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A closed subspace N of a Banach space V is called an L-summand if there is a closed subspace N' of V such that V is the 1_1-direct sum of N and N'. A closed subspace N of V is called an M-ideal if its annihilator $N^\perp$ in V^* is an L-summand. Among the predual L_1-spaces the G-spaces are characterized by the property that every point in the w^*-closure of the extreme points of the dual unit ball is a multiple of an extreme point. In this note we prove that if V is a separable predual L_1-space such that the intersection of any family of M-ideals is an M-ideal, then V is a G-space.

The notions of L-summands and M-ideals were introduced by Alfsen and Effros [1] who showed that they play a similar role for Banach spaces as ideals do for rings. The intersection of a finite family of M-ideals in a Banach space is an M-ideal, but as shown by Bunce [2] and Perdrizet [5] the intersection of an arbitrary family of M-ideals in a Banach space need not be an M-ideal. However, Gleit [3] has shown that if V is a separable simplex space, then V is a G-space if and only if the intersection of an arbitrary family of M-ideals is an M-ideal. Later on, Uttersrud [7] proved that in G-spaces intersections of arbitrary families of M-ideals are M-ideals. Then N. Roy [6] gave a partial converse when she proved that if in a separable predual V of L_1 the intersection of an arbitrary family of M-ideals is an M-ideal then V is a G-space. Here we present a short proof of this result.

THEOREM. *Let V be a separable predual L_1-space. Then V is a G-space if and only if the intersection of any family of M-ideals in V is an M-ideal.*

Proof. As already mentioned the only if part is proved in [7]. For the if part we will show that

$$\overline{\partial_e V_1^*} \subseteq [0,1]\,\partial_e V_1^*$$

where $\partial_e V_1^*$ denotes the set of extreme points in the unit ball V_1^* of V^*. It then follows from [4] that V is a G-space. To this end let $\{x_n^*\}_{n=1}^\infty$ be a convergent sequence of mutually disjoint extreme points in V_1^*, say $x_0^* = w^*\text{-lim } x_n^*$. For each n, let

$$N_n = \text{norm-closure lin}\{x_0^*, x_n^*, x_{n+1}^*, \ldots\}.$$

Let c denote the space of convergent sequences and define a linear operator $T\colon V \to c$ by

$$Tx = \left(x_n^*(x)\right)_{n=1}^{\infty}.$$

We identify c with the space of continuous functions on the one point compactification $\mathbf{N} \cup \{\infty\}$ of the natural numbers $\mathbf{N}$ and we let e_n^* be the point mass in n, e_0^* the point mass in ∞. Then

$$T^* e_n^* = x_n^*, \qquad n = 1, 2, \ldots.$$

And consequently

$$T^* e_0^* = x_0^*.$$

Since $(x_n^*)_{n=1}^{\infty}$ is equivalent with the usual basis of 1_1 we observe that for each n

$$T^*\left(\text{norm-closure } \operatorname{lin}\{e_0^*, e_n^*, e_{n+1}^*, \cdots\}\right) = N_n.$$

Since, by a well-known category argument, the range of a dual map is norm closed if and only if it is w^*-closed, it follows that N_n is w^*-closed for each n. Now the dual statement of our assumption gives that the w^*-closure of arbitrary sums of w^*-closed L-sumands is an L-summand, so since an extreme point in the unit ball of an L_1-space spans an L-summand we get that N_n is a w^*-closed L-summand. Therefore

$$\bigcap_{n=1}^{\infty} N_n = \operatorname{lin}\{x_0^*\}$$

is an L-summand. Hence $x_0^* = 0$ or $x_0^*/\|x_0^*\|$ is an extreme point, and the proof is complete.

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