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ON THE ITERATES OF DERIVATIONS OF PRIME RINGS

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In this paper we study properties of associative derivations whose iterates are related in rather special ways to the original derivation, or to the iterates of another derivation. An associative derivation $d\colon R \to R$ is an additive (or linear when appropriate) mapping on a ring R satisfying $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. A derivation $d\colon R \to R$ is called *inner* if $d(x) = (\mathrm{ad}\, a)(x)$ for some $a \in R$ where $(\mathrm{ad}\, a)(x) = [a, x] = ax - xa$. In particular we ask when can the iterate of an inner derivation be an inner derivation? When can the iterates of two derivations commute? More precisely, we characterize elements $a, b \in R$, R a prime ring, for which $(\mathrm{ad}\, a)^n(x) = (\mathrm{ad}\, b)(x)$ for all $x \in R$, and we characterize derivations $d\colon R \to R$, $\delta\colon R \to R$ for which $[d^n(x), \delta^n(y)] = 0$ for all $x, y \in R$, R prime. Applications are made to C^*-algebras.

1. Introduction. In **[15]** it was shown that if d_1 and d_2 are derivations of a prime ring not of characteristic 2 with $d_1 \circ d_2$ a derivation, then either $d_1 = 0$ or $d_2 = 0$. Consequently if $d^2(x) = 0$ for all x, d a derivation on such a ring, then $d = 0$. In **[6]** it was shown that if $(\mathrm{ad}\, a)^n(x) = 0$ for all x in a simple ring there exists a scalar λ such that $(a - \lambda e)^{[(n+1)/2]} = 0$. And in **[14]** it was shown that if $(\mathrm{ad}\, a)^3(x) = (\mathrm{ad}\, a)(x)$ for a self-adjoint a and all x in a von Neumann algebra then $(a - z)^2 = a - z$ for some central element z. In [7] it was shown that if $[d(x), d(y)] = 0$ for all $x, y \in R$ where R is a prime ring and $\mathrm{char}\, R \neq 2$, then $d = 0$ or R is a commutative integral domain. Our results show that if d and δ are as above, and R is prime of characteristic 0 then R is commutative, or $d^{3n-1} = 0$, or $\delta^{3n-1} = 0$. If $d = \delta = \mathrm{ad}\, b$ there exists λ in the extended centroid of R such that $a = b - \lambda$ satisfies $a^{[(2n+3)/3]} = 0$.

In §§2 and 3 we prove results in the full generality of prime rings. Crucial use is made of the notions of extended centroid and central closure of a prime ring and of a key result on tensor products of closed prime algebras. We summarize these constructions by quoting from **[11]**. Let R be a prime ring and let T be the totality of all right R-homomorphisms $f\colon U_R \to R_R$, where U ranges over the non-zero ideals of R. An equivalence relation $\sim$ is defined on T as follows: f (acting on U) $\sim g$ (acting on V) if $f = g$ on W where W is non-zero ideal contained in $U \cap V$. The set $Q = \{\hat{f}\}$ of all equivalence classes forms a ring under the operations induced by addition and composition of representatives of the equivalence classes. $R \subseteq Q$ via the map $a \to \hat{a}_l$ where a_l is left multiplication by a acting on R. The center C of Q is a field containing the centroid of R and is called the *extended centroid* of R. The C-algebra $A = RC + C$ is again a prime ring and is called the *central closure* of R. In general we define a prime algebra S over a field F to be a *closed prime algebra* over F if S is its own central closure, i.e., the extended center of S is just F itself. We list some examples of closed prime algebras which are important for the purposes of this paper:

(1) The central closure of a prime ring ([**12**], p. 503).

(2) $A \otimes_C F$, where A is a closed prime algebra over C and F is an extension field of C ([**2**], Theorem 3.5).

(3) Any 2-fold transitive algebra of linear transformations on a complex vector space ([**4**], Theorem 2.1.3 and [**11**], Theorem 12).

Finally, if A is an algebra over F we denote by A_l the algebra of left multiplications a_l of A determined by the elements of A and A_r, the algebra of right multiplications a_r determined by the elements of A. A key result on tensor products is

THEOREM 1. *If S is a closed prime algebra over F and S^0 is the opposite algebra of S then*

$$S \underset{F}{\otimes} S^0 \cong S_l S_r \text{ via the map } u \otimes v \to u_l v_r.$$

In §4 the results of §§2 and 3 are applied to C^*-algebras. Although C^*-algebras, in general, are not prime they have a complete set of (algebraically) irreducible representations and, in our case, a phenomenon which occurs in each of these representations can be translated to a corresponding result for the original algebra. For a full account of C^*-algebras we refer the reader to [**1**].

2. Iterates of an inner derivation.

LEMMA 1. *Let R be a prime ring of characteristic $> n > 1$, and suppose $(\operatorname{ad} a)^n = \operatorname{ad} b$ for some $a, b \in R$. Let C be the extended centroid of R, $A = RC + C$ the central closure of R, F the algebraic closure of C, and $S = A \otimes_C F$. Then a is algebraic over F of degree $\leq n$, and if $p(x) = (x - \lambda)^l g(x)$ is the minimum polynomial of a over F where $\lambda \in F$ is a root of multiplicity $l \geq 1$, then $b - \beta = c^n$, $\beta \in F$, and*

$$(1) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

holds in $S \otimes_F S^0$ with $c = a - \lambda$.

Proof. The condition $(\mathrm{ad}\, a)^n = \mathrm{ad}\, b$ clearly lifts to S, i.e., $(a_l - a_r)^n = b_l - b_r$ holds in the algebra $S_l S_r$. By Theorem 1, $S \otimes_F S^0 \cong S_l S_r$ via $u \otimes v \to u_l u_r$ since S is a closed prime algebra over F, and the condition further translates to

$$(2) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} \otimes a^k = b \otimes 1 - 1 \otimes b \text{ in } S \underset{F}{\otimes} S^0.$$

It is clear that $b \in \operatorname{span}\{1, a, \ldots, a^n\}$. Writing $b = \sum_{i=0}^{n} \beta_i a^i$, $\beta_i \in F$, and substituting this into (2) we see that

$$(3) \quad (a^n - b + \beta_0) \otimes 1 + \left(\beta_1 - n a^{n-1}\right) \otimes a + \cdots$$
$$+ \left(\beta_{n-1} + (-1)^{n-1} n a\right) \otimes a^{n-1} + \left(\beta_n + (-1)^n\right) \otimes a^n = 0.$$

If $1, a, \ldots, a^n$ are independent then, in particular, $\beta_{n-1} + (-1)^{n-1} n a = 0$, whence $a = \pm \beta_{n-1}/n \in F$ a contradiction. Therefore $\{1, a, \ldots, a^n\}$ is a dependent set and we have established that a is algebraic over F of degree $\leq n$. Hence we may set $p(x) = (x - \lambda)^l g(x)$ as indicated in the statement of the lemma. Setting $c = a - \lambda$ we then see that $m(x) = p(x + \lambda) = x^l q(x)$ is the minimum polynomial of c over F with $\deg m(x) = \deg p(x) \leq n$ and $q(0) \neq 0$. Since $\mathrm{ad}\, a = \mathrm{ad}\, c$ we may rewrite (2) as

$$(4) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = b \otimes 1 - 1 \otimes b.$$

We set $v = c^{l-1} q(c) \neq 0$, note that $cv = 0$, and multiply (4) on the right by $1 \otimes v$ to obtain

$$(5) \qquad c^n \otimes v = b \otimes v - 1 \otimes \beta v,$$

whence $(c^n - b + \beta) \otimes v = 0$. Thus $b = c^n + \beta$ and (4) becomes

$$(6) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

which completes the proof of the lemma. □

In the following Theorem, C, $A = RC + C$ and $S = A \otimes_C F$ have the same meaning as in Lemma 1.

THEOREM 2. *Let R be a prime ring of char. $> n > 1$, and suppose $(\mathrm{ad}\, a)^n = \mathrm{ad}\, b$ for some $a, b \in R$. If the minimum polynomial $p(x)$ of a over F (which necessarily exists in view of Lemma 1) contains a root $\lambda \in F$ of multiplicity $l > 1$, then $\lambda \in C$ and $(a - \lambda)^{[(n+1)/2]} = 0$.*

Proof. By Lemma 1 we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}c^{n-k}\otimes c^k = c^n\otimes 1 - 1\otimes c^n \tag{1}$$

holding in $S\otimes_F S^0$, where $c = a - \lambda$ and the minimum polynomial of c over F is $m(x) = x^l q(x)$, $l > 1$, $q(0) \neq 0$. We remark that $q(c)$, $cq(c),\cdots,c^{l-1}q(C)$ are F-independent. Indeed, one would have a dependency $a_{i_1}c^{i_1}q(c) + \cdots + a_{i_r}c^{i_r}q(c) = 0$, $\alpha_{i_j}\neq 0$, $i_1 < i_2 < \cdots < i_r < l$, with "length" $r > 1$ minimal. Multiplication of the dependency by c^{l-i_r} would result in a dependency of shorter "length", a contradiction.

Now we multiply (1) on the right by $1\otimes q(c)$ to obtain

$$\sum_{k=0}^{l-1}(-1)^k\binom{n}{k}c^{n-k}\otimes c^k q(c) = 0. \tag{7}$$

By the preceding remark $\{c^k q(c) \mid k = 1,\cdots,l-1\}$ is an independent set so in particular $c^{n-(l-1)} = 0$. Thus c is nilpotent, with $m(x) = x^l$ where $l \leq n-(l-1)$, i.e., $l\leq (n+1)/2$, and so $c^{[(n+1)/2]} = 0$. In terms of a this says that $(a-\lambda)^{[(n+1)/2]} = 0$ and $p(x) = (x-\lambda)^l$.

It remains to prove that $\lambda\in C$. Since $c = a\otimes 1 - 1\otimes\lambda \in A\otimes_C F$ (when written more precisely we see from $p(x) = (x-\lambda)^l$ that $0 = (a\otimes 1 - 1\otimes\lambda)^l = \sum_{k=0}^{l}(-1)^k\binom{l}{k}a^{l-k}\otimes\lambda^k$ holds in $A\otimes_C F$. It follows that a is algebraic over C of degree $\leq l$. Let $h(x)$ be the minimum polynomial of a over C. On the one hand we must have $\deg h(x) \leq \deg p(x) = l$, and so $h(x) = p(x)$. Therefore the coefficients of $p(x)$ lie in C and so from $p(x) = (x-\lambda)^l = x^l - l\lambda x^{l-1} + \cdots$ we have $l\lambda = \alpha \in C$, whence $\lambda = \alpha/l \in C$. □

COROLLARY 1. *Let R be a prime ring of char. $> n > 1$, and suppose* $(\mathrm{ad}\, a)^n = \mathrm{ad}\, b$ *for some* $a, b\in R$. *Then* $(a-\lambda)^{[(n+1)/2]} = 0$ *for some* λ *in the extended centroid C if either of the following conditions hold*:

(a) n is *even*;

(b) $b = 0$.

Proof. By Lemma 1 we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}c^{n-k}\otimes c^k = c^n\otimes 1 - 1\otimes c^n \tag{8}$$

where $c = a - \lambda$ has minimum polynomial $m(x) = x^l q(x)$, $q(0)\neq 0$. We show that if either (a) or (b) holds then c is nilpotent whence the conclusion follows from Theorem 2, since then $m(x) = x^l$ and $l > 1$. If we are given (a) then (8) becomes

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}c^{n-k}\otimes c^k = -2\otimes c^n. \tag{9}$$

Multiplication of (9) by $c^{l-1}q(c) \otimes 1$ on the left yields $0 = -2c^{l-1}q(c) \otimes c^n$ whence $c^n = 0$.

If we are given (b) then (8) reads

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = 0. \tag{10}$$

Multiplication of (10) by $1 \otimes c^{l-1}q(c)$ yields $c^n \otimes c^{l-1}q(c) = 0$ so again we obtain $c^n = 0$. □

REMARK. Corollary 1(b) was first proved for simple rings by Herstein [**6**] and conjectured for prime rings by Kovacs [**10**] and Herstein and the proof announced in [**13**].

The conclusion of Corollary 1 is false for prime rings in general. As an example, let $n = 5$, $R = M_2(\mathbf{C})$, the ring of 2×2 matrices over $\mathbf{C}$, and $a = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+\sqrt{-3}}{2} \end{bmatrix}$. One may verify that $(\operatorname{ad} a)^5 = \operatorname{ad}(a^5)$, but there is no $\lambda \in \mathbf{C}$ such that $(a - \lambda)$ is nilpotent. A more complicated example of the same nature can be constructed as follows:

Let $n = 5$, $R_j = M_2(\mathbf{C})$ for $j = 1, 2, \cdots, R = \oplus_{j=1}^{\infty} R_j$, $\{\theta_j\}_{j=1}^{\infty}$ a collection of distinct real numbers such that $0 < \theta_j < \pi/6$, $z_j = e^{i\theta_j}$ where $i^2 = -1$, $a_j = \begin{pmatrix} z_j & 0 \\ 0 & z_j\frac{1+\sqrt{-3}}{2} \end{pmatrix}$, and $a = \oplus a_j$. Then $(\operatorname{ad} a)^5 = \operatorname{ad}(a^5)$ and a has infinitely many distinct eigenvalues.

COROLLARY 2. *Let R be a 2-fold transitive algebra of linear transformations on a complex vector space H and suppose* $(\operatorname{ad} a)^n = \operatorname{ad} b$, $n > 1$, *for some $a, b \in R$. Then a is algebraic, and if the minimum polynomial $p(x)$ of a contains a repeated root λ (in particular if either n is even or $b = 0$) then* $(a - \lambda)^{[(n+1)/2]} = 0$.

The following corollary shows that the example following Corollary 1 is typical.

COROLLARY 3. *Let R be a 2-fold transitive algebra of linear transformations on a complex vector space H, and suppose* $(\operatorname{ad} a)^n = \operatorname{ad} b$ *for some $a, b \in R$ with n odd, $n > 1$. If the minimum polynomial $p(x)$ of a has distinct roots $\lambda_1, \cdots, \lambda_k$, $k \leq n$, there exist idempotents $p_1, \cdots, p_k$ with $p_i p_j = 0$ for $i \neq j$, $\sum_{i=1}^{k} p_i = 1$ and $a = \sum_{i=1}^{k} \lambda_i p_i$. If $k > 2$ and $0, \lambda, \mu$ are distinct eigenvalues of a, then $\lambda^n - \mu^n = (\lambda - \mu)^n$. Conversely, if R is an algebra over a field F of characteristic 0, with idempotents $p_1, \cdots, p_k$ such* *that* $1 = \sum_{i=1}^{k} p_i$, $p_i p_j = 0$ *and* $\lambda_1, \cdots, \lambda_k$ *distinct elements of* F *such that* $\lambda_i^n - \lambda_j^n = (\lambda_i - \lambda_j)^n$, $1 \le i, j \le n$, *and* n *odd, then* $(\operatorname{ad} a)^n = \operatorname{ad}(a^n)$ *and* $a = \sum_{i=1}^{k} \lambda_i p_i$.

Proof. Standard linear algebra gives the existence of the p_i with the desired properties. Equation (1) of Lemma 1 becomes $\sum_{k=0}^{n}(-1)^k\binom{n}{k}c^{n-k}xc^k = [c^n, x]$ for all $x \in R$ where $c = a - \lambda$. Since n is odd we have

$$0 = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k}xc^k \quad \text{for all } x \in R.$$

Let $v \neq 0$ be such that $av = \mu v$. Then $cv = (a - \lambda)v = (\mu - \lambda)v \neq 0$. Hence

$$\begin{aligned} 0 &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k}xc^k v \\ &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k}(\mu - \lambda)^k xv \quad \text{for all } x \in R. \end{aligned}$$

By transitivity,

$$0 = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k}(\mu - \lambda)^k = (c - (\mu - \lambda))^n - c^n + (\mu - \lambda)^n.$$

In terms of a this says $0 = (a - \mu)^n - (a - \lambda)^n + (\mu - \lambda)^n$. Therefore, the minimum polynomial $p(x)$ must divide $(x - \mu)^n - (x - \lambda)^n + (\mu - \lambda)^n$. Since 0 is an eigenvalue of a we have $(-\mu)^n - (-\lambda)^n + (\mu - \lambda)^n = 0$. The other part of the corollary is a straightforward calculation. □

3. **Commuting iterates of derivations.**

THEOREM 3. *Let* R *be a prime ring and let* d *and* δ *be derivations on* R *such that* $[d^n(x), \delta^n(y)] = 0$ *for all* $x, y \in R$. *Then either* R *is commutative, or* $d^{3n-1} = 0$, *or* $\delta^{3n-1} = 0$. *Furthermore, if* $n = 1$, *and the characteristic of* $R \neq 2$, *then either* R *is commutative, or* $d = 0$, *or* $\delta = 0$.

Proof. Let W be the subring generated by $\{d^n(x) \mid x \in R\}$ and note that $d(W) \subseteq W$. By the Leibniz formula we have

(1) $$d^n(xy) = \sum_{k=0}^{n} \binom{n}{k} d^k(x)d^{n-k}(y) \in W \quad \text{for all } x, y \in R.$$

For $l = 1, 2, \cdots, n$ we substitute $d^{l-1}(x)$ for x and $d^{2n-l}(y)$ for y in (1) to obtain

$$\sum_{k=0}^{n-l} \binom{n}{k} d^{k+l-1}(x) d^{3n-k-l}(y) \in W, \tag{2}$$

$$l = 1, 2, \cdots, n \text{ for all } x, y \in R.$$

For $l = n$, (2) reads

$$d^{n-1}(x) d^{2n}(y) \in W, \tag{3}$$

and for $l = n - 1$, (2) reads

$$d^{n-2}(x) d^{2n+1}(y) + \binom{n}{1} d^{n-1}(x) d^{2n}(y) \in W. \tag{4}$$

Together (3) and (4) imply that $d^{n-2}(x)d^{2n+1}(y) \in W$. Continuing in this fashion by comparing successive decreasing values of l from $l = n$ to $l = 1$ we have that $xd^{3n-1}(y) \in W$ for $x, y \in R$. Therefore, $Rd^{3n-1}(y) \subseteq W$ for all $y \in R$. Similarly for all x, $R\delta^{3n-1}(x) \subseteq V$, the subring generated by $\{\delta^n(t) \mid t \in R\}$. Since $[W, V] = 0$ by assumption, the left ideals $Rd^{3n-1}(y)$ and $R\delta^{3n-1}(x)$ commute for all $x, y \in R$. Without loss of generality we may assume that $d^{3n-1}(y) \neq 0$ for some y and that $\delta^{3n-1}(x) \neq 0$ for some x. Since we have two commuting non-zero left ideals $Rd^{3n-1}(y)$ and $R\delta^{3n-1}(x)$ in the prime ring R, R must be commutative.

For $n = 1$, if R is not commutative, we may assume $d^2 = 0$. From $d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y)$ we have $d(x)d(y) = 0$. In particular, $0 = d(xy)d(x) = [d(x)y + xd(y)]d(x)$ whence $d(x)yd(x) = 0$ for all $x, y \in R$. Since R is prime it follows that $d(x) = 0$ for all $x \in R$. □

THEOREM 4. *Let R be a prime ring of characteristic $\geq 3n$ and let $d = \mathrm{ad}\, b$ be an inner derivation of R satisfying $[d^n(x), d^n(y)] = 0$ for all $x, y \in R$. Then there exists an element λ in the extended centroid of R such that $a = b - \lambda$ satisfies $a^{[(2n+3)/3]} = 0$.*

Proof. We can assume, by [7], that $n > 1$. The condition on d clearly extends to the central closure $A = RC + C$ of R. By Theorem 3, $d^{3n-1} = 0$ and hence by Corollary 1(b) there exists $\lambda \in C$ such that $a = b - \lambda \in A$ satisfies $a^{[3n/2]} = 0$. If l is the degree of nilpotency of a we have

$$l \leq \frac{3n}{2} \tag{5}$$

and, assuming the theorem to be false, we may also suppose that

$$l > \frac{2n+3}{3}, \tag{6}$$

in other words, $3l - 2n - 4 \geq 0$. We then set

$$p = q = \frac{3l - 2n - 4}{2} \quad \text{if } l \text{ is even} \tag{7}$$

and

$$p = \frac{3l - 2n - 5}{2}, q = \frac{3l - 2n - 3}{2} \quad \text{if } l \text{ is odd.} \tag{8}$$

In either case $p + q = 3l - 2n - 4$, which we wish to view in the form $2n + p + q = 3(l - 1) - 1$.

Expansion of $[d^n(x), d^n(y)] = 0$ by the Leibniz formulas, followed by replacement of x by ax, yields

$$0 = a^p[d^n(ax), d^n(y)]a^q = g - h$$

where

$$g = \sum_{j,k=0}^{n} (-1)^{n-j+k} \binom{n}{j}\binom{n}{k} a^{p+j+1}xa^{2n-j-k}ya^{q+k}$$

and

$$h = \sum_{j,k=0}^{n} (-1)^{n-j+k} \binom{n}{j}\binom{n}{k} a^{p+j}ya^{2n+1-j-k}xa^{q+k}.$$

Since $2n + 1 + p + q = 3(l - 1)$ the only possible surviving summand of g occurs when $j + p + 1 = q + k = l - 1$; for h it occurs when $j + p = q + k = l - 1$. To see that these terms actually occur it is necessary to show that the j and k thus determined are indeed within the range $0 \leq j, k \leq n$. This means verifying

$$0 \leq l - p - 2 \leq n, \quad 0 \leq l - p - 1 \leq n, \quad 0 \leq l - q - 1 \leq n. \tag{9}$$

We leave it to the reader to check that the various substitutions of (7) and (8) in (9) lead to the following inequalities

$$0 \leq 2n - l + i \leq 2n, \qquad i = 0, 1, 2, 3 \tag{10}$$

where $i = 0, 2$ when l is even and $i = 1, 3$ when l is odd. But the inequalities (10) follow readily from (5) and (6), so that we have established

$$g = (-1)^{n+p-q+1}\binom{n}{l-p-2}\binom{n}{l-q-1} a^{l-1}xa^{l-1}ya^{l-1} \tag{11}$$

and

$$h = (-1)^{n+p-q}\binom{n}{l-p-1}\binom{n}{l-q-1} a^{l-1}ya^{l-1}xa^{l-1}. \tag{12}$$

Setting $x = y$ in (11) and (12), noting that the coefficients of $a^{l-1}xa^{l-1}xa^{l-1}$ in (11) and (12) are of opposite parity, and knowing $g = h$, we may conclude that $a^{l-1}xa^{l-1}xa^{l-1} = 0$ for all $x \in A$. This means that the non-zero right ideal $a^{l-1}A$ is nil of bounded degree which in view of [**5**], Lemma 1.1 provides a contradiction since A is a prime ring.

COROLLARY 4. *Let R be a 2-fold transitive ring of linear transformations on a complex vector space H and let $d = \operatorname{ad} b$ be an inner derivation of R satisfying $[d^n(x), d^n(y)] = 0$ for all $x, y \in R$. Then there exists a complex scalar λ such that $a = b - \lambda$ satisfies $a^{[(2n+3)/3]} = 0$.* □

4. Applications. If A is an algebraically irreducible algebra of operators on a complex Banach space H then A is 1-fold transitive (since it is irreducible) and hence by [**3**] it is m-fold transitive if H is infinite dimensional or is of dimension at least m. In particular, $\mathfrak{L}(H)$, the algebra of all bounded linear operators on H is 2-fold transitive so that the previous results apply.

Let A be a C^*-algebra of operators, containing the identity operator 1, acting on a complex Hilbert space H. Let $R = A''$ be the ultraweak closure of A and let M be the universal enveloping von Neumann algebra of R. If ϕ is any $*$-representation of R and π the natural injection of R into M, there exists a normal $*$-representation $\tilde{\phi}$ of M such that $\phi(x) = \tilde{\phi}(\pi(x))$. We have that $\tilde{\phi}(M) = \phi(R)''$. If ϕ is irreducible, $\tilde{\phi}(M) = \phi(R)'' = \mathfrak{L}(H_\phi)$, the ring of all bounded linear operators on H_ϕ, where H_ϕ is the representation Hilbert space. If $\tilde{\phi}$ is a normal homomorphism of M onto a von Neumann algebra N, there exists a central projection $c \in M$ and a $*$-isomorphism $\tilde{\psi}$ of M_c onto N such that $\tilde{\phi}(x) = \tilde{\psi}(xc)$ for all x in M.

THEOREM 5. *Let A be a C^*-algebra of operators acting on a complex Hilbert space H, and assume A contains the identity operator* 1. *Let $R = A''$, the ultraweak closure of A, and suppose* $(\operatorname{ad} a)^n(x) = (\operatorname{ad} b)(x)$ *for some $a, b \in A$ and all $x \in A$. If n is odd there exists a central projection $c \in R$ and a central element z in R such that $((a - z)c)^{(n+1)/2} = 0$, $1 - c = \Sigma_{\beta \in B'} d_\beta$, and $\operatorname{ad}_\beta = \Sigma_{i=1}^{j(\beta)} \lambda_i^\beta r_j^\beta$ where the λ_i^β are distinct complex numbers, the r_j^β are (not necessarily self-adjoint) orthogonal idempotents, and the d_β are orthogonal central projections. If n is even there exists a central element $z \in R$ such that $(a - z)^{n/2} = 0$.*

Proof. Let $\{\phi_\beta\}_{\beta \in B}$ be a complete set of irreducible representations of R. Then $\phi_\beta(x) = \tilde{\phi}_\beta(\pi(x))$ where $\tilde{\phi}_\beta$ is a normal $*$-homomorphism of M on $\mathfrak{L}(H_{\phi_\beta})$. As above, for each β, there exists a central projection c_β in M and a $*$-isomorphism $\tilde{\psi}_\beta$ of M_{c_β} on $\mathfrak{L}(H_{\phi_\beta})$ such that $\tilde{\phi}_\beta(x) = \tilde{\psi}_\beta(xc_\beta)$ for all $x \in M$. Now $(\text{ad } a)^n(x) = (\text{ad } b)(x)$ for all $x \in A$ implies $(\text{ad } \pi(a))^n(x) = (\text{ad } \pi(b))(x)$ for all $x \in M$ so that $(\text{ad } \tilde{\phi}_\beta(\pi(a)))^n(x) = (\text{ad } \tilde{\phi}_\beta(\pi(b)))(x)$. Since $\tilde{\phi}_\beta$ is irreducible, the results of Theorem 2 and Corollaries 2 and 3 apply. If n is odd then either (i) there exists $\lambda_\beta \in \mathbf{C}$ such that $(\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = 0$ (in the case that the minimum polynomial of $\tilde{\phi}_\beta(\pi(a))$ has a repeated root or $\tilde{\phi}_\beta(\pi(a))$ is central) or (ii) $\tilde{\phi}_\beta(\pi(a)) = \Sigma_{i=1}^{j(\beta)} \lambda_i^\beta p_i^\beta$ where the p_i^β are mutually orthogonal idemotents and the λ_i^β are distinct. Since $\{\phi_\beta\}_{\beta \in B}$ is complete we have LUB $c_\beta = 1$. Choose mutually orthogonal central projections $d'_\beta \in M$ such that $d'_\beta \leq c_\beta$ and $\Sigma_{p \in B} d'_\beta = 1$. Let $c' = \Sigma_{\beta \in B_1} d'_\beta$ where $B_1 = \{\beta \mid \text{(i) holds}\}$. Then $1 - c' = \Sigma_{\beta \in B \setminus B_1 = B'} d'_\beta$. If $\beta \in B_1$ then $0 = (\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = (\phi_\beta(a) - \lambda_\beta)^{(n+1)/2}$ so that $|\lambda_\beta| \leq \|\phi_\beta(a)\| \leq \|a\|$. Hence $z' = \Sigma_{\beta \in B_1} \lambda_\beta d'_\beta \in Z_M$. Moreover, $\beta \in B_1$ implies $0 = (\tilde{\phi}(\pi(a)) - \lambda_\beta)^{(n+1)/2} = (\tilde{\psi}_\beta((\pi(a) - \lambda_\beta)c_\beta))^{(n+1)/2}$ so that $0 = (\pi(a) - \lambda_\beta)^{(n+1)/2} c_\beta$ since $\tilde{\phi}_\beta$ is an isomorphism. $(\pi(a) - z')^{(n+1)/2} c' = ((\pi(a) - z')c')^{(n+1)/2} c_\beta d'_\beta = 0$ for each $\beta \in B_1$. Similarly if $\beta \in B \setminus B_1$, $\tilde{\phi}_\beta(\pi(a)) = \Sigma_{i=1}^{j(\beta)} \lambda_i^\beta p_i^\beta = \tilde{\psi}_\beta(\pi(a)c_\beta)$ so that $\pi(a)c_\beta = \Sigma_{i=1}^{j(\beta)} \lambda_i^\beta q_i^\beta$ and $\pi(a)d'_\beta = \Sigma_{i=1}^{j(\beta)} \lambda_i^\beta q_i^\beta d'_\beta$.

Let i: $R \to R$ be the identity map, $\tilde{i}$ the normal homomorphism of M on R for which $\tilde{i}(\pi(x)) = i(x)$ for all $x \in R$. Let c'' be a central projection in M and $\tilde{j}$ an isomorphism of $M_{c''}$ on R such that $\tilde{i}(\pi(x)) = \tilde{j}(\pi(x)c'')$. $\tilde{j}$ induces an isomorphism between $Z_{M_{c''}}$ the center of $M_{c''}$ and Z_R. Hence $c'c'' + (1 - c')c''$ which is the identity of $M_{c''}$ is sent by $\tilde{j}$ to 1, the identity of R. Let $c = \tilde{j}(c', c'')$ and $z = \tilde{j}(z', c'')$, $d_\beta = \tilde{j}(d'_\beta c'')$, and $r_j^\beta = \tilde{j}(q_i^\beta d'_\beta c'')$. □

THEOREM 6. *Let A be a C^*-algebra, d a derivation on A such that $[d^n(x), d^n(y)] = 0$ for all $x, y \in A$. There exists $s \in R$, $z \in Z_R$ the centre of R, such that $d(x) = [s, x]$ for all $x \in A$ and $(s - z)^{[(2n+3)/3]} = 0$.*

Proof. By [**16**: 4.1.7] there exists such an s. Moreover d extends in this way to a derivation on R. The result follows from Theorem 4 and an argument as in Theorem 5. □

We finish with a result which does not fit the title of the paper but which contains the same methods in its proof.

LEMMA [*See* 8: *Theorem*]. *If R is a prime ring not of characteristic* 2 *and d: $R \to R$ a derivation, then then either $d = 0$ or $\{x \in R \mid [x, d(r)] = 0$ for all $r \in R\} \subseteq Z_R$, the centre of R.* □

Proof. Let $b \in \{x \in R \mid [x, d(r)] = 0$ for all $r \in R\}$. Then $0 = [b, d(r)] = ((\text{ad } b) \circ d)(r)$ for all $r \in R$. By [**15**: Theorem 1] either $d = 0$ or ad $b = 0$. If ad $b = 0$ then $[b, r] = 0$ for all r so $b \in Z_R$. □

THEOREM 7. *Let A be a C^*-algebra with identity e, $R = A''$, and $d: A \to A$ a derivation. There exists a central projection $c \in Z_R$ such that $dc(a) = 0$ for all $a \in A$, and $\{a(e-c) \mid [a, d(b)](e-c) = 0$ for all $b \in A\} \subseteq Z_A(e-c)$.*

Proof. d extends to a derivation (denoted by d) from R to R. Let ϕ be an irreducible representation of R and consider $d_\phi: \phi(R) \to \phi(R)$ given by $d_\phi(\phi(r)) = \phi(d(r))$. Then d_ϕ is a derivation on the irreducible algebra $\phi(R)$. By the lemma either $d_\phi = 0$ or $\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0$ for all $s \in R\} \subseteq Z_{\phi(R)}$.

If $\tilde{\psi}$, π, and c are as in the beginning of this section,

$$\begin{aligned}\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0 \text{ for all } s \in R\}\\ = \{\tilde{\psi}(\pi(r)c) \mid [\tilde{\psi}(\pi(r)c), \tilde{\psi}(\pi(d(s))c)] = 0 \text{ for all } s \in R\}\\ = \{\tilde{\psi}(\pi(r)c) \mid [\pi(r)c, \pi(d(s))c] = 0 \text{ for all } s \in R\}.\end{aligned}$$

Since $\tilde{\psi}$ is a $*$-isomorphism from M_c onto $\mathfrak{L}(H_\phi)$ it carries centers to centers so that if $\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0$ for all $s \in R\} \subseteq Z_{\phi(R)}$ we must have $\{\pi(r)c \mid [\pi(r)c, \pi(d(s))c] = 0$ for all $s \in R\} \subseteq Z_{M_c}$.

Let $\{\phi_\beta\}$ be a complete set of irreducible $*$-representations of R and $d\phi_\beta$ as above. If $d\phi_\beta = 0$ there exists a central projection c_β in M such that $0 = d\phi_\beta(\phi_\beta(x)) = \phi_\beta(d(x)) = \tilde{\phi}_\beta(\pi(d(x))) = \tilde{\psi}_\beta(\pi(d(x))c_\beta)$ so that $\pi(d(x))c_\beta = 0$ for all $x \in R$. If $d\phi_\beta \neq 0$ there exists c_β in M such that $\{\pi(r)c_\beta \mid [\pi(r)c_\beta, \pi(d(s))c_\beta] = 0$ for all $s \in R\} \subseteq Z_{M_{c_\beta}}$.

Since $\{\phi_\beta\}$ is complete, $\text{LUB}\, c_\beta = e$, choose mutually orthogonal central projections c'_β in M such that $c'_\beta \le c_\beta$ and $\Sigma c'_\beta = e$. Let $c_0 = \Sigma c'_\beta$ where the sum is over all β such that $\pi(d(x))c_\beta = 0$ for all $x \in R$.

Let i, $\tilde{i}$ and $\tilde{j}$ be as above with c_1 a central projection in M such that $\tilde{j}$ is an isomorphism of M_{c_1} on R. There exists $c \in Z_R$ such that $\tilde{j}(c_0c_1) = c$. We have $\tilde{i}(c_0) = c$. Now $0 = \pi(d(r))c_0$ for all $r \in R$ so that $0 = \tilde{i}(\pi(d(r))c_0) = d(r)c$ for all $r \in R$. Moreover,

$$\begin{aligned}\{\pi(r)(e-c_0) \mid\mid \pi(r), \pi(d(s)) \mid (e-c_0) = 0 \text{ for all } s \in R\}\\ = \{\pi(r)(e-c_0) \mid \tilde{i}([\pi(r), \pi(d(s))](e-c_0)) = 0 \text{ for all } s \in R\}\\ = \{\pi(r)(e-c_0) \mid [r, d(s)](e-c) = 0 \text{ for all } s \in R\}.\end{aligned}$$

Hence $\{\pi(r)(e-c_0) \mid [\pi(r), \pi(d(s))](e-c_0) = 0$ for all $s \in R\} \subseteq Z_{M_{e-c_0}}$ implies $\{r(e-e) \mid [r, d(s)](e-c) = 0$ for all $s \in R\} \subseteq \tilde{i}(Z_{M_{e-c_0}}) = Z_{R_{e-c}}$.

Finally,

$$\{a(e-c) \mid [a, d(b)](e-c) = 0 \text{ for all } b \in A\}$$
$$\subseteq \{r(e-c) \mid [r, d(s)](e-c) = 0 \text{ for all } s \in R\}$$

by the ultra weak continuity of d. Therefore

$$\{a(e-c) \mid [a, d(b)](e-c) = 0 \text{ for all } b \in A\}$$
$$\subseteq A(e-c) \cap Z_R(e-c) = Z(e-c). \qquad \square$$

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