PERMUTATIONS AND CUBIC GRAPHS

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In studying maximal nonparabolic subgroups of the modular group, B. H. Neumann and later C. Tretkoff were led to study pairs of permutations $A$ and $B$ of an infinite set $\Omega$ such that $A^2 = B^3 = 1$ and that $C = AB$ is transitive on $\Omega$. Here we study such triples $(\Omega, A, B)$, but without the requirement that $\Omega$ be infinite. Our method is to associate with each such triple a graph $G^*(\Omega, A, B)$. Such graphs have been used before, especially by Stothers and by Cori.

Our central result here is that the graphs under consideration are precisely those that can be obtained by attaching trees, in certain simple specified ways, to finite or infinite graphs equipped with a reduced path that traverses each edge exactly once in each direction.

The classification or enumeration of all such cubic graphs appears to be a difficult question, to which we are able to contribute only a few remarks and examples. We also give a catalog of all triples of the kind described, for $\Omega$ of cardinality up to 12. In addition, we catalog all such Eulerian paths on cubic graphs that have no more than six vertices. As the cardinality of $\Omega$ increases, the complexity of the catalog increases rapidly.

In another paper [13] we show that several large classes of infinite planar graphs have Eulerian paths; these include the 1-skeletons of almost all regular tessellations of the plane (with Euclidean or hyperbolic metric). The results of the present paper are used in [10] to extend results of Neumann and Tretkoff. The central result here, which has been obtained earlier and independently by Stothers [39], is as follows, where we write $\infty$ for $\aleph_0$. We note that points (1) and (2) of this theorem had been observed by Tretkoff in the examples she constructed.

**THEOREM.** It is well known that the modular group $M = \text{PSL}(2, \mathbb{Z})$ has a free factorization $M = Z_2 * Z_3$, and that if $A$ is a generator for $Z_2$ and $B$ is a generator for $Z_3$, then the subgroup $P$ generated by $C = AB$ is a maximal parabolic subgroup of $M$. Let $N$ be a complement of $P$ in $M$, and let $\Omega$ be the family of right cosets of $N$ in $M$. Then $A$ and $B$ act on $N$ by right multiplication in such a way that $C = AB$ is transitive. Let $G^* = G^*(\Omega, A, B)$. Then (i) $N$ is the free product of $r_2$ groups of order 2, $r_3$ groups of order 3, and $r_\infty$ infinite cyclic groups; $0 \leq r_2, r_3, r_\infty \leq \infty$. (ii) $r_2$ is the number of fixed points of $A$, $r_3$ is the number of fixed points of $B$, and $r_\infty$ is the Betti number of $G^*$.

(iii) (1) $r_2 + r_3 + r_\infty = \infty$.
(2) If $r_\infty$ is finite, it is even.
Conversely, if three numbers $r_2$, $r_3$, $r_\infty$ with $0 \leq r_2, r_3, r_\infty \leq \infty$, satisfy conditions (1) and (2), then they are realized by some complement $N$ to $P$ in $M$.

In yet another paper [12] we have used the methods of the present article to discuss various similar problems. For example, we find all those permutations of a set that are the product of an involution and a transitive permutation. In other work [14] we have again used this method to reprove a theorem of G. A. Miller [31] that: given integers $a$, $b$, $c$ \(2 < a, b < c\), there exist permutations $A$, $B$ of a set of cardinality at most $c + 2$, such that $A$, $B$, and $C = AB$ have respective periods $a$, $b$, and $c$. A variant of Miller's theorem is established in [12], while in [14] we sharpen Miller's result to obtain in certain cases permutations of smallest degree $|\Omega|$.

2. Graphs. We consider directed graphs in which each edge has an inverse. Explicitly a graph $G$ consists of a set $V$ of vertices, a set $E$ of edges, a function $\alpha: E \to V$ assigning to each edge its initial point, and a function $\eta: E \to E$ assigning to each edge $e$ its inverse edge $e^{-1}$. It is required that $e^{-1} \neq e$ but $(e^{-1})^{-1} = e$. One says that the edge $e$ runs from $\alpha(e)$ to $\alpha(e^{-1})$.

One may view each pair \(\{e, e^{-1}\}\) of opposite edges as constituting an 'undirected edge' or line $|e|$, between $\alpha(e)$ and $\alpha(e^{-1})$, with $e$ and $e^{-1}$ its two directions or orientations. The set of vertices and lines then form a 1-complex $|G|$. We shall not always distinguish between $G$ and $|G|$. For example, if we say that $G$ is a tree, we really mean that $|G|$ is a tree. Similarly, the degree of a vertex $v$ of $G$ is really its degree in $|G|$, that is, the number of lines at $v$.

A path in $G$ is reduced if it contains no consecutive pair of inverse edges. We call a tree simply infinite if, at any point, there is exactly one (reduced) infinite path beginning at that point. We call a simply infinite path, beginning at some point, a ray at that point.

A cubic graph is one in which each vertex has degree 3. We call a graph cuboid if each vertex has degree at most 3. A Eulerian path in a cuboid graph (hence also in a cubic graph) is one that contains each edge exactly once, and which is reduced except at vertices of degree 1.

In speaking of triples $(\Omega, A, B)$ it is always understood that $A$ and $B$ are permutations of $\Omega$ such that $A^2 = B^3 = 1$. We call the triple transitive if $C = AB$ is transitive on $\Omega$. 
We associate a graph $G = G(\Omega, A, B)$ with each triple $(\Omega, A, B)$. The vertex set is $\Omega$, and there is an edge $e$ from $p$ to $q$ if and only if $p \neq q$ and $q$ is one of $pA$, $pB$, $pB^{-1}$. We call $e$ an $A$-edge, $B$-edge, or $B^{-1}$-edge accordingly. The inverse edge $e^{-1}$ is then an $A$-edge, $B^{-1}$-edge, or $B$-edge. (If, for example, $p \neq q$, and both $q = pA$ and $q = pB$, we require both an $A$-edge and a $B$-edge from $p$ to $q$. However, this cannot occur in the cases we consider, where $C = AB$ is transitive, since it would yield $qC = qAB = pB = q \neq p$.)

Evidently an $A$-edge $e$ meets no $A$-edge except $e$ and $e^{-1}$. Similarly, the $B$-edges fall into triples, forming oriented triangles, and any two distinct triangles of this type are disjoint. Conversely, it is easy to see that if the edges of a graph $G$ are divided into three categories: $A$-edges, $B$-edges, and $B^{-1}$-edges, the inverses of which are respectively $A^{-1}$-edges, $B^{-1}$-edges, and $B$-edges; and if these disjointness conditions are met, then there is a unique triple $(\Omega, X, B)$ such that $G = G(\Omega, A, B)$.

Let $G^* = G/B$ be the quotient graph of $G$ by $B$. Explicitly, the vertices of $G^*$ are the $B$-orbits of $\Omega$, and the edges of $G^*$ are in one-to-one correspondence with the $A$-edges of $G$ under the natural map from $G$ onto $G^*$. (We identify each $A$-edge $e$ with its image $e^*$ in $G^*$.) Clearly $G^*$ is a cuboid graph.

Let $W = (\ldots, p_{-1}, p_0, p_1, \ldots)$ be a $C$-orbit in $\Omega$, where (as usual) if $W$ has finite length $n$, the subscripts are to be taken modulo $n$. Since $p_iA$ lies in the image $p_{i+1}^*$ of $p_{i+1} = p_iAB$, the relation $p_iC = p_{i+1}$ determines an edge $e_i^*$ from $p_i^*$ to $p_{i+1}^*$, except that, in case $p_iA = p_i$, we have $p_i^* = p_{i+1}^*$. Thus $W$ determines a path $\pi$ in $G^*$, the successive edges of which are these $e_i^*$; $\pi$ is either a closed path or a doubly infinite path. Since a given $p_i$ occurs at most once in $W$, (the image of) a given $A$-edge $e_i$ occurs at most once in $\pi$. If $C$ is transitive on $\Omega$, then every vertex occurs in $W$, and hence every edge of $G^*$ occurs in $\pi$; in this case $\pi$ is a Eulerian path on $G^*$. 

We now suppose, conversely, that a connected cuboid graph $H$ is given, together with a Eulerian path $\pi$ on $H$, and we seek to construct a transitive triple $(\Omega, A, B)$ such that $H \simeq G^*$, in which $\pi$ is the path described above. We begin by constructing an undirected graph $\tilde{H}$ as follows. We replace each vertex $p$ of degree 2 or 3 in $H$, and some arbitrarily chosen set of vertices of degree 1 in $H$, by an unoriented triangle $(p_1, p_2, p_3)$. Each line $l$ of $H$, between vertices $p$ and $q$, is replaced by a line $\tilde{l}$ between one of the $p_i$ and one of the $q_j$ in such a way that the various lines $\tilde{l}$ are disjoint. We next orient the triangles in $\tilde{H}$, that is, we choose one of the two cyclic orders $(p_1, p_2, p_3)$ and $(p_3, p_2, p_1)$ to be the orientations of the $B$-edges. Suppose first that $p$ has degree 3 in $H$, and suppose the lines $l_1, l_2, l_3$ at $p$ have covers $\tilde{l}_1, \tilde{l}_2, \tilde{l}_3$ in $H$ that contain $p_1, p_2, p_3$. Let $e_1, e_2, e_3$ be the (directed) edges of $H$ that enter $p$ along $l_1, l_2, l_3$. If, Case 1, $\pi$ contains all three of $e_ie_{i+1}^{-1}$ (subscripts mod 3), we choose the orientation $(p_1, p_2, p_3)$. It is easy to see that in the contrary case, Case 2, $\pi$ must contain all three of $e_ie_{i-1}^{-1}$, in which case we choose the other orientation $(p_3, p_2, p_1)$. We orient the remaining triangles (which correspond to vertices of degrees 1 and 2 in $H$) arbitrarily.

The graph $\tilde{H}$, with the given orientations, now satisfies the conditions for it to be $\tilde{H} \simeq G = G(\Omega, A, B)$ for a certain triple $(\Omega, A, B)$, whence $H \simeq G^*$. Moreover, we have chosen the orientations in such a way that $\pi$ is the image of a $C$-orbit $W$ in $\Omega$. Since $\pi$ contains all the edges of $H$, $W$ must contain all the initial points of $A$-edges in $G$. If $C$ were not transitive, an orbit $W_0$ other than $W$ would then contain only points fixed by $A$, hence would be a $B$-orbit, mapping into an isolated point of $H$. This contradicts the assumption that $H$ is connected. We conclude that $C$ is indeed transitive. We have proved the following.

(2.1) Theorem. Each transitive triple $(\Omega, A, B)$ determines uniquely a connected cuboid graph $H$ together with a Eulerian path $\pi$ on $H$. Conversely each pair consisting of a connected cuboid graph $H$ with Eulerian path $\pi$ arises thus from some transitive triple $(\Omega, A, B)$; however, this triple fails of uniqueness in two respects:

(2.11) a vertex of degree 1 in $H$ may be the image of either a 1-cycle (fixed point) or of a 3-cycle of $B$,

(2.12) the orientation of any 3-cycle of $B$ mapping to a vertex of degree 2 in $H$ (that is, containing exactly one point fixed by $A$) may be chosen arbitrarily (that is, the cycle may be replaced by its inverse).
If $\Omega$ contains no fixed point of either $A$ or $B$, then $G^*$ will be a cubic graph. Conversely, if $H$ is a cubic graph, and $\pi$ a Eulerian path on $H$, then the ambiguities of (2.11, 2.12) do not arise, and a transitive triple $(\Omega, A, B)$ is determined uniquely; in this triple clearly neither $A$ nor $B$ has a fixed point.

(2.13) Corollary. The transitive triples $(\Omega, A, B)$ in which neither $A$ nor $B$ has a fixed point correspond one-to-one with the Eulerian paths of cubic graphs.

3. Reduction. We shall see that every cuboid graph with Eulerian path can be obtained in a straightforward way from a cubic graph with Eulerian path.

(3.1) Lemma. Let a cuboid graph $H$ be obtained from the disjoint union of graphs $H_1$ and $H_2$ by first subdividing the lines $l_1$ and $l_2$ of $H_1$ and $H_2$ by vertices $v_1$ and $v_2$, to obtain graphs $H'_1$, $H'_2$ and then adding a line $l$ between $v_1$ and $v_2$. Let $e$ be the (directed) edge along $l$ from $v_1$ to $v_2$. Let $\pi$ be a Eulerian path on $H$. Then after possible interchange of $H_1$ and $H_2$, the following holds.

\[ \pi = \pi_1 e \pi_2 e^{-1} \pi'_1 \] where $\pi_1, \pi'_1$ are in $H_1$ and $\pi_2$ in $H_2$, and (apart from subdivision of edges) $\pi_1 \pi'_1$ is a Eulerian path on $H_1$ and $\pi_2$ is an Eulerian path on $H_2$;

(3.12) $H_2$ is finite.
Conversely, if \( \pi \pi' \) is an Eulerian path for \( H' \) where \( \pi \) ends at \( v_1 \), and \( \pi_2 \) is an Eulerian path on \( H_2 \) beginning and ending at \( v_2 \), then \( \pi \), as given in (3.11), is an Eulerian path on \( H \).

These assertions are all clear from an inspection of the graphs, if one notes that, since \( \pi_2 \) is finite and contains all edges of \( H_2 \), \( H_2 \) must be finite.

(3.2) COROLLARY. If an infinite tree \( H \) has an Eulerian path, then \( H \) must be simply infinite.

The same argument shows that if \( H \) is an infinite graph with an Eulerian path and if deletion of some finite set of edges separates \( H \) into components \( H_1, \ldots, H_k \), then all but one of the \( H_i \) must be finite. In particular, if \( H \) is a locally finite planar graph with an Eulerian path, then the complement of \( H \) in the plane can have at most one infinite component.

Now suppose that \( H, H_1, H_2, \) and \( l \) are as above, and that \( H_2 \) is a tree. Then \( T = H_2 \cup l \) is a tree, and we say \( H \) is obtained from \( H_1 \) by attaching the tree \( T \) at the point \( v_1 \), or that \( H_1 \) is obtained from \( H \) by deleting the tree \( T \). We call \( T \) a branch of \( H \). It is convenient to incorporate here a degenerate case: if \( H \) consist merely of \( H_1 \) with the line \( l_1 \) subdivided by the vertex \( v_1 \), we regard \( H \) as obtained from \( H_1 \) by attaching the trivial branch consisting of the point \( v_1 \) alone, and \( H_1 \) as obtained from \( H \) by deleting this trivial branch.

From the above it is clear that if \( H \) has an Eulerian path and \( H_1 \) is obtained from \( H \) by deleting a finite branch, then \( H_1 \) has an Eulerian path. The same is true if \( H_1 \) is obtained by deleting a possibly infinite number of finite branches, attached at different points. We seek to delete all finite branches.

It is easy to see that if \( H \) contains an infinite ascending chain of finite trees, then \( H \) is in fact the union of this chain, and is an infinite tree. If \( H \) has an Eulerian path, then \( H \) is a simply infinite tree. Putting aside this case, every finite branch in \( H \) is contained in a maximal finite branch, and, after deleting all maximal finite branches, we arrive at a graph \( H_1 \) that contains no finite branch. If \( H \) has finite branches not contained in
any maximal finite branch, then \( H \) is a simply infinite tree, and, by deletion of finite branches, \( H \) can be reduced to a ray \( H \).

It may be that \( H \) contains an infinite branch. By (3.12) we see that \( H \) can contain at most one maximal infinite branch, which must then be simply infinite, and that, if it does, \( H \) is obtained by attaching this infinite branch to a finite subgraph \( H_0 \), which will have an Eulerian path. In this latter case, we say that \( H_0 \) is obtained from \( h_1 \) by deleting a simply infinite branch.

In all cases, by deleting branches we can pass from \( H \) to a subgraph \( H_0 \) that contains no branches, and which possesses an Eulerian path. In the corresponding triple \((\Omega_0, A_0, B_0)\), there can be no fixed point for \( B_0 \), else there would be a vertex of degree 1 in \( H_0 \). Suppose that \( H_0 \) is not a single point; then no \( B_0 \)-orbit can contain three fixed points of \( A_0 \). Since \( H_0 \) contains no vertex of degree 1, no \( B_0 \)-orbit can contain two fixed points of \( A_0 \). Finally, if some \( B_0 \)-orbit contained a single fixed point of \( A_0 \), it would give a vertex of degree 2 in \( H_0 \); this vertex can be removed by detaching a trivial branch. We conclude that either \( H_0 \) is a single point, or that \( A_0 \) and \( B_0 \) are without fixed points and \( H_0 \) is a cubic graph. Finally, if a single point \( H_0 \) is left after detaching branches from a larger graph \( H \), then \( H \) must in fact be a tree.

We have proved the following.

(3.3) THEOREM. Every cuboid graph \( H \) with Eulerian path is either a finite or simply infinite tree, or is obtained, from a cubic graph \( H_0 \) with Eulerian path, by attaching finite trees and, in case \( H_0 \) is finite, possibly one additional simply infinite tree.

4. Appendix A. Enumeration of the permutations \( A, B \) of a finite set such that \( A^2 = B^3 = 1 \) and \( AB \) is transitive.

In this appendix we give some lemmas concerning the enumeration problem in the title. We raise some questions that we cannot solve. Especially, we cannot give a reasonable algorithm for the counting problem. In the last part of this Appendix (Section 5), we do enumerate all solutions of the conditions

\[
A^2 = 1, \quad B^3 = 1, \quad C = AB \text{ transitive on } \Omega,
\]

for \( \Omega \) of cardinality \( c = |\Omega| \) no greater than 12.

If \((A, B)\) is a solution of (4.01) and \( P \in \text{Sym} \Omega \), then \((A', B') = (A^P, B^P)\), where \( X^P = P^{-1}XP \), is also a solution, conjugate to \((A, B)\). If
$C_0$ is any cycle transitive on $\Omega$, for example, $C_0 = (1, 2, \ldots, c)$, then clearly every solution for (4.01) is conjugate to a solution of

\begin{equation}
A^2 = 1, \quad B^3 = 1, \quad \text{and} \quad AB = C_0.
\end{equation}

Since $\text{Sym}\, \Omega$ contains exactly $(c - 1)!$ cycles $C$ of length $c$, the number of solutions of (4.01) is $(c - 1)!$ times the number of solutions of (4.011).

If $(A, B)$ and $(A^p, B^p)$ are conjugate solutions of (4.011) then $C_0^p = C_0$, that is, $P$ commutes with $C_0$. Moreover, $(A, B) = (A^p, B^p)$ if and only if $P$ commutes with $A$ and $B$ as well. In this case $P$ effects an automorphism of the graph $H$, and every automorphism of $H$ is of this kind. Now $P$ commutes with $C_0$ if and only if $P$ is a power of $C_0$, whence the group $\text{Aut}\, H$ of all automorphisms of $H$ is a subgroup of the cyclic group, of order $c$, generated by $C_0$. If $\text{Aut}\, H$ has order $s$, then $s$ divides $c = rs$, where $C_0^r$ generates $\text{Aut}\, H$, and $r$ is the number of solutions of (4.011) that are conjugate to $(A, B)$.

We find that the values $s = 1, 2, 3, 6$ occur as the number of symmetries of finite cubic graphs $H$ with Eulerian paths. We conjectured that no other values could occur, and this has been proved by Bianchi and Cori [2], indeed with the orders 2 and 3 replaced by any pair of distinct primes. Obvious examples (see below) exhibit the values $s = 1, 2, 3$. The case $s = 6$ is illustrated in (4.05) below, with $c = 6$. The value $s = 6$ occurs for infinitely many graphs $H$; the construction of such graphs is illustrated by Figure 4.051, in which $A$ and $B$ are without fixed points, and $c = 2.3.13 = 78$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure4.05.png}
\caption{Figure 4.05}
\end{figure}
In this Appendix, the Cauchy-Jordan lemma is used repeatedly. This lemma states that if $P$, $Q$ are cycles, then the transposition of an element from $P$ with an element from $Q$ will link them into a single cycle.

In the notation of the earlier sections, we are addressing the problem of enumerating all pairs $(A, B)$ of permutations of a finite set $\Omega$ such that $A^2 = 1$, $B^3 = 1$, and such that $C = AB$ is transitive on $\Omega$. If $\Omega$ has cardinal $|\Omega| = c$, we may suppose that $\Omega = \{1, 2, \ldots, c\}$ and that $C = (1, 2, \ldots, c)$. To avoid trivialities, we assume that $c \geq 3$.

Two solutions $(A, B)$ and $(A', B')$ are conjugate, or lie in the same conjugacy class, if $A' = P^{-1}AP$ and $B' = P^{-1}BP$ for some permutation $P$ of $\Omega$. Thus $P$ must be a power of $C$, and all conjugates of a solution $(A, B)$ are obtained by repeated conjugation by $C$. This implies the following.

\[(4.03)\]

The number of solutions in any conjugacy class is a divisor of $c$.

We next examine the possible number $r$ of solutions in a conjugacy class.

\[(4.04)\]

A conjugacy class consists of a single solution only in the case $c = 6$ and $A = (14)(25)(36)$, $B = (135)(246)$. 

\[\text{Unshaded triangles are oriented +; shaded } - .\]
Proof. We have assumed that \( C = AB = (1, 2, \ldots, c) \). Suppose that \((A, B)\) is the sole member of its conjugacy class. Then conjugation by \( C \) leaves \((A, B)\) unchanged, that is, \( C \) commutes with \( A \) and \( B \). Since \( C = AB \), this implies that \( A \) and \( B \) commute. Since \( c \geq 3 \), the permutation \( A \neq 1 \) must exchange two elements \( p \) and \( p' \) of \( \Omega \). Now \( B \) cannot fix both \( p \) and \( p' \). Therefore some element \( p \) belongs both to a non-trivial \( A \)-orbit \((p, p')\) and to a non-trivial \( B \)-orbit \((p, q, r)\). Since \( A \) and \( B \) commute, \((pA, qA, rA) = (p', q', r')\) is also a \( B \)-orbit, whence it follows that \((q, q')\) and \((r, r')\) are distinct \( A \)-orbits. But now the set \( \{p, q, r, p', q', r'\} \) is invariant under both \( A \) and \( B \), hence must be all of \( \Omega \). We find that \( C = (p, q', r, p', q, r') \) which, up to notation, gives the conclusion. \( \square \)

The associated graph \( G = G(A, B) \) is the 'prismatic' graph shown in Figure 4.05.

\[(4.06)\]

If \( c \) is a prime, then every conjugacy class contains exactly \( c \) solutions.

We next observe that \( r \) is determined by the order of the automorphism group of the graph \( G = G(A, B) \).

The automorphism group of \( G(A, B) \) is cyclic, of order \( s \), where \( c = sr \), and is generated by \( S = C' \).

\[(4.07)\]

Thus \( r = c \) if and only if \( G \) admits no non-trivial automorphism. In particular, if \( c \) is a prime, then \( G \) admits no nontrivial automorphism.

An automorphism \( P \) of \( H \) fixes a vertex, a link, or a triangle only if \( P = 1 \), \( P^2 = 1 \), or \( P^3 = 1 \), respectively.

\[(4.08)\]

Proof. Since \( P \) is a power of \( C \), a transitive cycle on the vertices, \( P \) can fix an element only if \( P = 1 \). If \( P \) fixes a link, then \( P^2 \) must fix the end points of the link, whence \( P^2 = 1 \). Similarly, if \( P \) fixes a triangle, then \( P^3 \) fixes the vertices of the triangle, whence \( P^3 = 1 \).

We call a graph \( G \) strongly planar if it can be embedded in the plane in such a way that all non-trivial \( B \)-orbits have counterclockwise orientation. Since \( C \) is transitive, \( G \) must then be simply connected, and the graph \( H = G/B \) must be a tree. (For example, a necklace of triangles does not correspond to a transitive permutation \( C \), since \( C \) will have one orbit on the outside of the necklace and another on the inside.) Conversely, if \( H = G/B \), as abstract graph, is a tree, then \( H \) can be embedded in the
plane, and, by arranging the edges at each vertex of $H$ in the proper cyclic order, we derive a strong embedding of $G$ in the plane. We have shown that

\begin{equation}
G \text{ is strongly planar if and only if } H = G/B \text{ is a tree.}
\end{equation}

Every automorphism of a finite tree has either a fixed edge or a fixed vertex. If $T$ is a non-trivial automorphism of $H = G/B$, a tree, then either $T$ fixes an edge, interchanging its end points, or $T$ fixes a vertex. By virtue of (4.08), $T$ has order 2 or 3. This establishes

\begin{equation}
\text{If } G \text{ is strongly planar, then } s = 1, 2, \text{ or } 3; \text{ thus the number of solutions conjugate to } (A, B) \text{ is } c, c/2, \text{ or } c/3.
\end{equation}

It is easy to see that if $s = 2$, then $G$ can be embedded in the plane in such a way that the symmetry $S$ is effected by a rotation through $\pi$ about the midpoint of an edge; if $s = 3$, $G$ can be embedded in such a way that $S$ is effected by a rotation through $2\pi/3$ about the center of a triangle.

\textbf{Remark.} The 'prismatic' graph of Figure 4.05 is planar, as Figure 4.12 shows, but it is not strongly planar.

![Graphs](image)

\textbf{Figure 4.12}

This graph has $s = 6$, as we have seen, but it is exceptional in that it is the only graph with $s = c$. We note that there exist planar cubic graphs with arbitrarily large symmetry groups. For example, the graph of Figure 4.13, where there are $n$ radial lines, evidently admits (even as orientation preserving maps, if we orient the triangles properly) the dihedral group of order $2n$ as symmetry group. But this graph does not have any Eulerian trail. We are left with the following question.
(4.14) **Problem.** With the exception of the 'prismatic graph' does there exist any graph \( G = G(A, B) \) with symmetry group of order \( s > 3 \)? Equivalently, does there exist any pair consisting of a cubic graph \( H \) on more than 2 vertices and an Eulerian trail \( \tau \) on \( H \), such that the automorphism group of the pair \( (H, \tau) \) has order \( s > 3 \)? (Solved in Fig. 4.051.)

(4.15) **Theorem.** For every value of \( c \), \( c \geq 12 \), if 2 divides \( c \) there is a class of solutions for which \( s = 2 \), and if 3 divides \( c \) there is a class for which \( s = 3 \). In these cases, \( r = c/s < c \).

*Proof.* If \( c \equiv 0 \) (mod 6), we obtain \( s = 2 \) with a chain of triangles as shown in Figure 4.16.

![Figure 4.16](image)

If \( c \equiv 2, 4 \) (mod 6), we modify these figures by attaching spurs. For \( s = 3 \) we construct graphs with rotational symmetry of order 3, as shown in Figure 4.17, again possibly attaching spurs.

![Figure 4.17](image)
We turn now to a catalog of all solutions $G = G(A, B)$ of (5.01) for small values of $c$. We exhibit a representative for each conjugacy class, and also indicate the value of $s$, that is, the size $r = c/s$ of that conjugacy class. The classes are listed according to the number of nontrivial orbits in $B$, and, for two classes that agree in this attribute, precedence is given to the classes with the fewest nontrivial orbits in $A$. (For the graphs, this means, first, according to the number of triangles, and second, according to the number of spurs, that is, links attached to only a single triangle.) For each value of $c$, we explain why the list is exhaustive.

We have carried our catalog only as far as $c = 12$. Already at $c = 6$ we encounter the prismatic graph, which is not strongly planar. However, we encounter no graph that is not planar, in the usual sense, for the reason that the first such graph occurs for $c = 18$. Such a graph $G$ is shown in Figure 5.131; note that the associated quotient graph $H = G/B$ is cubic.

5. Appendix A, continued. Catalog of classes for $1 \leq c \leq 12$. In this section each class, $1 \leq c \leq 12$ is represented by a graph. All these graphs are planar. Nonplanar graphs first arise for $c > 12$, but for $c = 6, 9, 10, 11, 12$ some of the graphs are not strongly planar. The size $s$ of the automorphism group of the graph is always indicated ($s = c/r$ is tabulated). Thus the number of solutions of $A^2 = 1, B^3 = 1, AB = (1 2 \ldots c)$ is obtainable. For example when $c = 12$, the number of solutions is $\sum r_i = \sum c/s_i = 232$, and the number of solutions of $A^2 = 1, AB =$ some 12-cycle is 232 (11!).

When the orientation of a triangle is not indicated, orientation is counterclockwise; shaded triangles are oriented clockwise.

<table>
<thead>
<tr>
<th>$c$</th>
<th>Description</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>One class; trivial.</td>
<td>$1$</td>
</tr>
<tr>
<td>$2$</td>
<td>One class</td>
<td>$2$</td>
</tr>
<tr>
<td>$3$</td>
<td>One class</td>
<td>$3$</td>
</tr>
<tr>
<td>$4$</td>
<td>One class</td>
<td>$1$.</td>
</tr>
</tbody>
</table>

(5.041) If the triangle is oriented clockwise, turn it over. Then rotate the entire figure to obtain 5.041.
\( c = 5 \) One class

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

The argument is the same.

\( c = 6 \) Three classes. The first class has a single triangle.

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

The automorphisms result from plane rotations. The second class has two triangles joined by a single link.

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

If either triangle is oriented clockwise, twist it and turn it. The automorphisms are plane rotations.

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

The graph is not strongly planar. Transitivity requires that one triangle be oriented clockwise, and the other counterclockwise. After a plane rotation, we may suppose that it is the left triangle that is oriented clockwise.

In all the remaining examples, \( 7 \leq c \leq 12 \), all automorphisms are also plane rotations.

\( c = 7 \) Two classes. \( B \) has two nontrivial orbits, \( A \) has two nontrivial orbits. The graphs show that \( A \) cannot have four nontrivial orbits.

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

\( s = 1 \).

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

\( s = 1 \).

Any triangle oriented clockwise can be twisted. Then the entire figure can be rotated in the plane so that the triangle to which the first spur is attached is the left triangle.
**c = 8** Three classes. Two triangles, two spurs.

(5.081) \[ s = 1. \]

(5.082) \[ s = 2. \]

(5.083) \[ s = 1. \]

Any other graph with two triangles can be rotated or twisted to have one of these three shapes.

**c = 9** Four Classes.

(5.091) \[ s = 1. \]

(5.092) \[ s = 1. \]

Any graph with two triangles can be rotated and twisted into one of 5.091, 5.092. But if the right triangle is twisted, its orientation is reversed.

(5.093) \[ s = 1. \]

These two pictures represent the same class, since one can be rotated into the other.

Suppose now that there are four links (rather than two). By Lemma 3.09, if \( D = (\alpha \beta)(\gamma \delta)C \) is a 9-cycle, then so also is \( (\alpha \beta)(\gamma \delta)D = C \). Thus two of the four links (and the triangles) are disposed as in 5.093.

Suppose \( w \) to be the vertex to which none of the four links is attached. Every vertex except \( w \) has degree 3. Thus every triangle is connected to every other triangle, and the graph may be redrawn so that \( w \) is on the left-hand triangle (with the two extra links not yet attached). It is now clear that \( v \), the vertex before \( x \) on the left-hand triangle, must be attached as in 5.094; the other possible attachments result in an ineligible (intransitive) graph.
We remark that 5.094 is planar (in the usual sense); the embedding is obtained by twisting the right-hand triangle:

\[ s = 1. \]

\[ \text{c} = 10 \quad \text{Seven classes. If there are only two triangles, there must be four spurs ending at four additional vertices:} \]

\[ s = 1. \]

In every other case, there are three triangles and one spur. This spur has a unique end point, so that \( s = 1 \) for these cases. Deletion of the spur contracts the graph to 5.093 or 5.094. From 5.093, five classes arise. They are

\[ s = 1. \]

\[ s = 1. \]

\[ s = 1. \]

\[ s = 1. \]

\[ s = 1. \]

From 5.094, only one class arises, since there is only one place to attach the spur.

\[ s = 1. \]

\[ \text{c} = 11 \quad \text{Ten classes. There must be three nontrivial } B\text{-orbits, hence three triangles, so that two spurs are attached to 5.093.} \]
Since there are 5 (free) vertices of degree 2 in 5.093, there are $C_2^5 = 10$ classes, all strongly planar, and all with $s = 1$.

$c = 12$ Twenty-two classes, of which fourteen are strongly planar.

If there are three triangles, there must be three spurs. The number of classes in this case is $C_3^5 = 10$; in each case $s = 1$.

If there are four triangles, there are several classes with three links each; they are shown. Their automorphisms result from plane rotations.

\[(5.1211)\] \[(5.1212)\] \[s = 3.\]
\[s = 1.\]
\[s = 2.\]
\[s = 2.\]

More classes arise if there are four triangles and five links. First consider the number of ways of adding two links to the class \((5.1211)\). After the links are added, either: (i) each of three triangles has three attached links (one class, not strongly planar); or (ii) two triangles have three links, and the other two have two links (four classes, not strongly planar).

\[(5.1215)\]

Next, consider the number of ways of attaching more links in Fig. 5.1212. One way to do this is to attach both links at the four vertices of the end triangles.
(5.1220) This gives a new class (with $s = 1$). It is not strongly planar. Any other way of adding two links to 5.1212 either results in an intransitive $AB$, or else gives a figure that is isomorphic to one of the preceding.

The same remarks apply to Figures 5.1213, 5.1214.

(5.1221) $s = 2$

(5.1222) $s = 2$.

The automorphisms of all these classes can be exhibited as plane rotations; the canonical form of each skeleton is obtained by flipping, rotating, and twisting.

The number of classes for the cases $c = 13, 14$ is easily obtained from the pictures for $c = 12$.

We now exhibit some graphs (with $C$ transitive) for $c = 18, 24$.

**Figure 5.131**
A nonplanar graph with $c = 18$, $C = AB$ transitive

**Figure 5.132**
A nonplanar graph with $c = 24$, $C = AB$ transitive

**Figure 5.133**
A planar graph with $c = 18$, not strongly planar, $C = AB$ transitive.
To see that 5.131 is nonplanar, note that it can be contracted to $K_{3,3}$, which is known to be nonplanar [5, pp. 135–156].

Note further that example 5.131 can be generalized as follows. Let $k$ be odd, and attach $2k$ equally spaced triangles to a circumference, all facing inward. Connect opposite triangles, and orient all triangles (with one exception) clockwise. 5.131 is the case $k = 3$; 5.063 is the case $k = 1$. The cubic graph so obtained has a Eulerian circuit. For $k \geq 3$, the graph is nonplanar. We have proved:

5.14 Theorem. There exists a cubic graph with $6k$ vertices that has a Eulerian trail, if $k$ is odd.

5.15. If $k$ is even, there cannot be a Eulerian trail. For $C$ would have to be an odd permutation, that is, a cycle on $6k$ symbols. However, $AB = C$ is impossible, since both $A$ and $B$ are even permutations ($A$ is the product of $3k$ transpositions.)

5.16 Theorem. If (and only if) $k$ is odd, $k \geq 1$, there exists a cubic graph with $V = 2k$ vertices.

Proof of $\Rightarrow$. To construct $H$, let $T$ be a linear tree with $2k$ vertices, hence $2k - 1$ edges. Form $T^*$ by replacing (in one way or another) each vertex of $T$ by a triangle, positively oriented. Now $C_{T^*} = A_{T^*}B_{T^*}$ is transitive on the vertices of $T^*$. A free vertex is a vertex (of a triangle) not on a link. Let the $2k + 2$ free vertices of $C_{T^*}$ be labeled $v_1, \ldots, v_{2k+2}$, in the order in which they occur in $C_{T^*}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{construction}
\caption{The construction.}
\end{figure}

Since $k$ is odd, 4 divides $2k + 2$. Thus the free vertices can be divided into $\frac{1}{2}(k + 1)$ blocks of four vertices each: $v_{4i+1}, \ldots, v_{4i+4}$ ($0 \leq i \leq \frac{1}{2}(k + 1)$). Form $H$ by drawing links $(v_{4i+1}, v_{4i+3}), (v_{4i+2}, v_{4i+4})$.

The final step is to contract this graph at every triangle (compare Figure 5.133).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{contracted_graph}
\caption{The contracted graph.}
\end{figure}
5.17 REMARK. The same idea works if $T$ is any tree (not necessarily a linear tree) with $2k$ vertices.

6. Appendix B. Cubic Graphs with an Eulerian path. In §5 we noted that if a cubic graph $H$ possesses an Eulerian path $\pi$, then $H$ must have $v = 2k$ vertices, for some odd natural number $k$. We indicate how to enumerate all such $H$ and $\pi$ for $v = 2$ and for $v = 6$. We do this partly to indicate the abundance of such pairs (giving a method for finding them) and partly to illustrate the apparent difficulty of obtaining any overall view of the totality of cubic graphs with Eulerian paths. Note that 5.161 is not the only possible cubic graph with $2k$ vertices and with a Eulerian path.

Our basic method is that of adding a line. A set $\Sigma$ of (reduced) paths on a finite cubic graph $H$ will be called an Eulerian system if (i) no path occurs in $\Sigma$ if its inverse occurs in $\Sigma$, (ii) each directed edge of $H$ occurs exactly once in some path $\pi$ of the set $\Sigma$. For our inductive construction, it is technically convenient to count the circle $H_0$ as a cubic graph with $v = 0$ vertices, and with an Eulerian system consisting of two paths, which describe the circle in its two senses. Now let $H$ be any finite nonempty cubic graph, not necessarily connected. We mark any two distinct points $p_1$ and $p_2$ interior to two (not necessarily distinct) lines $l_1$ and $l_2$ of $H$, and form a new graph $H'$ by introducing a new line $l$ joining $p_1$ and $p_2$. Then $H'$ is a cubic graph with two more vertices than $H$. Moreover, every finite nonempty cubic graph can be obtained from a disjoint union of replicas of the circle graph $H_0$ by iteration of this construction.

If $\pi_1$ and $\pi_2$ are paths (not necessarily distinct) in an Eulerian system $\Sigma$ for $H$, we obtain an Eulerian system $\Sigma'$ for $H'$ by modifying $\pi_1$ and $\pi_2$ in the manner shown in Figure 6.01.
(We do not spell out a formal description of this construction, since it will be clear what is being done in all the special cases that arise below.) By the familiar argument of Cauchy-Jordan in the theory of permutation cycles, we see that if \( \pi_1 \) and \( \pi_2 \) are distinct, they are replaced by a single path in \( \Sigma' \), while, if \( \pi_1 = \pi_2 \), then this path is replaced by two paths in \( \Sigma' \). In any case, \( \Sigma' \) has cardinality \( |\Sigma'| = |\Sigma| \pm 1 \). This establishes the following.

6.02 Proposition. Let \( \Sigma \) be an Eulerian system on a cubic graph \( H \) with \( v = 2k \) vertices, where \( k \geq 0 \). Then \( 1 \leq |\Sigma| \leq k + 2 \), and \( |\Sigma| \equiv k \) (modulo 2).

It is easy to see that every Eulerian system \( \Sigma \) on \( H' \) can be obtained from a system \( \Sigma \) on \( H \) by the construction suggested by the figure.

We state another consequence of the method of adding a line.

6.03 Proposition. Let \( K \) be any finite cubic graph. Then, by successively adding lines, \( K \) can be embedded in a finite cubic graph \( H \) that possesses an Eulerian path. In particular, if \( K \) has the property that deletion of no pair of lines separates \( K \) into two or more components, then \( H \) can be chosen with the same property.

Two Eulerian systems \( \Sigma_1 \) and \( \Sigma_2 \) on the same cubic graph \( H \) will be called equivalent if some automorphism (preserving incidences) of \( H \) carries one into the other. We shall enumerate only equivalence classes \([\Sigma]\) of Eulerian systems. With each path \( \pi \) in an Eulerian system we associate the cyclically ordered sequence of the vertices on \( \pi \), in the cyclic order in which they occur on \( \pi \). The set of cycles \( \sigma \) associated with the paths of a (fully labeled) Eulerian system \( \Sigma \) on \( H \) evidently determines the graph \( H \) uniquely, and determines the paths \( \pi \) of \( \Sigma \) uniquely except for the choice of edges that connect a pair of vertices whenever there are two or three such edges. In any case, the set of cycles \( \sigma \) determines the equivalence class \([\Sigma]\) uniquely.

Evidently two systems \( \Sigma_1 \) and \( \Sigma_2 \) are equivalent if and only if there exists a permutation of the vertices carrying the set of cycles associated with \( \Sigma_1 \) into the set associated with \( \Sigma_2 \).
If an Eulerian system $\Sigma$ consists of a single path $\pi$, that is, if $\pi$ is an Eulerian path, we call $\Sigma$ or $\pi$ symmetric if $\pi$ is equivalent to its inverse $\pi^{-1}$.

We now turn to the cubic graphs $D$ with $v = 2$ vertices, and with no component $H_0$. There are in fact only two such graphs,

![Diagram of two cubic graphs](image)

By inspection we see that $D_1$ admits only a single class $[\Sigma]$, with $|\Sigma| = 3$, while $D_2$ admits two classes $[\Sigma_1], [\Sigma_2]$, with $|\Sigma_1| = 3$ and $|\Sigma_2| = 1$. In particular, $D_2$ is the only cubic graph with two vertices that admits a (reduced) Eulerian path $\pi$, and the class of this path is described by the cycle $\sigma = (a\ b\ a\ b\ a\ b)$.

The cubic graphs $Q$ with four vertices and no component $H_0$ are of three sorts:

(i) a disjoint union of copies of $D_i$ and $D_j$, $i, j \in \{1, 2\}$;
(ii) those obtained by adding a line to the disjoint union of a copy of $D_i$ and a copy of $H_0$, $i \in \{1, 2\}$;
(iii) those obtained by adding a line to a copy of $D_i$, $i \in \{1, 2\}$.

Since our only interest in graphs with four vertices is to use these for constructing graphs with six vertices that admit an Eulerian path, we confine attention to those 4-vertex graphs $Q$ that possess an Eulerian system $\Sigma$ with $|\Sigma| = 2$. Inspection shows that these are as follows:

![Diagram of four 4-vertex graphs](image)

The lettering of the vertices is for later reference. It is easy to see that each of $Q_1$, $Q_2$, and $Q_3$ admits only a single class of Eulerian systems $\Sigma_i$ with
$|\Sigma_i| = 2$. These have cycles as follows:

\[\Sigma_1: (a b a b a b), (c d c d c d);\]
\[\Sigma_2: (c d d e a b a c b a b), (d);\]
\[\Sigma_3: (a c d e a b), (b d c d b a).\]

For $Q_4$ there are exactly two classes of Eulerian systems, $\Sigma_4$ and $\Sigma'_4$ with $|\Sigma_4| = |\Sigma'_4| = 2$. To see this observe first that if $\Sigma = \{\pi_1, \pi_2\}$, then $|\sigma_1| + |\sigma_2| = 12$. By symmetry, we may suppose that $|\sigma_1| \leq 6$. Now $|\sigma_1| = 0, 1, \text{ or } 2$ is clearly impossible. If $|\sigma_1| = 3$, we may suppose that $\sigma_1 = (a b c)$; this uniquely determines that $\sigma_2 = (a c d b a d c b d)$. If $|\sigma_1| = 4$, we may suppose that $\sigma_1 = (a b c d)$; this uniquely determines that $\sigma_2 = (a d b a c b d c)$. Next, $|\sigma_1| = 5$ is impossible. To see this, we may suppose that $\sigma_1 = (a b c x y)$. Since $x \neq a, b, c$, we must have $x = d$ and $\sigma_1 = (a b c d y)$. But now $y$ cannot be any one of $a, b, c, d$. Also, $|\sigma_1| = 6$ is impossible. To see this we may suppose that $\sigma_1 = (a b c x y z)$. Now $x \neq b, c, y \neq a, c$, and $z \neq a, b$. Moreover, $x, y, \text{ and } z$ must be distinct. This implies that at most one of $x, y, z$ is $d$, and otherwise that $x = a, y = b, \text{ and } z = c$. But then $\sigma_1$ must contain two segments $ab, bc, \text{ or } ca$, impossible.

In summary, we have two classes $[\Sigma_4]$ and $[\Sigma'_4]$, as follows:

\[\Sigma_4: (a b c), (a c d b a d c b d);\]
\[\Sigma'_4: (a b c d), (a d b a c b d c).\]

Now every cubic graph $H$ with six vertices and an Eulerian path $\pi$ must be obtainable from one of $Q_1, Q_2, Q_3, Q_4$ by adding a line. There is a fairly large number of ways of adding a line to one of these $Q_i$ but, with some effort, one can see that those of the resulting graphs that admit an Eulerian path are of only five types, as follows:
These five graphs are nonisomorphic. $H_1$ is the only one that can be disconnected by deleting a single edge. $H_5$ is the only one that cannot be embedded in the plane. Of the remaining three, $H_2$ contains exactly one pair of lines with the same end points, while $H_3$ contains two such pairs and $H_4$ contains no such pair.

It is easy to see that $H_1$ admits only one class of Eulerian paths $\pi$, with cycle

$$\sigma = (cabacbabcdedefefedf)$$

This cycle is necessarily equivalent to its inverse: that is, the class $[\pi]$ is symmetric.

In $H_2$ it is easy to see that if an Eulerian path contained a segment $cefd$, it could not be completed to contain an additional segment $ef$ and two segments $fe$. We conclude that $\pi$ must contain segments $cefe$ and, similarly, $dfed$. Now deletion of the loops $efe$ and $fef$ from these segments reduces $\pi$ to $\pi'$ with cycle $\sigma' = (adbacbdac)\sigma$, or an equivalent cycle. To regain $\pi$ from $\pi'$, either occurrence of $c$ may be replaced by $cefe$, and either occurrence of $d$ may be replaced by $dfed$. These replacements may be made either at occurrences of $c$ and $d$ that are separated in $\pi'$ by a single letter, or at occurrences that are separated by two letters. We thus have, within equivalence, two cycles:

$$\sigma_1 = (adfeadbcefecebacdab);$$
$$\sigma_2 = (adfeadbcaacdacefe);$$

where $\sigma_1$ is not equivalent to $\sigma_2$. On the other hand, the permutation $(cd)(ef)$ carries $\sigma_1$ to $\sigma_1^{-1}$, and the permutation $(ab)(cd)(ef)$ carries $\sigma_2$ to $\sigma_2^{-1}$. In summary, we have two symmetric classes on $H_2$.

In $H_3$, by the argument above, an Eulerian path $\pi$ must contain segments $aehea$, $bfefb$, $acdc$, and $bdcd$. Now one of the displayed $a$'s must be followed by one of the displayed $b$'s, whence we must have either $aefeabf$, $aefeabdc$, or a segment equivalent under the permutation $(ce)(df)$. Each of these two segments admits a unique completion, as follows:

$$\sigma_1 = (aefeabfебacbdca);$$
$$\sigma_2 = (aefeabdcabdacb).$$

In $\sigma_2$ the occurrences of the parts $cdca$, $dcda$, are separated by those of $efe$ and $fef$, while in $\sigma_1$ they are not. Thus $\sigma_1$ is not equivalent to $\sigma_2$ or $\sigma_2^{-1}$. Here $(ab)(ce)(de)$ carries $\sigma_1$ to $\sigma_1^{-1}$, and $(ab)(cd)(ef)$ carries $\sigma_2$ to $\sigma_2^{-1}$. As with $H_2$, we have two symmetric classes on $H_3$. 
We now look for Eulerian paths $\pi$ on $H_4$. Observe that $H_4$ can be obtained from $Q_3$ by adding a line $l$ between vertices $e$ and $f$.

We denote by $Q'_3$ the graph:

If $\pi$ is an Eulerian path on $H_4$ it must contain edges $ef$ and $fe$, hence have the form (1) $\pi = (\alpha_1 ef \alpha_2 fe) = (e \alpha_1 ef \alpha_2 f)$ where $(\alpha_1)$ and $(\alpha_2 e)$ constitute an Eulerian system on $Q'_3$. The corresponding paths $(\beta_1)$ and $(\beta_2)$ will then constitute an Eulerian system $B$ on $Q_3$. Since $|B| = 2$, $B$ is equivalent to $\Sigma_3$ under some automorphism of $Q_3$. This implies that under some element of the fours group $V = \{1, (ab), (cd), (ab)(cd)\}$, the cycles $\pi_1$, $\pi_2$ associated with $(\beta_1)$, $(\beta_2)$ go into the cycles $\alpha_1, \alpha_2$ of $\Sigma_3$, and, evidently, with $\tau_1 \mapsto \sigma_1$, $\tau_2 \mapsto \sigma_2$. Since every automorphism in $V$ can be lifted to $H_4$, after replacing $\pi$ by an equivalent path we may assume that $\tau_1 = \sigma_1$, $\tau_2 = \sigma_2$, except that now, in (1), $e$ and $f$ may be exchanged.

We have then

$$(\beta_1) = (a c d e a b), \quad (\beta_2) = (b d e d b a).$$

The corresponding paths $(\beta'_1)$, $(\beta'_2)$ in $Q'_3$ will be obtained by insertion of an $e$ and an $f$ in each of $(\beta_1)$, $(\beta_2)$, in such a way that $(\beta'_1), (\beta'_2)$ constitute an Eulerian system on $Q'_3$. There are four possibilities:

- **EE**: $(a c f d c a e b) (b d f c d b e a)$
- **EL**: $(a c f d c a e b) (b d c f d b e a)$
- **LE**: $(a c f d c a b e) (b d f c d b a e)$
- **LL**: $(a c d f c a b e) (b d c f d b a e)$
The notation $EE$ means, for example, that, in $(\beta_1')$, $e$ is introduced early into the part $aba$, and $f$ is introduced early into the part $cdc$. Replacing $\pi$ by $\pi^{-1}$ interchanges $EE$ with $LL$ and $EL$ with $LE$. Thus it suffices to consider cases $EE$ and $EL$ (with the reservation that all inverses of resulting $\pi$ must be considered).

We consider the case $EE$. Here $\pi$ must contain a segment $ebacfdcae$ or $fdcaebaef$; the complementary segment is then fully determined. We obtain

$$\pi_1 = (ebacfdcae \cdot fcdbeabdf)$$
$$\pi_2 = (fdcaebaef \cdot eabdfcdbe).$$

From $EL$ we obtain similarly

$$\pi_3 = (ebacdfcae \cdot fdbabdcf)$$
$$\pi_4 = (fcabcadfe \cdot abdfcde).$$

The underlined blocks are the only parts of the form $xyuyyx$. In $\pi_1$ and $\pi_2$ the blocks are separated by two parts of length 3, while in $\pi_3$ and $\pi_4$ by parts of lengths 2 and 4. Therefore $\{\pi_1^\pm, \pi_2^\pm\}$ is disjoint from $\{\pi_3^\pm, \pi_4^\pm\}$.

Evidently the permutation $(ac)(bd)(ef)$ carries $\pi_1$ to $\pi_2$. Now $\pi_1$ is not equivalent to $\pi_1^{-1}$. To see this, note that any permutation carrying $\pi_1$ to $\pi_1^{-1}$ would have to carry the block $acfdca$ of $\pi_1$ to one of the two blocks in $\pi_1^{-1}$. In each case, this is incompatible with its action on the three letters $e$, $f$, and $c$.

The permutation $(ab)(cd)$ carries $\pi_3$ to $\pi_4$. Moreover, the permutation $(ad)(bd)(cf)$ carries $\pi_3$ to $\pi_3^{-1}$.

In summary, on $H_4$ there exist three classes of Eulerian paths:

$$[\pi_1], [\pi_1^{-1}], [\pi_3] = [\pi_3^{-1}].$$

We now look at the Eulerian paths on $H_5$. The explanations are analogous to those for $H_4$, and we present only an outline of the calculations.

**Case I.** $\pi$ is obtained by linking some $\Sigma$ automorphic in $Q_4$ to $\Sigma_4 = \{\rho_1, \rho_2\}$ by a line $l$ between points $e$ and $f$ on disjoint lines of $Q$. We obtain equivalent $\pi'$ by taking $\Sigma_4$ in place of $\Sigma$, and $l$ joining some other points $e'$ and $f'$ on disjoint lines in $Q$. We may suppose $e'$ is on an arc-line in the figure, and $f'$ on the opposite spoke. We now drop primes.
We have $\Sigma_4 = \{\rho_1, \rho_2\}$ where

$$\rho_1 = (abc), \quad \rho_2 = (acbdacbd)$$

Since $\Sigma_4$ has rotational symmetry, we may suppose that $e$ is on $ab$ and $f$ on $cd$, yielding

$$Q'$$

Passing to $Q'$, $\rho_1$ and $\rho_2$ become

$$\rho'_1 = (aebc), \quad \rho'_2 = (acdfbeacbd).$$

Now $\pi$ must contain a part $ebcae$, and there are 2 ways to complete it. These are

$$\pi_1 = (ebcae fcbdacfbdbeadf)$$

$$\pi_2 = (ebcae fdbedfcbdacf).$$

But the permutation $(ab)$ carries $\pi_1$ to $\pi_2^{-1}$, while $(abf)(cd)$ carries $\pi_1$ to $\pi_2$. Thus there is exactly one (symmetric) class.

Case II. $\pi$ is obtained by adding a line $l$ between $e$ and $f$ from some system $\Sigma = \{\rho_1, \rho_2\}$ on $Q$. Then some automorphism of $Q$ carries $\Sigma$ to $\Sigma_4$, $e$ and $f$ to points $e'$, $f'$, and $\pi$ to some $\pi'$ equivalent to $\pi$. Changing notation, and replacing $\pi$ by an equivalent path, we may suppose that $\pi$ is obtained from $\Sigma_4$ by linking two points $e$ and $f$ on disjoint lines of $Q$. We may suppose $e$ is on an arc-line in the figure and $f$ on the opposite spoke.
Now \( \Sigma' = \{ \pi_1, \pi_2 \} \) where
\[
\rho_1 = (abcd), \quad \rho_2 = (abdacbdc).
\]

It is not possible that \( e \) is on the bottom arc, since in this case neither \( e \) nor \( f \) would be on \( \rho_1 \), and \( l \) would not link \( \rho_1 \) and \( \rho_2 \).

If \( e \) is on the second of the consecutive arcs of \( \rho_1 \), we replace \( \pi, \rho_1, \rho_2 \) by \( \pi^{-1}, \rho_1^{-1}, \rho_2^{-1} \). Now \( e \) is on the first arc of \( \rho_1^{-1} \). Thus we may assume that \( e \) is on the arc \( ab \). Now \( H_5 \) is as previously shown, with \( \rho_1 \) and \( \rho_2 \) as above.

In \( Q' \), \( \rho_1 \) and \( \rho_2 \) become
\[
\rho'_1 = (aebcfd), \quad \rho'_2 = (adbeacdfb).
\]

As in Case I, we obtain 2 possibilities:
\[
\pi_1 = (ebcfdae fcadbeacdf)
\]
\[
\pi_2 = (bdaebcf eacbdfcadbe).
\]
Here we use the fact that each of $\pi_1$ and $\pi_2$ has exactly 3 blocks of the form $x***x***x$, with successive blocks overlapping in a word of length 3. This severely limits permutations $\theta$ carrying $\pi_1$ to $\pi_2$.

We find that the permutation $(ac)(bd)(ef)$ carries $\pi_1$ to $\pi_2$. We find similarly that $(bf)(cd)$ carries $\pi_1$ to $\pi_1^{-1}$.

Again we have a single symmetric class. But $(ae)(bdfe)$ carries this class into the class $\pi_1$ of Case I. Thus there is just one class of Eulerian paths in $H_5$. In all of $H_1, \ldots, H_5$ we have a total of 9 classes, consisting of 7 symmetric classes and one pair of inverse classes.

We conclude with two observations. First, there are many finite cubic graphs that admit no Eulerian path. For example any graph with a spur of the form $\rightarrow$ fails to have a (reduced) Eulerian path. Examples are

(Chemists are familiar with these patterns.) Another example, with 6 vertices, is $\bigcirc \bigcirc$. This graph is 2-arc-connected. A 3-arc-connected cubic graph with no (reduced) Eulerian path can be constructed by adding, to $H_4$, an extra line from the upper ellipse to the lower ellipse. Clearly, there is no need to look for a 4-arc-connected cubic graph of any sort: it does not exist.

Second, any classification of Eulerian paths $\pi$ on a cubic graph $H$ with $v$ vertices amounts to a classification of the (isomorphism types of) triples $(\Omega, A, B)$ where $A^2 = B^3 = 1$, $A$ and $B$ have no fixed points, $AB$ is transitive, and $|\Omega| = v$. For $|\Omega| = 6$, there are nine isomorphism classes of such triples. For 7 of these classes, $(\Omega, A, B) \simeq (\Omega, A, B^{-1})$, while the remaining 2 are nonisomorphic, but of the form $(\Omega, A_1, B_1)$ and $(\Omega, A_1, B_1^{-1})$ for certain $A_1, B_1$.

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