CENTRALIZERS OF IRREGULAR ELEMENTS IN REDUCTIVE ALGEBRAIC GROUPS

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Let G be a reductive linear algebraic group defined over an algebraically closed field K. An element $x \in G$ is called regular if $\dim Z_G(x)$ is the rank of G — which is the smallest possible dimension for a centralizer — otherwise x is called irregular. T. A. Springer has shown that if $x \in G$ is regular, then $Z_G(x)^0$ is abelian. In this paper, we show that when char K is good for G, the converse is also true — if $x \in G$ is irregular, then $Z_G(x)^0$ is nonabelian. In the course of the proof of this, we show that if G is a classical group or G_2 (char K good), then $\dim Z(Z_G(x)^0)$ is at most rank G. Further, if $G = Sp_{2n}(K)$ (char $K \neq 2$), then $Z(Z_G(x)^0)$ consists of polynomials in x.

Introduction and statement of results. In this paper, all groups will be linear algebraic groups defined over an algebraically closed field K. If G is a group, then G^0 denotes its identity component and $\mathfrak{g}$ or $\mathfrak{L}(G)$ denotes the Lie algebra of G. If G is reductive (we require that reductive groups be connected), then by "the simple components of G", we mean the simple components of the semisimple group $G/\operatorname{Rad} G$ where $\operatorname{Rad} G$ is the radical of G.

Let G be a reductive group. An element $x \in G$ is called regular if $\dim Z_G(x)$ is minimal in $\{\dim Z_G(y) \mid y \in G\}$; otherwise, x is called irregular. The regular elements are dense in G; the identity element is always irregular. In $GL_n(K)$, a diagonalizable element is regular if and only if all its eigenvalues are distinct. It is well known that if $x \in G$ is regular, then

$$\dim Z_G(x) = \operatorname{rank} G$$

and if $y \in G$ is irregular, then

$$\dim Z_G(y) \geq 2 + \operatorname{rank} G$$

(cf. [17]). If

$$\dim Z_G(y) = 2 + \operatorname{rank} G,$$

then y is called subregular.

T. A. Springer has shown that if $x \in G$ is regular, then $Z_G(x)^0$ is abelian. [13] Under some very mild restrictions on char K (char $K = 0$ or char $K \geq 7$), we shall prove the converse; namely, that if $Z_G(x)^0$ is

abelian, then x is regular — or, equivalently, if $x \in G$ is irregular, then $Z_G(x)^0$ is nonabelian. This last statement is the one that we will actually prove.

Our proof uses the classification of nilpotent conjugacy classes in a simple Lie algebra, and since this classification is only valid in good characteristics, we must require that char K be good for G. The usual way of defining good characteristic for a reductive group G (or Lie algebra $\mathfrak{g}$) is to define a bad prime as one that divides a coefficient in the highest root. Then all other primes and zero are called good. If G is of type A_n, then all characteristics are good. So char K is good for G

(a) if G has a component of type B_n, C_n, or D_n, then char $K \neq 2$

(b) if G has a component of type G_2, F_4, E_6, or E_7, then char $K \neq 2$ or 3

(c) if G has a component of type E_8, then char $K \neq 2$, 3, or 5.
Hence, if char $K = 0$ or char $K \geq 7$, then char K is always good.

THEOREM A. *Let G be a reductive group with* char K *good for G. Then $x \in G$ is regular if and only if $Z_G(x)^0$ is abelian.*

Proving Theorem A is the main object of this paper. We will first make some reductions (§2), then handle the classical groups (§3) and finally the exceptional groups (§§4 and 5). In the course of this, we will derive some stronger results for the classical groups. We say that a reductive group G is of classical type if none of its components is an exceptional group.

THEOREM B. 1. *Let G be a group of classical type with* char $K \neq 2$. *Let $x \in G$. Then*

$$\dim Z(Z_G(x)^0) \leq \operatorname{rank} G.$$

Equality holds if and only if x is regular.

2. *Let $G = G_2$ with* char $K \neq 2$ or 3 *and let $x \in G$. Then*

$$\dim Z(Z_G(x)^0) \leq \operatorname{rank} G.$$

Equality holds only in two cases.

(i) *x is regular, or*

(ii) *x is subregular, but not semisimple.*

The first part is proved in §3; the second part in §5. Note that $\dim Z(Z_G(x)^0)$ can equal rank G even if x is irregular (we suspect that G_2

is the only simple group in which this can happen). For the symplectic groups, we can improve Theorem B:

THEOREM C. *Let* $G = Sp_{2n}(K)$ *with* char $K \neq 2$ *and let* $x \in G$. *Then* $Z(Z_G(x)^0)$ *contains only polynomials in* x.

This is proved in §3. If $G = PGL_2(\mathbf{C})$ and

$$x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we see that x is regular, but $Z_G(x)$ is nonabelian — of course $Z_G(x)^0$ is abelian. This happens because $Z_G(x)$ is not connected, and that happens because G is not simply connected (as x is semisimple). Now if G is semisimple and simply connected, and if $x \in G$ is regular semisimple, then $Z_G(x)$ is connected and thus abelian. In her dissertation [9], B. Lou proved that if u is a regular unipotent element in a semisimple group G, then $Z_G(u)$ is abelian (char K arbitrary). Therefore, if G is a semisimple, simply connected group, and if $x \in G$ is regular, then $Z_G(x)$ is abelian. So as a corollary to Theorem A we have:

THEOREM D. *Let* G *be a semisimple, simply connected group with* char K *good for* G. *Then* $x \in G$ *is regular if and only if* $Z_G(x)$ *is abelian.*

As general references to algebraic groups and Lie algebras, we cite Humphreys' books [7, 7a] which we follow for basic definitions and conventions. For conjugacy classes and centralizers, the reader is invited to consult the Springer-Steinberg article [16] or Steinberg's Tata notes [18]. The classification of unipotent and nilpotent conjugacy classes is described in papers of Dynkin [4], Elkington [6], and Bala-Carter [1, 2]. The extension of the Dynkin classification to all good characteristics is described by Springer-Steinberg [16] for the classical groups, by Jeurissen and Stuhler [19] for G_2, and by Pommerening [10, 11] for the other exceptional groups. Dynkin, Elkington, and Bala-Carter also have information about the centralizers of nilpotent elements, which can also be found in an article of Elasvili [5].

2. **Reductions.** In this section, we show that in order to prove Theorem A, it suffices to prove:

THEOREM a. *Let* $\mathfrak{g}$ *be a simple Lie algebra with* char K *good for* $\mathfrak{g}$. *If* $X \in \mathfrak{g}$ *is nilpotent and irregular, then* $\mathfrak{z}_{\mathfrak{g}}(X)$ *is nonabelian.*

The rest of this paper will then be devoted to proving Theorem α. The first thing we notice is that if G is reductive and $t \in G$ is semisimple, then $Z_G(t)^0$ is well in hand:

PROPOSITION 2.1. *Let G be a reductive group and let $t \in G$ be semisimple. Let T be a maximal torus for G containing t and let Ψ be the resulting set of roots. Then*

1. $Z_G(t)^0$ *is generated by T and the one parameter unipotent subgroups U_α for those $\alpha \in \Psi$ for which $\alpha(t) = 1$. Moreover, $Z_G(t)^0$ is reductive and has the same rank as G.*

2. $Z_G(t)$ *is generated by $Z_G(t)^0$ and the $n \in N_G(T)/T$ such that $ntn^{-1} = t$.*

3. *t is regular if and only if $\alpha(t) \neq 1$ for all $\alpha \in \Psi$*

4. $\dim Z(Z_G(t)^0) \leq \operatorname{rank} G$
Equality occurs if and only if t is regular.

This is rather well known and the proof can be found in [18] and [3]. If we were dealing with real compact semisimple Lie groups, then we would be finished here. It is the unipotent elements of G that are the obstacle.

If G is reductive, and $x \in G$ is arbitrary, then we can write $x = su$ with s and u commuting semisimple and unipotent elements of G respectively (Jordan decomposition). If we let $H = Z_G(s)^0$, we see that $u \in H$ and $Z_G(x)^0 = Z_H(u)^0$. Because the ranks of G and H are the same, we see that x is regular in G if and only if u is regular in H. So we have:

REMARK 2.2. To prove Theorem A, it suffices to assume that the element x is unipotent.

The next step is to reduce the problem to the case where the group G is simple. Further we will only have to look at isogeny classes of simple groups instead of isomorphism classes.

PROPOSITION 2.3. *Let G be a reductive group, R a closed normal subgroup contained in the center, and $x \in G$. Let $\pi: G \to G/R = H$ be the canonical quotient map with $\pi(x) = \bar{x}$. Then*

1. *$R \cap (G, G)$ is finite and R^0 is a torus.*

2. $\dim Z_G(x) = \dim R + \dim Z_H(\bar{x})$.
Thus x is regular in G if and only if $\bar{x}$ is regular in H.

3. $\dim Z(Z_G(x)^0) = \dim R + \dim Z(Z_H(\bar{x})^0)$.
Thus the conclusions of Theorems A and B hold for G if and only if they hold for H.

Proof. 1. This is standard; see [7a], for example.

2. Let $\bar{C} = Z_H(\bar{x})$ and $C = \pi^{-1}(\bar{C})$. We have to show that $\dim C = \dim Z_G(x)$. Now $y \in C$ if and only if $xyx^{-1}y^{-1} \in R$. As $R \cap (G, G)$ is finite, so is $R \cap \{xyx^{-1}y^{-1} \mid y \in G\}$. Let $\{r_1, \ldots, r_p\}$ be that set and choose $y_i \in G$ so that $xy_ix^{-1}y_i^{-1} = r_i$. Now $xyx^{-1}y^{-1} = xzx^{-1}z^{-1}$ precisely when $z^{-1}y \in Z_G(x)$, so the y_i are determined mod $Z_G(x)$. Therefore, $C = \bigcup_{i=1}^{p} y_i Z_G(x)$ and $C^0 = Z_G(x)^0$ as desired.

Let $\bar{Z}_1 = Z(Z_H(\bar{x})^0)$ and $Z_1 = \pi^{-1}(\bar{Z}_1)^0$. Clearly

$$Z\big(Z_G(x)^0\big)^0 \subseteq Z_1 \subseteq Z_G(x)^0.$$

We want to show that the first inclusion is an equality. Choose $y \in Z_1$. Now $(y, Z_G(x)^0)$ is a finite set, contains 1 (as y commutes with x) and is connected. Hence $y \in Z(Z_G(x)^0)$. As Z_1 is connected, we have $y \in Z(Z_G(x)^0)^0$ as desired. $\qquad\square$

REMARK 2.4. 1. Let G be reductive. Then the conclusions of Theorems A and B hold for G if and only if they hold for $G/\operatorname{Rad} G$. So we can assume in Theorems A and B that G is semisimple.

2. Let G and H be isogenous semisimple groups. Then the conclusions of Theorems A and B hold for G if and only if they hold for H.

3. To prove Theorems A and B, we may assume that G is simple and x is unipotent. Further, it is enough to prove these theorems for one group in each simple isogeny class.

Proof. 1. Let $R = \operatorname{Rad} G$ and apply Proposition 2.3.

2. Let $\pi_1 \colon \tilde{G} \to G$ and $\pi_2 \colon \tilde{G} \to H$ be the universal covers of G and H. Now apply Proposition 2.3.

3. We can assume that G is adjoint and semisimple. Then $G = \prod_{i=1}^{r} G_i$ is a direct product of simple groups. If $x \in G$, then $x = x_1 \cdots x_r$ with $x_i \in G_i$. So $Z_G(x)^0 = \prod_{i=1}^{r} Z_{G_i}(x_i)^0$. $\qquad\square$

The next task is to transfer all this to the Lie algebra. If $x \in G$ and $X \in \mathfrak{g}$, we want to use results about $\mathfrak{z}_{\mathfrak{g}}(X)$ to understand $Z_G(x)^0$. If $\operatorname{char} K = 0$, then $\mathcal{L}(Z_G(x)^0) = \mathfrak{z}_{\mathfrak{g}}(x)$ and $\mathcal{L}(Z_G(X)^0) = \mathfrak{z}_{\mathfrak{g}}(X)$ — if x and X are semisimple, then these hold no matter what $\operatorname{char} K$ is. When $K = \mathbf{C}$, then there are maps (log and exp) from G to $\mathfrak{g}$ and vice versa which are power series and local isomorphisms of manifolds. So when $K = \mathbf{C}$ and $\log x = X$, then we have $\mathcal{L}(Z_G(x)^0) = \mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(X) = \mathcal{L}(Z_G(X)^0)$ which is as nice a relation among centralizers as one could want. We would like this to happen in our case — G reductive and x unipotent. Because we have room in the isogeny classes to skip around and because $\operatorname{char} K$ is good (by assumption), we can actually do this.

PROPOSITION 2.5 (*Richardson*). *Let G be a reductive group with* char K *good for G. Let $x \in G$ and $X \in \mathfrak{g} = \mathcal{L}(G)$. Then*

$$\mathcal{L}\left(Z_G(x)^0\right) = \mathfrak{z}_{\mathfrak{g}}(x)$$

and

$$\mathcal{L}\left(Z_G(X)^0\right) = \mathfrak{z}_{\mathfrak{g}}(X).$$

The proof is in [**12**]. Next, we let $U = \{x \in G \mid x \text{ is unipotent}\}$, the unipotent variety, and $\mathfrak{n} = \{X \in \mathfrak{g} \mid X \text{ is nilpotent}\}$, the nilpotent variety.

PROPOSITION 2.6. (*Springer*). *Let G be a semisimple, simply connected group with* char K *good for G. Then there is a G-equivariant bijective morphism $f: V \to \mathfrak{n}$ which topologically is a homeomorphism.*

The proof is in [**15**]. By G-equivariant, we mean that for $g \in G$ and $v \in V$, $\operatorname{Ad} g(f(v)) = f(gvg^{-1})$. Thus f preserves centralizers and if $f(x) = X$, then $\mathfrak{z}_{\mathfrak{g}}(x) = \mathcal{L}(Z_G(x)^0) = \mathcal{L}(Z_G(X)^0) = \mathfrak{z}_{\mathfrak{g}}(X)$. It now makes sense to define regular, subregular, and irregular elements in reductive Lie algebras. In good characteristics, the results are the same as in reductive groups.

The final thing we have to notice is that in going from a group to its Lie algebra, the center can only get larger. That is, $\mathcal{L}(Z(G)^0) \subseteq \mathfrak{z}(\mathfrak{g})$. Thus, in our case, if $\mathfrak{z}_{\mathfrak{g}}(X)$ is nonabelian, then so is $Z_G(X)^0$ and $Z_G(x)^0$.

REMARK 2.7. To prove Theorem A, it is sufficient to prove Theorem a.

PROPOSITION 2.8. (*Steinberg, Springer*). 1. *Let G be a reductive group with* char K *arbitrary. Let $x \in G$ be unipotent and let Ψ be a set of roots for G. Choose a base for Ψ so that x is supported by positive roots*, i.e. $x = \prod_{\alpha > 0} \varepsilon_\alpha(c_\alpha)$ *where $c_\alpha \in K$. Then x is regular if and only if $c_\alpha \neq 0$ for α simple. Further, all regular unipotent elements are conjugate.*

2. *Let $\mathfrak{g}$ be a reductive Lie algebra with* char K *good for $\mathfrak{g}$. Let $X \in \mathfrak{g}$ be nilpotent and let Ψ be a set of roots for $\mathfrak{g}$. Choose a base for Ψ so that X is supported by positive roots*, i.e. $X = \Sigma_{\alpha > 0} b_\alpha e_\alpha$ *with $b_\alpha \in K$. Then X is regular if and only if $b_\alpha \neq 0$ for α simple. All regular nilpotent elements are conjugate.*

Part 1 is proved in [**17**]; part 2 in [**14**]. In particular $\prod_{\alpha \in \Delta} \varepsilon_\alpha(1)$ and $\Sigma_{\alpha \in \Delta} \varepsilon_\alpha$ are regular (Δ is a base of Ψ). These are called the standard regular and nilpotent elements respectively.

3. The classical groups. In this section, we will prove Theorems A, B, and C for the classical groups. The next result is well-known.

PROPOSITION 3.1. *Let* $G = GL_n(K)$ *or* $SL_n(K)$ *and let* $x \in G$. *Then*,
1. $Z(Z_G(x)) = \{$ *polynomials in* $x\} \cap G$. *Therefore*,

$$\dim Z(Z_G(x)) \leq \operatorname{rank} G$$

and equality holds if and only if x *is regular.*
2. $Z(Z_G(x)^0)$ *consists of polynomials in* x. *Therefore*,

$$\dim Z(Z_G(x)^0) \leq \operatorname{rank} G$$

and equality holds if and only if x *is regular.*
3. *The following statements are equivalent*:
(a) x *is regular.*
(b) $Z_G(x)$ *consists of polynomials in* x.
(c) $Z_G(x)^0$ *consists of polynomials in* x.
(d) $Z_G(x)$ *is abelian.*
(e) $Z_G(x)^0$ *is abelian.*
(f) *The minimal polynomial of* x *is the characteristic polynomial of* x.
(g) *Different blocks have different eigenvalues in the Jordan form of* x.

This more than takes care of groups of type A_n. For the rest of this section, we will be concerned with the orthogonal and symplectic groups, so we assume that char $K \neq 2$. At this point, we want to describe explicitly the map of Proposition 2.6. Let $G = SO_n(K)$ or $Sp_n(K)$ and $\mathfrak{g} = \mathfrak{so}_n(K)$ or $\mathfrak{sp}_n(K)$, respectively, Consider

$$f(x) = (1 - x)(1 + x)^{-1}$$

(1 is the $n \times n$ identity matrix). This map takes unipotent elements of G to nilpotent elements of $\mathfrak{g}$ ([20]). Its inverse is formally the same:

$$f^{-1}(X) = (1 - X)(1 + X)^{-1}$$

If x is unipotent and X nilpotent, then f and f^{-1} are polynomials in x and X, respectively. Thus there is no problem in going from G to $\mathfrak{g}$ and vice versa.

PROPOSITION 3.2. *Let* $\mathfrak{g} = \mathfrak{so}_n(K)$ *or* $\mathfrak{sp}_n(K)$ *with* char $K \neq 2$. *Let* $X \in \mathfrak{g}$ (*not necessarily nilpotent*) *and let* k *be the degree of the minimal polynomial of* x. *Then*

$$\dim (\{ \text{ polynomials in } x\} \cap \mathfrak{g}) = \left[\frac{k}{2}\right]$$

where $[n]$ *is the greatest integer in* n.

Proof. If s is the form, then $X \in \mathfrak{g}$ means that

$$X = -s^{-1}X^t s$$

so

$$X^i = (-1)^i (X^i)^t s$$

and $X^i \in \mathfrak{g}$ if and only if i is odd. $\square$

PROPOSITION 3.3. 1. *Let* $\mathfrak{g} = \mathfrak{sp}_{2n}(K)$ *or* $\mathfrak{so}_{2n+1}(K)$ *with* char $K \neq 2$. *If* $X \in \mathfrak{g}$ *is a regular nilpotent element, then* X *is regular in* $\mathfrak{sl}_{2n}(K)$ *or* $\mathfrak{sl}_{2n+1}(K)$, *respectively. Therefore,*

$$\mathfrak{z}_{\mathfrak{g}}(X) = \{ \textit{polynomials in } X \} \cap \mathfrak{g}.$$

2. *Let* $\mathfrak{g} = \mathfrak{so}_{2n}(K)$ *with* char $K \neq 2$. *If* $X \in \mathfrak{g}$ *is a regular nilpotent element, then* X *is subregular in* $\mathfrak{sl}_{2n}(K)$. *Hence,* dim $(\{ \textit{polynomials in } X \} \cap \mathfrak{g}) = n - 1 <$ rank $\mathfrak{g}$ *and* $\{ \textit{polynomials in } X \} \cap \mathfrak{g} < \mathfrak{z}_{\mathfrak{g}}(X)$.

Proof. We use e_{ij} to denote the matrix with 1 in the (i, j) position and zeroes elsewhere. The argument for the first part consists of writing down the standard regular nilpotent element and seeing that it is regular in $\mathfrak{sl}_{2n}(K)$ in the symplectic case and regular in $\mathfrak{sl}_{2n+1}(K)$ in the orthogonal case. Then, Proposition 3.2 shows that $\{$polynomials in $X\} \cap \mathfrak{g}$ is the whole of $\mathfrak{z}_{\mathfrak{g}}(X)$. For part 2, let $\mathfrak{g} = \mathfrak{so}_{2n}(K)$ and let X be the standard regular nilpotent element, i.e.

$$X = \sum_{i=1}^{n-1} (e_{i,i+1} - e_{n+i+1,n+i}) + e_{n-1,2n} - e_{n,2n-1}.$$

Now the rank of X is $2n - 2$ and if we look at the $n + 1$st basis vector, we see that t^{2n-1} is the minimal polynomial of X, thus X is subregular in $\mathfrak{sl}_{2n}(K)$. $\square$

If $\mathfrak{g} = \mathfrak{so}_{2n}(K)$ and X is the standard regular nilpotent element, then an example of a $Y \in \mathfrak{z}_{\mathfrak{g}}(X)$ which is not a polynomial in X is:

$$Y = e_{1,n} - e_{2n,n+1} + e_{n,n+1} - e_{1,2n}.$$

Then $XY = YX = 0$. Thus is Y where a polynomial in X, it would have to be a multiple of X^{2n-2}, which cannot happen for two reasons: First, $Y^2 \neq 0$; and second, $X^{2n-2} \notin \mathfrak{g}$.

.   If $X \in \mathfrak{sl}_n(K)$ is nilpotent, then the conjugacy class of X (via $SL_n(K)$) is determined by the Jordan form of X, and thus by the sizes of the Jordan blocks. When we say that

$$X \sim k_1, k_2, \ldots, k_r$$

we mean that the Jordan form of X has blocks of sizes $k_1 \times k_1$, $k_2 \times k_2$, etc. The usual conventions is that $k_i \geq k_{i+1}$ for all i. Then a nilpotent conjugacy class in $\mathfrak{sl}_n(K)$ is uniquely represented by a partition of n. The regular class is n, and the subregular class is $n - 1, 1$.

Let $X \in \mathfrak{sl}_n(K)$ be nilpotent with Jordan form $k_1, k_2, \cdots, k_s$, and let r_j be the number of k_i that are equal to j. Springer and Steinberg have shown that X is conjugate (via $SL_n(K)$) to an element of $\mathfrak{so}_n(K)$ if and only if r_j is even when j is even, and X is conjugate to an element of $\mathfrak{sp}_n(K)$ if and only if r_j is even when j is odd. In $\mathfrak{sp}_n(K)$ and $\mathfrak{so}_{2n+1}(K)$ the Jordan form determines the conjugacy class; in $\mathfrak{so}_{2n}(K)$, two different conjugacy classes can have the same Jordan form, but then they are conjugate via $O_{2n}(K)$ [16]. So to investigate $\mathfrak{z}_\mathfrak{g}(X)$ for X nilpotent and $\mathfrak{g}$ orthogonal or symplectic it is sufficient to exhaust the possible Jordan forms for X.

LEMMA 3.4. *Let V be a finite dimensional vector space over K (char $K \neq 2$), and let $\mathfrak{g} = \mathfrak{sp}(V)$ or $\mathfrak{so}(V)$. If $X \in \mathfrak{g}$ is nilpotent, then we can orthogonally decompose V and X:*

$$V = V_1 \dotplus V_2 \cdots \dotplus V_r,$$

$$X = X_1 \dotplus X_2 + \cdots \dotplus X_r,$$

where $X_i \in \mathfrak{g}_i = \mathfrak{sp}(V_i)$ or $\mathfrak{so}(V_i)$ respectively and either
  (a) X_i is regular in $\mathfrak{sl}(V_i)$ and $\mathfrak{g}_i$, or
  (b) X_i has Jordan form λ, λ where λ is odd in the symplectic case and even in the orthogonal case.
Further, $\mathfrak{z}(\mathfrak{z}_{\mathfrak{g}_i}(X_i)) = \{$ polynomials in $X_i\} \cap \mathfrak{g}_i$.

PROPOSITION 3.5. *Let $\mathfrak{g} = \mathfrak{so}_n(K)$ or $\mathfrak{sp}_n(K)$ with char $K \neq 2$ and let $X \in \mathfrak{g}$ be nilpotent. Then*

$$\mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) = \{ \text{polynomials in } X\} \cap \mathfrak{g}$$

except in the case where $\mathfrak{g} = \mathfrak{so}_{2m}(K)$ and X has Jordan form λ, μ where λ and μ are unequal and odd. In that case, $\{$ polynomials in $X\} \cap \mathfrak{g}$ has codimension one in $\mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X))$. In any event,

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) \leq \operatorname{rank} \mathfrak{g}$$

and equality holds if and only if X is regular.

The proofs of 3.4 and 3.5 are straightforward. Note that 3.5 is sufficient to prove Theorem A for the classical groups. Let us say something about the exceptional case — which includes the case where $X \in \mathfrak{so}_{2m}(K)$ is regular. For convenience, we change the form to sum of

squares. Let

$$X = \left[\begin{array}{c|c} X_1 & 0 \\ \hline 0 & X_2 \end{array} \right]$$

where $X_1 \in \mathfrak{so}_\lambda(K)$ and $X_2 \in \mathfrak{so}_\mu(K)$ are regular nilpotent. Let

$$W = \left[\begin{array}{c|c} 0 & A \\ \hline -A^t & 0 \end{array} \right]$$

where $X_1 A = 0$ and $AX_2 = 0$. Then $W \in \mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X))$ but W is not a polynomial in X.

THEOREM 3.7. *Assume* char $K \neq 2$

1. *If* $G = Sp_n(K)$ *and* $x \in G$, *then* $Z(Z_G(x)^0)$ *consists of polynomials in* x.

2. *If* $G = SO_n(K)$ *or* $Sp_n(K)$ *and* $x \in G$, *then*

$$\dim Z\big(Z_G(x)^0\big) \leq \text{rank } G$$

and equality holds if and only if x *is regular. Therefore* $x \in G$ *is regular if and only if* $Z_G(x)^0$ *is abelian.*

Proof. Let $x = su$ be the Jordan decomposition of x, $H = Z_G(s)^0$, $\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(s)$, f the Cayley map, and $N = f(u)$.

1. $G = Sp_n(K)$, $\mathfrak{g} = \mathfrak{sp}_n(K)$.

Then $V = K^n = V_1 + V_2 + \cdots + V_r + V_1' + V_2' + \cdots + V_r'$ where $s|_{V_i} = \lambda_i I$ and $s|_{V_i'} = \lambda_i^{-1} I$.

(a) If no λ_i is a λ_j', then

$$\mathfrak{h} = \mathfrak{z}_\mathfrak{g}(s) \cong \mathfrak{gl}(V_1) + \cdots + \mathfrak{gl}(V_r)$$

Now $Z_G(x)^0 = Z_H(N)^0$ and the Lie algebra of this group is $\mathfrak{z}_\mathfrak{h}(N)$. So $\mathfrak{z}(\mathfrak{z}_\mathfrak{h}(N)) = \mathfrak{z}(h) \oplus (\{\text{polynomials in } N\} \cap \mathfrak{h}) = (\{\text{polynomials in } s\} \cup \{\text{polynomials in } N\}) \cap \mathfrak{h}$. Therefore $Z(Z_G(x)^0)$ consists of polynomials in x, because s and N are polynomials in x, and because for any affine group A,

$$\mathcal{L}(Z(A)) \subseteq \mathfrak{z}(\mathcal{L}(A)).$$

(b) If some $\lambda_i = \lambda_j^{-1}$, then one gets a pair of symplectic algebras in $\mathfrak{z}_\mathfrak{g}(s)$ — if $\lambda_i = \pm 1$, then one gets one symplectic algebra. Proposition 2.6 allows us to use the same argument as above. This completes the symplectic case.

2. $G = SO_n(K)$. The situation here is similar to the symplectic case — everything is easy unless $\lambda_i = \lambda_j^{-1}$ and then orthogonal algebras occur. For then if N restricted to one of these algebras has just two unequal odd blocks, then — and only then — an element can appear in $Z(Z_G(x)^0)$ which is not a polynomial in x. Clearly the only time that dim $Z(Z_G(x)^0)$ could equal rank G is when x is regular. $\qquad\square$

So now we have proved Theorem C and Theorems A and B for the classical groups. To finish the proofs of Theorems A and B, we need only work with the exceptional groups and algebras — this will be done in the next two sections.

4. Semiregular elements. Let G be a reductive group and $\mathfrak{g}$ its Lie algebra. A unipotent element $x \in G$ (nilpotent element $X \in \mathfrak{g}$) is called semiregular if whenever a semisimple element $t \in G$ centralizes x (or X), then $t \in Z(G)$. Regular elements are semiregular; however, we shall use semiregular mean semiregular and irregular. If char K is good, then semiregular elements occur only in groups of type D_n, E_6, E_7, and E_8. In this section we will prove that if $\mathfrak{g}$ is E_6, E_7, or E_8, char K is good for $\mathfrak{g}$, and $X \in \mathfrak{g}$ is semiregular, then

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) < \text{rank } \mathfrak{g}.$$

Thus, $\mathfrak{z}_\mathfrak{g}(X)$ is nonabelian. Unfortunately, our only proof of this is computational, so we omit the details which can be found in [8].

E_6 has only one semiregular class, while E_7 and E_8 each have two. These classes are usually denoted by $E_6(a_1)$, $E_7(a_1)$, $E_7(a_2)$, $E_8(a_1)$, and $E_8(a_2)$. We use Elkington's paper [6] for representatives of these classes. For our numbering of the roots, the reader should take a glance at Table 1.

First, we handle $E_6(a_1)$. Here we take

$$X = e_{\alpha_1} + e_{\alpha_2 + \alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_3 + \alpha_6} + e_{\alpha_6}.$$

PROPOSITION 4.1. *Let* $\mathfrak{g} = E_6$ *with* char $K \neq 2$ *or* 3 *and let* X *be as above. Then,*

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) = 3 < \text{rank } \mathfrak{g}$$

Next, we move to E_7. A representative for $E_7(a_1)$ is

$$X = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3 + \alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_4 + \alpha_7} + e_{\alpha_7}.$$

This is the subregular class. A representative for $E_7(a_2)$ is

$$Y = e_{\alpha_1} + e_{\alpha_2 + \alpha_3 + \alpha_4} + e_{\alpha_3} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_4 + \alpha_7}$$

$\dim \mathfrak{z}_\mathfrak{g}(Y) = 11$ [4].

PROPOSITION 4.2. *Let* $\mathfrak{g} = E_7$ *with* char $K \neq 2$ *or* 3 *and let* X *and* Y *be as above. Then,*

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) = 5 < \text{rank } \mathfrak{g}$$

and

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(Y)\big) = 4 < \text{rank } \mathfrak{g}.$$

Finally, $\mathfrak{g} = E_8$. A representative for $E_8(\alpha_1)$ is

$$X = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4 + \alpha_5} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8} + e_{\alpha_5 + \alpha_8};$$

X is subregular. A representative for $E_8(\alpha_2)$ is

$$Y = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3 + \alpha_4 + \alpha_5} + e_{\alpha_4} + e_{\alpha_5 + \alpha_8} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8};$$

$\dim \mathfrak{z}_\mathfrak{g}(Y) = 12$ [4].

PROPOSITION 4.3. *Let* $\mathfrak{g} = E_8$ *with* char $K \neq 2, 3$, *or* 5 *and let* X *and* Y *be as above. Then*

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) = 5 < \text{rank } \mathfrak{g}$$

and

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(Y)\big) = 4 < \text{rank } \mathfrak{g}.$$

This completes the case of the semiregular elements. The next section deals with the rest of the irregular elements.

5. Non-semiregular elements. In this section we shall prove that if $\mathfrak{g}$ is an exceptional Lie algebra with char K good for $\mathfrak{g}$ and if $X \in \mathfrak{g}$ is nilpotent and neither regular non semiregular, then $\mathfrak{z}_\mathfrak{g}(X)$ is nonabelian. This will complete the proof of Theorem A. We will also prove Theorem B for G_2, which will complete the proof of Theorem B.

Let Ψ be a root system. A subset $\Sigma \subseteq \Psi$ is called an integrally closed subsystem of Ψ if Σ itself is a root system and if whenever $n_1 \alpha_1 + \cdots + n_r \alpha_r \in \Psi$ where $n_i \in \mathbf{Z}$ and $\alpha_i \in \Sigma$, then $n_1 \alpha_1 + \cdots + n_r \alpha_r \in \Sigma$.

Let G be a reductive group and G_1 a reductive subgroup of G such that G_1 contains a maximal torus T of G. Then we call G_1 an integrally closed subgroup of G if the root system of G_1 relative to T is an integrally closed subsystem of the root system of G relative to T. We define integrally closed subalgebra in the analogous manner, although we must be careful in extending the roots to a larger Cartan subalgebra. Further, we need char K to be good for the Lie algebra $\mathfrak{g}$.

If Σ is an integrally closed subsystem of Ψ, then Σ is called a maximal subsystem if there are no integrally closed subsystems properly between Σ and Ψ. From this, we define maximal integrally closed reductive subgroup and subalgebra.

PROPOSITION 5.1. *Let G be a reductive group and let $x \in G$ be unipotent. Then there is a reductive integrally closed subgroup G_1 of G with the same rank as G such that x is regular or semiregular in G_1.*

The proof is in [16] — note that it holds in all characteristics. The Lie algebra analogue is:

PROPOSITION 5.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra with char K good for $\mathfrak{g}$ and let $X \in \mathfrak{g}$ be nilpotent. Then there is a semisimple integrally closed subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ such that X is regular or semiregular in $\mathfrak{g}_1$. Further,*

$$\text{rank } \mathfrak{g}_1 \leq \text{rank } \mathfrak{g}.$$

The problem we will run into is the case where the ranks are the same. Anyway, the problem of determining nilpotent conjugacy classes in $\mathfrak{g}$ takes on a tractable form. First determine all the integrally closed subsystems of a given root system. Then determine which root systems can have semiregular elements. Finally, solve the conjugacy problem: Given X_i regular or semiregular in $\mathfrak{g}_i$ ($i = 1, 2$) integrally closed subalgebras of $\mathfrak{g}$, then when are X_1 and X_2 conjugate in $\mathfrak{g}$?

This was first done by Dynkin [4] in characteristic zero. The task of showing that the Dynkin results hold in all good characteristics has been completed by Pommerening [10, 11] (for a more complete list of citations, see §1).

Dynkin's method for determining the integrally closed subsystems of a root system Ψ is as follows. Take the lowest root, $-\mu$, and attach it to the Dynkin diagram for Ψ in the usual manner. That is, if α is a simple root, then α and $-\mu$ are joined by

$$\frac{4(-\mu, \alpha)^2}{(-\mu, -\mu)(\alpha, \alpha)}$$

lines. If $-\mu$ and α have different lengths, then one draws an arrow from the longer root to the shorter root. This gives the extended Dynkin diagram of Ψ. Table 1 shows the extended diagrams for all the irreducible root systems. From the extended diagram, one deletes one or more nodes. This gives the diagram of an integrally closed subsystem of Ψ. Repeating this process ultimately yields all the integrally closed subsystems of Ψ.

PROPOSITION 5.3. (*Dynkin*). *All the maximal integrally closed subsystems are obtained by omitting a root whose coefficient in the highest root is either one or a prime.*

We proceed on the inductive assumption that we have proved Theorem $\mathfrak{a}$ for simpler systems — or if one prefers, induction is by dimension. Thus is X is regular nilpotent in $\mathfrak{g}_1$ where $\mathfrak{g}_1$ is an integrally closed subsystem of $\mathfrak{g}$ which is not maximal, then we know that there is a semisimple subalgebra $\mathfrak{g}'$ such that $\mathfrak{g}_1 \subset \mathfrak{g}' \subset \mathfrak{g}$ is a chain of integrally closed subalgebras. By induction, $\mathfrak{z}_{\mathfrak{g}'}(X)$ is nonabelian. Of course if X semiregular in $\mathfrak{g}$, then we already know (by §§3 and 4) that

$$\dim \mathfrak{z}\big(\mathfrak{z}_{\mathfrak{g}}(X)\big) < \operatorname{rank} \mathfrak{g}.$$

So we are looking at the situation of a simple Lie algebra $\mathfrak{g}$ (with char K good) and $X \in \mathfrak{g}_1 \subset \mathfrak{g}$ where X is regular nilpotent in $\mathfrak{g}_1$ an integrally closed subalgebra of $\mathfrak{g}$.

PROPOSITION 5.4. *Let $\mathfrak{g}$ be a simple Lie algebra with char K good for $\mathfrak{g}$. Suppose $X \in \mathfrak{g}$ is nilpotent such that X is regular or semiregular in $\mathfrak{g}_1$, an integrally closed subalgebra of $\mathfrak{g}$. Suppose further that $\operatorname{rank} \mathfrak{g}_1 < \operatorname{rank} \mathfrak{g}$. Then*

$$\mathfrak{z}\big(\mathfrak{z}_{\mathfrak{g}}(X)\big) \subseteq \mathfrak{z}_{\mathfrak{g}_1}(X)$$

Therefore $\dim \mathfrak{z}(\mathfrak{z}_{\mathfrak{g}}(X)) < \operatorname{rank} \mathfrak{g}$.

Proof. We have to show that if $Y \in \mathfrak{g} - \mathfrak{g}_1$ centralizes X, then there is a $W \in \mathfrak{z}_{\mathfrak{g}}(X)$ such that $[Y, W] \neq 0$. Let Ψ be the root system for $\mathfrak{g}_1$. Since $\mathfrak{z}_{\mathfrak{g}_1}(X)$ is nilpotent, we can choose orderings for Ψ and Ψ_1 such that

$$\mathfrak{z}_{\mathfrak{g}_1}(X) \subseteq \mathfrak{g}_1^+ = \operatorname{span}\{e_\gamma \mid \gamma \in \Psi_1^+\}$$

and $\Psi_1^+ \subset \Psi^+$. We use Δ to denote the base of Ψ in this ordering, and we write

$$Y = \sum_{\beta < 0} a_\beta e_\beta + \sum_{\alpha \in \Delta} b_\alpha H_\alpha + \sum_{\beta > 0} a_\beta e_\beta$$

with $a_\beta, b_\alpha \in K$ and we label the three pieces of Y as Y^-, Y^0, and Y^+.

If there is a root $\delta \in \Psi - \Psi_1$ such that $a_\delta \neq 0$, then there is an $H \in \mathfrak{h}$ (the Cartan subalgebra for $\mathfrak{g}$ spanned by the H_α) such that $[H, e_\delta] = e_\delta$ and $[H, \mathfrak{g}_1] = 0$. Then H will commute with X but not with Y.

That leaves us with the case that both Y^- and $Y^+ \in \mathfrak{g}_1$. As the Killing form is nondegenerate, we can write

$$Y^0 = H_1 + H_2$$

where $H_t \in \mathfrak{h}$, $H_1 \in \mathfrak{g}_1$ and $[H_2, \mathfrak{g}_1] = 0$. But then $Y^- + H_1 + Y^+ \in \mathfrak{z}_{\mathfrak{g}_1}(X) \subseteq \mathfrak{g}_1^+$, so $Y^- = H_1 = 0$. So what is really the case is that

$$Y = Y^+ + H_2$$

where $Y^+ \in \mathfrak{g}_1^+$ and H_2 is orthogonal to $\mathfrak{g}_1$. Because $H_2 \neq 0$ (otherwise Y belongs to $\mathfrak{g}_1$), there is a positive root $\delta \in \Psi - \Psi_1$ such that δ is not orthogonal to H_2. We take δ to have maximal height; that is
(i) $[H_2, e_\delta] = \lambda e_\delta \neq 0$ and
(ii) If ht $\nu >$ ht δ, then $[H_2, e_\nu] = 0$.
We now claim that e_δ is the W that we want. First we must show that e_δ centralizes X:

$$[H_2, [e_\delta, X]] = 0$$

by maximality of δ and the fact that X is supported by positive roots. But

$$[H_2, [e_\delta, X]] = -[e_\delta, [X, H_2]] - [X, [H_2, e_\delta]] = [X, \lambda e_\delta]$$

because H_2 centralizes X. Therefore $[X, \lambda e_\delta] = 0$ and so $e_\delta \in \mathfrak{z}_\mathfrak{g}(X)$.

Next we show $[e_\delta, Y] \neq 0$. $[e_\delta, Y] = [e_\delta, H_2 + Y^+] = -\lambda e_\delta +$ higher root vectors. Hence $[e_\delta, Y] \neq 0$ and so $Y \notin \mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X))$.

Finally, if X is regular in $\mathfrak{g}_1$, then

$$\dim \mathfrak{z}\big(\mathfrak{z}_{\mathfrak{g}_1}(X)\big) = \text{rank } \mathfrak{g}_1$$

and if X is semiregular in $\mathfrak{g}_1$, then

$$\dim \mathfrak{z}\big(\mathfrak{z}_{\mathfrak{g}_1}(X)\big) < \text{rank } \mathfrak{g}_1$$

In any event, then

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) < \text{rank } \mathfrak{g}. \qquad \square$$

The case where $\mathfrak{g}_1$ and $\mathfrak{g}$ have the same rank is much harder. In fact, it does not seem to be true that $\mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X)) \subseteq \mathfrak{g}_1$ in that case. However we are still hopeful that it will turn out that $\dim \mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X)) \leq \text{rank } \mathfrak{g}$. What we did was to compute just enough of $\mathfrak{z}_\mathfrak{g}(X)$ to show that it was nonabelian. Before we get to that, there is still one last thing that can be done.

PROPOSITION 5.5. *Let* $\mathfrak{g}$ *be a simple Lie algebra with* char K *good for* $\mathfrak{g}$. *Suppose* $X \in \mathfrak{g}$ *is irregular and nilpotent such that* X *is regular in a maximal integrally closed semisimple subalgebra* $\mathfrak{g}'$ *which is not simple. Then* $\mathfrak{z}_{\mathfrak{g}}(X)$ *is nonabelian.*

Proof. Let Ψ be a root system for $\mathfrak{g}$ with base Δ. Then a base Δ' for Ψ', the root system for $\mathfrak{g}'$ is $\{-\mu\} \cup (\Delta - \{\beta\})$ for some simple root β (μ is the highest root). As Ψ' is reducible, there are at least two simple roots α_1 and α_2 that are connected to β in the extended diagram (the case where one of these is $-\mu$ is permitted). So ignoring root lengths and possibly other roots connected to β, we have

$$\underset{\alpha_1 \quad \beta \quad \alpha_2}{\bigcirc - \bigcirc - \bigcirc}$$

It does no harm to assume that X is standard regular in $\mathfrak{g}'$, that is,

$$X = e_{-\mu} + \sum_{\alpha \in \Delta - \{\beta\}} e_\alpha.$$

As Δ' is reducible, $\Delta' = \Delta'_1 \overset{.}{\cup} \Delta'_2$ with $\alpha_i \in \Delta'_i$. Write $X = X_1 + X_2$ where $X_i = \sum_{\alpha \in \Delta'_i} e_\alpha$.

Now let $Y = e_{-\alpha_1 - \beta} + a e_{-\alpha_2 - \beta}$ where $a = -(N_{\alpha_1, -\alpha_1 - \beta} / N_{\alpha_2, -\alpha_2 - \beta}) \neq 0$. Then $[X, Y] = 0$ and clearly $[X, X_i] = 0$ $(i = 1, 2)$.

But $[X_1, Y] = N_{\alpha_1, -\alpha_1 - \beta} e_{-\beta} \neq 0$.

So $\mathfrak{z}_{\mathfrak{g}}(X)$ is nonabelian.   □

What remains is the following case: $\mathfrak{g}$ is a simple Lie algebra (char K good), $\mathfrak{g}_1 \subset \mathfrak{g}$ is a maximal integrally closed subalgebra of $\mathfrak{g}$ with $\mathfrak{g}_1$ simple; X is regular nilpotent in $\mathfrak{g}_1$ such that X is not conjugate to an element in a "smaller" subalgebra (i.e. one that is integrally closed but not maximal). These are the possibilities: (cf. Table 1)

(a) $\mathfrak{g} = G_2$
        $X = e_{-\mu} + e_\beta$ (type A_2)
(b) $\mathfrak{g} = F_4$
        $X = e_{-\mu} + e_\alpha + e_\beta + e_\gamma$ (type B_4)
(c) $\mathfrak{g} = E_6$
Any irreducible integrally closed subsystem which is maximal is E_6, so nothing happens here.
(d) $\mathfrak{g} = E_7$
        $X = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{-\mu}$ (type A_7)
We will show that X is conjugate to an element semiregular in a system of type E_6, so nothing happens here either.

(e) $\mathfrak{g} = E_8$

   (i) $X = e_{-\mu} + e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7}$ (type A_8)

   (ii) $X = e_{-\mu} + e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_8}$ (type D_8).

First we tackle $\mathfrak{g} = G_2$. $X = e_{-\mu} + e_{\beta}$. Dynkin has shown that X is subregular, so $\dim \mathfrak{z}_\mathfrak{g}(X) = 4$. Now X is conjugate to $Y = e_{-\mu} + e_{\alpha}$ to which we could apply Proposition 5.5, but that would not help us with Theorem B. The following result can be checked by computation.

PROPOSITION 5.6. *Let* $\mathfrak{g} = G_2$ *with* char $K \neq 2$ *or* 3, *and let* $X = e_{-\mu} + e_{\beta}$. *Then a basis for* $\mathfrak{z}_\mathfrak{g}(X)$ *is* $\{X, e_{-\alpha}, e_{-2\alpha-\beta}, e_{-3\alpha-\beta}\}$ *and a basis for* $\mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X))$ *is* $\{X, e_{-3\alpha-\beta}\}$.

In G_2, there are no irregular semiregular elements. The only other integrally closed subsystem of rank 2 is $A_1 + A_1$ whose regular element is the Y above. By Dynkin's tables X and Y are conjugate, so $\dim \mathfrak{z}(\mathfrak{z}_\mathfrak{g}(Y)) = 2$. Note that for X and Y, the double centralizer has dimension 2 which equals the rank.

PROPOSITION 5.7. *Let* $G = G_2$ *with* char $K \neq 2$ *or* 3. *If* $x \in G$, *then*

$$\dim Z\big(Z_G(x)^0\big) \leq \operatorname{rank} G = 2$$

Equality can occur only when
   1. *x is regular, or*
   2. *x is subregular, but not semisimple.*

Proof. If x is subregular semisimple, then $Z_G(x)^0$ is reductive of rank 2 and dimension 4, hence $\dim Z(Z_G(x)^0) = 1$. If $x = su$ is the Jordan decomposition of a subregular element x which is neither unipotent nor semisimple, then s and u belong to $Z(Z_G(x)^0)$ and so $\dim Z(Z_G(x)^0) = 2$. Proposition 5.6 handles subregular unipotent elements and Proposition 5.4 handles everything else. $\qquad\qquad\square$

This finishes the proof of Theorem B and proves Theorem A for G_2. We now move to $F_4 = \mathfrak{g}$. The element X regular in a system of type B_4 is subregular in F_4. Thus $\dim \mathfrak{z}_\mathfrak{g}(X) = 6$.

PROPOSITION 5.8. *Let* $\mathfrak{g} = F_4$ *with* char $K \neq 2$ *or* 3 *and let*

$$X = e_{-\mu} + e_{\alpha} + e_{\beta} + e_{\gamma}.$$

Then the following six elements form a basis for $\mathfrak{z}_\mathfrak{g}(X)$:

$$X, e_{-\delta}, e_{-\beta-2\gamma-2\delta}, e_{\alpha+\beta+\gamma} + ae_{\beta+2\gamma} + be_{-\mu},$$

$$e_{\alpha+2\beta+2\gamma} + ce_{-\alpha-2\beta-2\gamma-2\delta}, \quad and \quad e_{-\alpha-\beta-\gamma-\delta} + de_{-\beta-2\gamma-\delta},$$

for appropriate a, b, c, d ∈ K. Moreover,

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) \le 3 < \operatorname{rank} \mathfrak{g}.$$

The proof is computational. To find the exact values of a, b, c, and d (and then the exact dimension of $\mathfrak{z}(\mathfrak{z}_\mathfrak{g}(X))$), one must compute a number of structure constants. In any event, $a = \pm 2$ and b, c, and d are ± 1. More detail can be found in [8]. This proves Theorem A for F_4.

As we have said, E_6 has no irreducible maximal integrally closed subsystems, so the proof of Theorem A is complete for E_6 as well. To handle E_7, we have to describe the Dynkin diagram of a nilpotent element in a simple Lie algebra $\mathfrak{g}$.

If $\mathfrak{g}$ is simple and $X \in \mathfrak{g}$ is nilpotent, then there is a $\{X, Y, H\} \subseteq \mathfrak{g}$ with X and Y nilpotent and H semisimple so that $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. By taking the Dynkin diagram for $\mathfrak{g}$ and putting the number $\alpha_i(H)$ at the α_i node, we get the Dynkin diagram for X. By applying the Weyl group W to the base, we can always arrange matters so that the numbers $\alpha_i(H)$ are 0, 1, or 2. A nilpotent element is regular if and only if it has a 2 at every node in its diagram.

In our case, $\mathfrak{g} = E_7$ (char $K \ne 2$ or 3), and X is regular in a subsystem of type A_7, with base α_1, α_2, α_3, α_4, α_5, α_6, $-\mu$. Thus $\alpha_1(H) = \alpha_2(H) = \cdots = -\mu(H) = 2$. Since μ has an α_7 term, we can see that $\alpha_7(H) = -16$. If we write the diagram of X with just the numbers $\alpha_i(H)$, we have

$$2\ 2\ 2\ 2\ 2\ 2$$
$$-16$$

If w_i is reflection in α_i, then by applying w_7, we get

$$2\ 2\ 2 - 14\ 2\ 2$$
$$16$$

Continuing, we eventually get

$$0\ 2\ 0\ 2\ 0\ 2$$
$$0$$

Consulting the Dynkin tables (in [4] or [6]), we see that this is the diagram for $E_6(a_1)$. That is, X is conjugate to an element which is semiregular in a subsystem of type E_6. So by Propositions 4.1 and 5.4, we see that

$$\dim \mathfrak{z}\big(\mathfrak{z}_\mathfrak{g}(X)\big) \le 3.$$

Thus we have proven:

PROPOSITION 5.9. *Let* $\mathfrak{g} = E_6$ *or* E_7 *with char* $K \ne 2$ *or* 3. *If* $X \in \mathfrak{g}$ *is nilpotent and irregular, then* $\mathfrak{z}_\mathfrak{g}(X)$ *is nonabelian.*

That finishes the proof of Theorem A for E_6 and E_7, leaving us only with $\mathfrak{g} = E_8$. Unfortunately, the regular elements in the A_8 and D_8 subsystems are not conjugate to anything in a smaller subsystem. Hence, what we do is compute a large enough part of the centralizers to show that they are nonabelian. This is presented in [8] with all the details.

To simplify notation, we will use the 8-tuple $(n_1 n_2 n_3 n_4 n_5 n_6 n_7 n_8)$ to denote the root $\beta = \sum_{i=1}^{8} n_i \alpha_i$ when $n_i \geq 0$. For $-\beta$, we write $-(n_1 n_2 n_3 n_4 n_5 n_6 n_7 n_8)$. Since n_i is at most 6, this is unambiguous. The proofs of the next two results are simply computations.

First we handle the A_8 subsystem. So

$$X = e_{-\mu} + e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7}.$$

Dynkin's tables tell us that $\dim \mathfrak{z}_\mathfrak{g}(X) = 24$.

PROPOSITION 5.10. *Let* $\mathfrak{g} = E_8$ *with* char $K \neq 2, 3,$ *or* 5 *and let* X *be as above. Let*

$$Y = e_{-(12345423)} + e_{(11111100)} - e_{(01111110)}$$

and

$$Z = e_{(12345422)} - e_{-(11111101)} + 2e_{-(01112101)} - e_{-(01111111)}$$
$$- 3e_{-(00122101)} + e_{-(00112111)} - e_{-(00012211)}.$$

Then Y *and* Z *both centralize* X *but not each other and so* $\mathfrak{z}_\mathfrak{g}(X)$ *is nonabelian.*

Lastly, we have the D_8 case. So

$$X = e_{-\mu} + e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_8}$$

From Dynkin's tables, we see that $\dim \mathfrak{z}_\mathfrak{g}(X) = 16$ ([4] or [6]).

PROPOSITION 5.11. *Let* $\mathfrak{g} = E_8$ *with* char $K \neq 2, 3,$ *or* 5 *and let* X *be as above. Let*

$$Y = e_{-(12345422)} - e_{(11111101)} + e_{(01112101)} - e_{(00122101)}$$
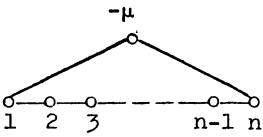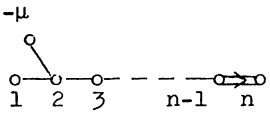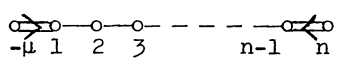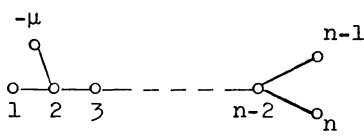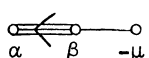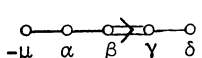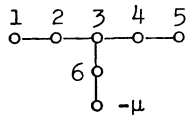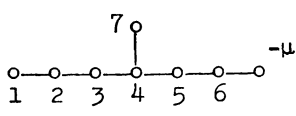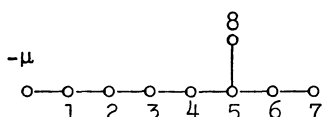
and

$$Z = e_{-(12344322)} - e_{(11122101)} + e_{(01222101)}.$$

Then Y *and* Z *both centralize* X, *but not each other. Therefore,* $\mathfrak{z}_\mathfrak{g}(X)$ *is nonabelian.*

This finishes the proof of Theorem A.

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TABLE 1
Extended Dynkin diagrams for the irreducible root systems

$A_n (n \geq 1)$

$$\mu = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

$B_n (n \geq 2)$

$$\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-1} + 2\alpha_n$$

$C_n (n \geq 3)$

$$\mu = 2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$$

$D_n (n \geq 4)$

$$\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$$

G_2

$$\mu = 3\alpha + 2\beta$$

F_4

$$\mu = 2\alpha + 3\beta + 4\gamma + 2\delta$$

E_6

$$\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$$

E_7

$$\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7$$

E_8

$$\mu = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

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