 \pmod{32}$ & $d \equiv 4 \pmod{8}$ & & $+b/a$ & $p = 113$ \\
\hline
$b \equiv 4 \pmod{8}$ & $d \equiv 2 \pmod{4}$ & $b \equiv 12 \pmod{32}$ & $\frac{-(a+b)d}{ac}$ & $p = 193$ \\
$b$ chosen $\equiv 12 \pmod{16}$ & $d$ chosen $\equiv 2 \pmod{8}$ & $b \equiv 28 \pmod{32}$ & $\frac{(a+b)d}{ac}$ & $p = 17$ \\
\hline
\end{tabular}
\end{center}
\end{table}
<table>
<thead>
<tr>
<th></th>
<th>$b \equiv 0(32), d \equiv 0(8)$</th>
<th>$(b, d, u + w) \equiv (0, 0, 0), (0, 8, 4), (32, 0, 4), (32, 8, 0)$</th>
<th>$2^{(p-1)/32}$ (mod $p$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Examples $p = 47713, 10657, 31649, 50753$</td>
<td>+1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(b, d, u + w) \equiv (0, 0, 4), (0, 8, 0), (32, 0, 0), (32, 8, 4)$</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Examples $p = 25121, 18593, 51137, 2113$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>$b \equiv 16(32), d \equiv 4(8)$</td>
<td>$u + w \equiv 4$</td>
<td>+1</td>
</tr>
<tr>
<td></td>
<td>Choose</td>
<td>Example $p = 2593$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b \equiv 16(64), d \equiv 4(16)$</td>
<td>$u + w \equiv 0$</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>$bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2)$ (mod $p$)</td>
<td>Example $p = 257$</td>
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<tr>
<td>III</td>
<td>$b \equiv 0(32), d \equiv 4(8)$</td>
<td>$(b, u) \equiv (0, 6), (32, 2)$</td>
<td>$-b/a$</td>
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<td></td>
<td>Choose</td>
<td>Examples $p = 10337, 1249$</td>
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<td></td>
<td>$d \equiv 4(16), w \equiv 2(8)$</td>
<td>$(b, u) \equiv (0, 2), (32, 6)$</td>
<td>$+b/a$</td>
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<td></td>
<td>$bd(x^2 - 2v^2) \equiv ac(u^2 + 2uw - w^2)$ (mod $p$)</td>
<td>Examples $p = 10337, 1249$</td>
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<tr>
<td>IV</td>
<td>$b \equiv 16(32), d \equiv 0(8)$</td>
<td>$(d, u + w) \equiv (0, 0), (8, 4)$</td>
<td>$-b/a$</td>
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<td>Choose</td>
<td>Examples $p = 14753, 1217$</td>
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<td></td>
<td>$b \equiv 16(64)$</td>
<td>$(d, u + w) \equiv (0, 4), (8, 0)$</td>
<td>$+b/a$</td>
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<td>Examples $p = 4481, 11329$</td>
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<tr>
<td>$b = 8(16), d = 4(8)$</td>
<td>$b = 24(32), d = 4(16)$</td>
<td>$b = 8(32), w = 2(8)$</td>
<td>$b = 4(8), d = 2(8)$</td>
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<td><strong>Examples</strong> $p = 15361, 1889$</td>
<td><strong>Examples</strong> $p = 9377, 577$</td>
<td><strong>Examples</strong> $p = 2273, 353, 1601, 13921$</td>
<td><strong>Examples</strong> $p = 7, 1409, 3041, 97$</td>
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References


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Nestor Edgardo Aguilera and Eleonor Ofelia Harboure de Aguilera, On
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