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INTERSECTIONS OF M -IDEALS AND G -SPACES

ÅSVALD LIMA, G. H. OLSEN AND U. UTTERSUD

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A closed subspace N of a Banach space V is called an L -summand if there is a closed subspace N' of V such that V is the l_1 -direct sum of N and N' . A closed subspace N of V is called an M -ideal if its annihilator N^\perp in V^* is an L -summand. Among the predual L_1 -spaces the G -spaces are characterized by the property that every point in the w^* -closure of the extreme points of the dual unit ball is a multiple of an extreme point. In this note we prove that if V is a separable predual L_1 -space such that the intersection of any family of M -ideals is an M -ideal, then V is a G -space.

The notions of L -summands and M -ideals were introduced by Alfsen and Effros [1] who showed that they play a similar role for Banach spaces as ideals do for rings. The intersection of a finite family of M -ideals in a Banach space is an M -ideal, but as shown by Bunce [2] and Perdrizet [5] the intersection of an arbitrary family of M -ideals in a Banach space need not be an M -ideal. However, Gleit [3] has shown that if V is a separable simplex space, then V is a G -space if and only if the intersection of an arbitrary family of M -ideals is an M -ideal. Later on, Uttersrud [7] proved that in G -spaces intersections of arbitrary families of M -ideals are M -ideals. Then N. Roy [6] gave a partial converse when she proved that if in a separable predual V of L_1 the intersection of an arbitrary family of M -ideals is an M -ideal then V is a G -space. Here we present a short proof of this result.

THEOREM. *Let V be a separable predual L_1 -space. Then V is a G -space if and only if the intersection of any family of M -ideals in V is an M -ideal.*

Proof. As already mentioned the only if part is proved in [7]. For the if part we will show that

$$\overline{\partial_e V_1^*} \subseteq [0, 1] \partial_e V_1^*$$

where $\partial_e V_1^*$ denotes the set of extreme points in the unit ball V_1^* of V^* . It then follows from [4] that V is a G -space. To this end let $\{x_n^*\}_{n=1}^\infty$ be a convergent sequence of mutually disjoint extreme points in V_1^* , say $x_0^* = w^*\text{-lim } x_n^*$. For each n , let

$$N_n = \text{norm-closure lin}\{x_0^*, x_n^*, x_{n+1}^*, \dots\}.$$

Let c denote the space of convergent sequences and define a linear operator $T: V \rightarrow c$ by

$$Tx = (x_n^*(x))_{n=1}^{\infty}.$$

We identify c with the space of continuous functions on the one point compactification $\mathbf{N} \cup \{\infty\}$ of the natural numbers \mathbf{N} and we let e_n^* be the point mass in n , e_0^* the point mass in ∞ . Then

$$T^*e_n^* = x_n^*, \quad n = 1, 2, \dots$$

And consequently

$$T^*e_0^* = x_0^*.$$

Since $(x_n^*)_{n=1}^{\infty}$ is equivalent with the usual basis of l_1 we observe that for each n

$$T^*(\text{norm-closure } \text{lin}\{e_0^*, e_n^*, e_{n+1}^*, \dots\}) = N_n.$$

Since, by a well-known category argument, the range of a dual map is norm closed if and only if it is w^* -closed, it follows that N_n is w^* -closed for each n . Now the dual statement of our assumption gives that the w^* -closure of arbitrary sums of w^* -closed L -summands is an L -summand, so since an extreme point in the unit ball of an L_1 -space spans an L -summand we get that N_n is a w^* -closed L -summand. Therefore

$$\bigcap_{n=1}^{\infty} N_n = \text{lin}\{x_0^*\}$$

is an L -summand. Hence $x_0^* = 0$ or $x_0^*/\|x_0^*\|$ is an extreme point, and the proof is complete.

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