

# Pacific Journal of Mathematics

**INTERSECTIONS OF  $M$ -IDEALS AND  $G$ -SPACES**

ÅSVALD LIMA, G. H. OLSEN AND U. UTTERSUD

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A closed subspace  $N$  of a Banach space  $V$  is called an  $L$ -summand if there is a closed subspace  $N'$  of  $V$  such that  $V$  is the  $1_1$ -direct sum of  $N$  and  $N'$ . A closed subspace  $N$  of  $V$  is called an  $M$ -ideal if its annihilator  $N^\perp$  in  $V^*$  is an  $L$ -summand. Among the predual  $L_1$ -spaces the  $G$ -spaces are characterized by the property that every point in the  $w^*$ -closure of the extreme points of the dual unit ball is a multiple of an extreme point. In this note we prove that if  $V$  is a separable predual  $L_1$ -space such that the intersection of any family of  $M$ -ideals is an  $M$ -ideal, then  $V$  is a  $G$ -space.

The notions of  $L$ -summands and  $M$ -ideals were introduced by Alfsen and Effros [1] who showed that they play a similar role for Banach spaces as ideals do for rings. The intersection of a finite family of  $M$ -ideals in a Banach space is an  $M$ -ideal, but as shown by Bunce [2] and Perdrizet [5] the intersection of an arbitrary family of  $M$ -ideals in a Banach space need not be an  $M$ -ideal. However, Gleit [3] has shown that if  $V$  is a separable simplex space, then  $V$  is a  $G$ -space if and only if the intersection of an arbitrary family of  $M$ -ideals is an  $M$ -ideal. Later on, Uttersrud [7] proved that in  $G$ -spaces intersections of arbitrary families of  $M$ -ideals are  $M$ -ideals. Then N. Roy [6] gave a partial converse when she proved that if in a separable predual  $V$  of  $L_1$  the intersection of an arbitrary family of  $M$ -ideals is an  $M$ -ideal then  $V$  is a  $G$ -space. Here we present a short proof of this result.

**THEOREM.** *Let  $V$  be a separable predual  $L_1$ -space. Then  $V$  is a  $G$ -space if and only if the intersection of any family of  $M$ -ideals in  $V$  is an  $M$ -ideal.*

*Proof.* As already mentioned the only if part is proved in [7]. For the if part we will show that

$$\overline{\partial_e V_1^*} \subseteq [0, 1] \partial_e V_1^*$$

where  $\partial_e V_1^*$  denotes the set of extreme points in the unit ball  $V_1^*$  of  $V^*$ . It then follows from [4] that  $V$  is a  $G$ -space. To this end let  $\{x_n^*\}_{n=1}^\infty$  be a convergent sequence of mutually disjoint extreme points in  $V_1^*$ , say  $x_0^* = w^*\text{-lim } x_n^*$ . For each  $n$ , let

$$N_n = \text{norm-closure lin}\{x_0^*, x_n^*, x_{n+1}^*, \dots\}.$$

Let  $c$  denote the space of convergent sequences and define a linear operator  $T: V \rightarrow c$  by

$$Tx = (x_n^*(x))_{n=1}^{\infty}.$$

We identify  $c$  with the space of continuous functions on the one point compactification  $\mathbf{N} \cup \{\infty\}$  of the natural numbers  $\mathbf{N}$  and we let  $e_n^*$  be the point mass in  $n$ ,  $e_0^*$  the point mass in  $\infty$ . Then

$$T^*e_n^* = x_n^*, \quad n = 1, 2, \dots$$

And consequently

$$T^*e_0^* = x_0^*.$$

Since  $(x_n^*)_{n=1}^{\infty}$  is equivalent with the usual basis of  $l_1$ , we observe that for each  $n$

$$T^*(\text{norm-closure lin}\{e_0^*, e_n^*, e_{n+1}^*, \dots\}) = N_n.$$

Since, by a well-known category argument, the range of a dual map is norm closed if and only if it is  $w^*$ -closed, it follows that  $N_n$  is  $w^*$ -closed for each  $n$ . Now the dual statement of our assumption gives that the  $w^*$ -closure of arbitrary sums of  $w^*$ -closed  $L$ -summands is an  $L$ -summand, so since an extreme point in the unit ball of an  $L_1$ -space spans an  $L$ -summand we get that  $N_n$  is a  $w^*$ -closed  $L$ -summand. Therefore

$$\bigcap_{n=1}^{\infty} N_n = \text{lin}\{x_0^*\}$$

is an  $L$ -summand. Hence  $x_0^* = 0$  or  $x_0^*/\|x_0^*\|$  is an extreme point, and the proof is complete.

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