INTERSECTIONS OF $M$-IDEALS AND $G$-SPACES

Ásvald Lima, G. H. Olsen and U. Uttersrud
A closed subspace $N$ of a Banach space $V$ is called an $L$-summand if there is a closed subspace $N'$ of $V$ such that $V$ is the $1_1$-direct sum of $N$ and $N'$. A closed subspace $N$ of $V$ is called an $M$-ideal if its annihilator $N^\perp$ in $V^*$ is an $L$-summand. Among the predual $L_1$-spaces the $G$-spaces are characterized by the property that every point in the $w^*$-closure of the extreme points of the dual unit ball is a multiple of an extreme point. In this note we prove that if $V$ is a separable predual $L_1$-space such that the intersection of any family of $M$-ideals is an $M$-ideal, then $V$ is a $G$-space.

The notions of $L$-summands and $M$-ideals were introduced by Alfsen and Effros [1] who showed that they play a similar role for Banach spaces as ideals do for rings. The intersection of a finite family of $M$-ideals in a Banach space is an $M$-ideal, but as shown by Bunce [2] and Perdrizet [5] the intersection of an arbitrary family of $M$-ideals in a Banach space need not be an $M$-ideal. However, Gleit [3] has shown that if $V$ is a separable simplex space, then $V$ is a $G$-space if and only if the intersection of an arbitrary family of $M$-ideals is an $M$-ideal. Later on, Uttersrud [7] proved that in $G$-spaces intersections of arbitrary families of $M$-ideals are $M$-ideals. Then N. Roy [6] gave a partial converse when she proved that if in a separable predual $V$ of $L_1$ the intersection of an arbitrary family of $M$-ideals is an $M$-ideal then $V$ is a $G$-space. Here we present a short proof of this result.

**Theorem.** Let $V$ be a separable predual $L_1$-space. Then $V$ is a $G$-space if and only if the intersection of any family of $M$-ideals in $V$ is an $M$-ideal.

**Proof.** As already mentioned the only if part is proved in [7]. For the if part we will show that

$$\partial_e V_1^* \subseteq [0, 1] \partial_e V_1^*$$

where $\partial_e V_1^*$ denotes the set of extreme points in the unit ball $V_1^*$ of $V^*$. It then follows from [4] that $V$ is a $G$-space. To this end let $\{x_n^*\}_{n=1}^\infty$ be a convergent sequence of mutually disjoint extreme points in $V_1^*$, say $x_0^* = w^*-\lim x_n^*$. For each $n$, let

$$N_n = \text{norm-closure } \text{lin}\{x_0^*, x_n^*, x_{n+1}^*, \ldots\}.$$
Let $c$ denote the space of convergent sequences and define a linear operator $T: V \to c$ by

$$Tx = \left(x_n^*(x)\right)_{n=1}^\infty.$$ 

We identify $c$ with the space of continuous functions on the one point compactification $\mathbb{N} \cup \{\infty\}$ of the natural numbers $\mathbb{N}$ and we let $e_n^*$ be the point mass in $n$, $e_0^*$ the point mass in $\infty$. Then

$$T^*e_n^* = x_n^*, \quad n = 1, 2, \ldots$$

And consequently

$$T^*e_0^* = x_0^*.$$ 

Since $(x_n^*)_{n=1}^\infty$ is equivalent with the usual basis of $l_1$ we observe that for each $n$

$$T^*(\text{norm-closure lin}\{e_0^*, e_n^*, e_{n+1}^*, \ldots\}) = N_n.$$ 

Since, by a well-known category argument, the range of a dual map is norm closed if and only if it is $w^*$-closed, it follows that $N_n$ is $w^*$-closed for each $n$. Now the dual statement of our assumption gives that the $w^*$-closure of arbitrary sums of $w^*$-closed $L$-sumands is an $L$-summand, so since an extreme point in the unit ball of an $L_1$-space spans an $L$-summand we get that $N_n$ is a $w^*$-closed $L$-summand. Therefore

$$\bigcap_{n=1}^\infty N_n = \text{lin}\{x_0^*\}$$

is an $L$-summand. Hence $x_0^* = 0$ or $x_0^*/\|x_0^*\|$ is an extreme point, and the proof is complete.

**References**

Received February 10, 1981.

AGRICULTURAL UNIVERSITY OF NORWAY
1432 AAS-NLH, NORWAY

AND

TELEMARK DH-SKOLE
3800 BØ I TELEMARK, NORWAY
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