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MOSES GLASNER AND MITSURU NAKAI

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In this paper it is shown that there exists a Riemann surface R and a nonnegative 2-form P on R such that the space of energy finite solutions of $d*du = uP$ on R is properly contained in the space of Dirichlet finite solutions yet the subspaces of bounded functions in these two spaces coincide.

Consider a nonnegative locally Hölder continuous 2-form P on a hyperbolic Riemann surface R . Let $PX(R)$ denote the space of solutions of $d*du = uP$ on R satisfying a certain boundedness property X , e.g. D (finite Dirichlet integral $\int_R du \wedge *du$), E (finite energy integral $\int_R du \wedge *du + u^2P$), B (finite supremum norm) or the combinations BD and BE . The reduction operator T_X is defined to be the linear injection of the space $PX(R)$ into the space $HX(R)$ such that for each $u \in PX(R)$ there is a potential p_u on R with $|u - T_X u| \leq p_u$. The unique existence of T_X for the cases $X = B, D, E$ was established in [5] together with the representations

$$T_X u(z) = u(z) + \frac{1}{2\pi} \int_R G_R(z, \zeta) u(\zeta) P(\zeta).$$

where $G_R(\cdot, \zeta)$ is the Green's function for T with pole at ζ .

One of the central questions concerning reduction operators is whether

- (1) T_{BX} is surjective implies that T_X is surjective,

$X = D, E$. Since $PBX(R)$ is dense in $PX(R)$ in the same fashion as $HBD(R)$ is dense in $HD(R)$ (cf. [1], [4]), it is natural to conjecture that the implication (1) holds. Surprisingly, in [12] and [7] it was shown that (1) is false for $X = D, E$. Even the stronger conditions $\int_R P < +\infty$, $\int_{R \times R} G_R(z, \zeta) P(z) P(\zeta) < +\infty$ do not imply the surjectiveness of T_E and T_D respectively as was shown in [8], [9], [10].

In this connection we raise the question whether the fact that (1) does not hold for $X = E$ by itself implies that (1) does not hold for $X = D$. This is closely related to the following: Is it true that $PBD(R) = PBE(R)$ implies that $PD(R) = PE(R)$? We shall show here that the answer to the latter question is no even under the stronger assumption that $PBD(R) = PBE(R) \cong HBD(R)$ which is a consequence of the surjectiveness of T_{BE} . Therefore the former question will also be settled in the negative.

1. We freely use the machinery of the Royden ideal boundary theory and the equation $d*du = uP$, as well as some of the techniques introduced in [2]. Let $W \subset R$ be an open set with ∂W consisting of analytic curves. The extremization operator μ_X^P , $X = D, E, BD, BE$, is the linear injection of the relative class $PX(W; \partial W)$ into $PX(R)$ characterized by the property that for each $u \in PX(W; \partial W)$ there is a potential p_u on R with $|u - \mu_X^P u| \leq p_u$. An alternate characterization of μ_X^P is the condition that $u - \mu_X^P u | \Delta = 0$ for each $u \in PX(W; \partial W)$. We also consider the reduction operator $T_{X,W}: PX(W) \cap \tilde{M}(R) \rightarrow HD(W)$, which can be characterized by the condition $u - T_{X,W}u | \partial W \cup bW = 0$, where $bW = (\overline{W} \cap \Delta) \setminus \partial W$. In particular, $T_{X,W}: PX(W; \partial W) \rightarrow HD(W; \partial W)$ is determined by $u - T_{X,W}u | bW = 0$.

In [2] we completely characterized the functions in $HD(R)$ that are in the image of μ_D . In view of the following result, which we will make essential use of here, the corresponding problem for $PX(R)$ is also settled.

THEOREM. *Let u be in $PX(R)$ and set $h = T_X u$. Then $u \in \mu_X^P(PX(W; \partial W))$ if and only if $h \in \mu_D(HD(W; \partial W))$, $X = D$ or E .*

The necessity is simple. Let $s \in PX(W; \partial W)$ such that $\mu_X^P s = u$. Define $v = T_{X,W} s$. Since $v | \Delta = s | \Delta = u | \Delta = h | \Delta$, we conclude that $h = \mu_D v$. Conversely, assume that $h = \mu_D v$, $v \in HD(W; \partial W)$. It suffices to establish the sufficiency in case $v \geq 0$. We begin by showing that the assertion holds for $X = D$. For each positive integer k , set $\psi_k = (h \cap k) \cup k^{-1}$ and $v_k = \prod_{R \setminus W} (\psi_k - k^{-1})$, the harmonic projection of $\psi_k - k^{-1}$ on W . The sequence $\{v_k\}$ is easily seen to have the following properties (cf. [2]): $v_k \in HBD(W; \partial W)$, $v_k \leq v_{k+1} \leq v$, $v = CD\text{-lim } v_k$, $\text{Supp}(v_k | \Delta) \subset \{p \in \Delta \mid v(p) > 0\}$, $\lim(v_k | \Delta) = v | \Delta$ and

$$(3) \quad D_W(v_k) \leq D_W(v).$$

In view of $v | \Delta = h | \Delta = u | \Delta$ and $u \in PD(R)$ we have that $v | \Delta \setminus \Delta_p = 0$. Thus $\text{Supp}(v_k | \Delta) \subset bW \cap \Delta_p$ and consequently there is a function $s_k \in PBD(W; \partial W)$ such that $s_k | \Delta = v_k | \Delta$. By the maximum principle, $s_k \leq s_{k+1} \leq v_{k+1} \leq v$ on R and then by the Harnack principle $s = C\text{-lim } s_k$ exists on W . Since $v | R \setminus W = 0$, we actually have $s = C\text{-lim } s_k$ on R with $s | R \setminus W = 0$, in particular, $s \in P(W; \partial W)$.

We show that in fact $s \in PD(W; \partial W)$. In view of the identity $D_W(u) = D_W(T_D, W u) + \langle u, u \rangle_W^P$ (cf. [5]), we have $\langle u, u \rangle_W^P < +\infty$. Comparing boundary values shows that $s_k \leq u$ on W . This together with (3) gives the following bound on $D_W(s_k)$:

$$D_W(s_k) = D_W(v_k) + \langle s_k, s_k \rangle_W^P \leq D_W(v) + \langle u, u \rangle_W^P < +\infty.$$

By Fatou's lemma we obtain $D_W(s) < +\infty$.

In view of $s \leq u$, we have $s | \Delta \leq u | \Delta$. On the other hand,

$$s | \Delta \geq \lim(s_k | \Delta) = \lim(v_k | \Delta) = v | \Delta = u | \Delta.$$

We conclude that $\mu_D^P s = u$, establishing the sufficiency in case $X = D$. If in addition $u \in PE(R)$, then $s \leq u$ gives $\int_W s^2 P \leq \int_W u^2 P < +\infty$, i.e. $s \in PE(W; \partial W)$, which completes the proof.

2. Let T be a hyperbolic Riemann surface such that $HBD(R)$ consists only of the constant functions. Examples of such surfaces are the Tôki surface and the Tôki covering surfaces (cf. [11]). The harmonic boundary of T consists of a single point which we denote by p^* . Fix $q_0 \in T$ and consider the polar coordinate differentials $(dr, d\theta)$ on T defined by

$$\frac{dr(z)}{r(z)} = -dG_T(z, q_0), \quad d\theta(z) = -*dG_T(z, q_0).$$

Fix α such that \bar{U}_α is homeomorphic to a closed disk, where $U_\alpha = \{p \in T \mid r(p) < e^{-\alpha}\}$. Also fix $\beta \leq \alpha/2$. For each $\lambda > 0$ define a rotation free 2-form $P_\lambda = \varphi_\lambda(r) r dr \wedge d\theta$, where φ_λ is the nonnegative Hölder continuous function on $[0, 1)$ determined by the following conditions:

$$\varphi_\lambda(t) = \begin{cases} (1-t)^{-\lambda}, & \text{if } e^{-\beta} \leq t < 1, \\ \text{linear}, & \text{if } e^{-2\beta} \leq t < e^{-\beta}, \\ 0, & \text{if } 0 \leq t < e^{-2\beta}. \end{cases}$$

According to the main results of [6] we have

$$P_\lambda BD(T) = \{0\} \text{ if and only if } \lambda \in [3/2, +\infty),$$

$$P_\lambda BE(T) = \{0\} \text{ if and only if } \lambda \in [1, +\infty).$$

By our choice of T here it follows that $\dim P_\lambda X(T) = \dim P_\lambda BX(T) \leq 1$ for $X = D, E$. Furthermore, it can be seen from [6] that

$$(4) \quad \int_T G_T(q_0, \cdot) P_\lambda < +\infty, \text{ if } \lambda < 2,$$

$$(5) \quad \langle 1, 1 \rangle_T^{P_\lambda} < +\infty, \text{ if } \lambda < \frac{3}{2},$$

$$(6) \quad \int_T P_\lambda < +\infty, \text{ if } \lambda < 1.$$

Set $W_\alpha = T \setminus \bar{U}_\alpha$. Then $P_\lambda X(W_\alpha; \partial W_\alpha) = P_\lambda BX(W_\alpha; \partial W_\alpha)$ is isometric to $P_\lambda X(T) = P_\lambda BX(T)$, $X = D, E$. So for each $\lambda \in (0, 1]$ we may choose $w_\lambda \in P_\lambda D(W_\alpha; \partial W_\alpha)$ with $w_\lambda(p^*) = 1$. Then w_λ spans $P_\lambda D(W_\alpha; \partial W_\alpha)$. For

$\lambda \in (0, 1)$, $w_\lambda \in P_\lambda E(W_\alpha; \partial W_\alpha)$ whereas $\int_{W_\alpha} w_1^2 P_1 = +\infty$. Define

$$\Phi(\lambda) = E_T^{P_\lambda}(w_\lambda).$$

LEMMA. Φ is a continuous function on $(0, 1)$ and $\liminf_{\lambda \uparrow 1} \Phi(\lambda) = +\infty$.

Fix $\lambda_0 \in (0, 1)$. We first show that Φ is continuous from the right at λ_0 . Let $\lambda_0 \leq \lambda < \nu < 1$. Since $P_\lambda \leq P_\nu$, we see that w_λ is a supersolution with respect to $d * du = u P_\nu$ and thus $w_\nu \leq w_\lambda \leq w_{\lambda_0} < 1$. The function $\hat{w} = \lim_{\lambda \downarrow \lambda_0} w_\lambda$ exists and $\hat{w} \leq w_{\lambda_0}$ on T . The Harnack inequality applied to (4) implies $\int_T G_T(\zeta, \cdot) P_1 < +\infty$, for any $\zeta \in W_\alpha$ and hence $\int_{W_\alpha} G_{W_\alpha}(\zeta, \cdot) P_1 < +\infty$. Set

$$\tau^{P_\lambda} \varphi(\zeta) = \frac{1}{2\pi} \int_{W_\alpha} G_{W_\alpha}(\zeta, z) \varphi(z) P_\lambda(z),$$

for a suitable function φ on W_α . Since $w_\lambda P_\lambda < P_1$ on W_α , the Lebesgue dominated convergence theorem gives $\lim_{\lambda \downarrow \lambda_0} \tau^{P_\lambda} w_\lambda = \tau^{P_{\lambda_0}} \hat{w}$. Note that for any λ , $T_{D, W_\alpha} w_\lambda$ is the same function $v \in HBD(W_\alpha; \partial W_\alpha)$. Therefore,

$$\begin{aligned} v &= w_{\lambda_0} + \tau^{P_{\lambda_0}} w_{\lambda_0} \geq \hat{w} + \tau^{P_{\lambda_0}} \hat{w} \\ &= \lim_{\lambda \uparrow \lambda_0} (w_\lambda + \tau^{P_\lambda} w_\lambda) = v \end{aligned}$$

on W_α . This implies that $w_{\lambda_0} = \hat{w}$ on W_α and consequently on T . By Dini's theorem we arrive at $w_{\lambda_0} = B\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$ on T .

We continue with $\lambda_0 \leq \lambda < \nu < 1$. Note that the function $w_\lambda - w_\nu$ is P_λ energy finite. Indeed, $w_\lambda - w_\nu$ is clearly Dirichlet finite and the inequality $0 \leq w_\lambda - w_\nu \leq w_\lambda$ gives $\int_T (w_\lambda - w_\nu)^2 P_\lambda \leq \int_T w_\lambda^2 P_\lambda < +\infty$. Since $w_\lambda - w_\nu$ vanishes at p^* we may choose a sequence $\{f_n\} \subset M_0(T)$ such that $w_\lambda - w_\nu = BE^{P_\lambda}\text{-}\lim f_n$. Moreover, the sequence $\{f_n\}$ may be chosen with $f_n|_{U_\alpha} = 0$ since $w_\lambda - w_\nu$ has this property. Thus

$$E_T^{P_\lambda}(w_\lambda - w_\nu, w_\lambda) = \lim_n E_{W_\alpha}^{P_\lambda}(f_n, w_\lambda) = 0$$

and consequently

$$\begin{aligned} 0 &\leq D_T(w_\nu - w_\lambda) \leq E_T^{P_\lambda}(w_\nu - w_\lambda) \\ &= E_T^{P_\lambda}(w_\nu) - E_T^{P_\lambda}(w_\lambda) \leq \Phi(\nu) - \Phi(\lambda). \end{aligned}$$

This shows that $\lim_{\lambda \downarrow \lambda_0} \Phi(\lambda)$ exists which in turn implies that $\{w_\lambda\}$ is D -Cauchy. By Kawamura's lemma we arrive at $w_{\lambda_0} = BD\text{-}\lim_{\lambda \downarrow \lambda_0} w_\lambda$, and in particular, $\lim_{\lambda \downarrow \lambda_0} D_T(w_\lambda) = D_T(w_{\lambda_0})$. By (6), $\int_T P_\nu < +\infty$ and we apply the Lebesgue dominated convergence theorem to obtain $\lim_{\lambda \downarrow \lambda_0} \int_T w_\lambda^2 P_\lambda = \int_T w_{\lambda_0}^2 P_{\lambda_0}$. This completes the proof of $\lim_{\lambda \downarrow \lambda_0} \Phi(\lambda) = \Phi(\lambda_0)$.

We now consider $\lambda_0 \in (0, 1]$ and show that $w_{\lambda_0} = BD\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$. Let $0 < \nu < \lambda \leq \lambda_0$ and note that $w_{\lambda_0} \leq w_\lambda < w_\nu < 1$. Thus $\lim_{\lambda \uparrow \lambda_0} w_\lambda$ exists and by an argument analogous to the one above we see that actually $w_{\lambda_0} = B\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$. Since $w_\nu - w_\lambda$ vanishes at p^* we can find a sequence $\{f_n\} \subset M_0(T)$ such that $w_\nu - w_\lambda = BD\text{-}\lim f_n$. We choose $\{f_n\}$ with the additional properties $f_n \geq 0, f_n|_{U_\alpha} = 0$. Thus

$$\begin{aligned} D_T(w_\nu - w_\lambda, w_\nu) &= \lim_n D_{W_\alpha}(f_n, w_\nu) \\ &= -\lim_n \int_{W_\alpha} f_n d * dw_\nu \leq 0, \end{aligned}$$

which implies that

$$0 \leq D_T(w_\lambda - w_\nu) \leq D_T(w_\lambda) - D_T(w_\nu).$$

Thus $D_T(w_\lambda)$ increases as λ increases and is bounded above by $D_T(w_{\lambda_0})$. Therefore $\{w_\lambda\}$ is D -Cauchy and by Kawamura's lemma $w_{\lambda_0} = BD\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$.

In case $\lambda_0 \in (0, 1)$, as before we see that $\lim_{\lambda \uparrow \lambda_0} \int_T w_\lambda^2 P_\lambda = \int_T w_{\lambda_0}^2 P_{\lambda_0}$. We arrive at $\lim_{\lambda \uparrow \lambda_0} \Phi(\lambda) = \Phi(\lambda_0)$ and the continuity of Φ at λ_0 is established. In case $\lambda_0 = 1$ we apply Fatou's lemma to conclude that $+\infty = \int_T w_1^2 P_1 \leq \liminf_{\lambda \uparrow 1} \int_T w_\lambda^2 P_\lambda \leq \liminf_{\lambda \uparrow 1} \Phi(\lambda)$.

3. Recall that the definition of P_λ involved a parameter β . We now adopt the notations $P_\lambda^{(\beta)}, w_\lambda^{(\beta)}$ to indicate the dependence of P_λ, w_λ on β . Set $a = D_{W_\alpha}(v)$, where v is the function in $HBD(W_\alpha; \partial W_\alpha)$ determined by $v(p^*) = 1$.

LEMMA. *Let b, c be given such that $a < b < c$. It is possible to choose $\beta \in (0, \alpha/2), \lambda \in (0, 1)$ such that*

$$(7) \quad D_{W_\alpha}(w_\lambda^{(\beta)}) < b,$$

$$(8) \quad E_{W_\alpha}^{P_\lambda^{(\beta)}}(w_\lambda^{(\beta)}) = c.$$

Note that for $\beta \leq \beta'$ we have $P_\lambda^{(\beta)} \leq P_\lambda^{(\beta')}$ and that $\lim_{\beta \downarrow 0} P_\lambda^{(\beta)} = 0$. Thus in view of (4), (5) we have

$$\lim_{\beta \downarrow 0} \langle 1, 1 \rangle_{W_\alpha}^{P_\lambda^{(\beta)}} = 0, \lim_{\beta \downarrow 0} \int_{W_\alpha} P_{1/2}^{(\beta)} = 0.$$

We therefore may choose β such that

$$\langle 1, 1 \rangle_{W_\alpha}^{P_\lambda^{(\beta)}} < \frac{b - a}{2}$$

and

$$(9) \quad \int_{W_\alpha} P_{1/2}^{(\beta)} < \frac{b - a}{2}.$$

For any $\lambda \in (0, 1]$ we have $T_{D, W_\alpha} w_\lambda^{(\beta)} = v$ and hence

$$\begin{aligned} D_{W_\alpha}(w_\lambda^{(\beta)}) &= D_{W_\alpha}(v) + \langle w_\lambda^{(\beta)}, w_\lambda^{(\beta)} \rangle_{W_\alpha}^{P(\beta)} \\ &\leq a + \langle 1, 1 \rangle_{W_\alpha}^{P(\beta)} < \frac{a + b}{2}, \end{aligned}$$

which shows that (7) holds for this β and any $\lambda \in (0, 1]$. By this and (9) we obtain

$$E_{W_\alpha}^{P_{1/2}^{(\beta)}}(w_{1/2}^{(\beta)}) = D_{W_\alpha}(w_{1/2}^{(\beta)}) + \int_{W_\alpha} (w_{1/2}^{(\beta)})^2 P_{1/2}^{(\beta)} < \frac{a + b}{2} + \frac{b - a}{2} = b.$$

In view of Lemma 2 we can choose $\lambda \in (\frac{1}{2}, 1)$ so that (8) also holds.

4. We use the notation v_α to indicate the dependence of the function $v \in HBD(W_\alpha; \partial W_\alpha)$ with $v(p^*) = 1$ on α . We claim that

$$(10) \quad D_{W_\alpha}(v_\alpha) = \frac{2\pi}{\pi}.$$

In fact, $v_\alpha | W_\alpha = 1 - \alpha^{-1} G_T(\cdot, q_0) | W_\alpha$ and hence (10) follows from the formula $D_{W_\alpha}(G_T(\cdot, q_0)) = 2\pi\alpha^{-1}$ (cf. [6]). Define $\alpha_n = 4^{n+1}\pi$, $n = 1, 2, \dots$. Then by (10) we have

$$(11) \quad D_{W_{\alpha_n}}(v_{\alpha_n}) = \frac{1}{2 \cdot 4^n}.$$

According to Lemma 3 we may choose λ_n, β_n such that

$$(12) \quad \delta_n = D_{W_{\alpha_n}}(w_{\lambda_n}^{(\beta_n)}) < \frac{1}{4^n}$$

and

$$(13) \quad \varepsilon_n = E_{W_{\alpha_n}}^{P_{\lambda_n}}(w_{\lambda_n}^{(\beta_n)}) = \frac{1}{2^n},$$

for $n = 1, 2, \dots$. Consider $W_{2\alpha_n} = \{p \in T \mid r(p) > e^{-2\alpha_n}\}$ and $v_{2\alpha_n} \in HBD(W_{2\alpha_n}; \partial W_{2\alpha_n})$ such that $v_{2\alpha_n}(p_n^*) = 1$. It can easily be seen that

$$(14) \quad v_{2\alpha_n} | \partial W_{\alpha_n} = \frac{1}{2}.$$

We prepare infinitely many copies T_n of T , $n = 1, 2, \dots$ and view $W_{2\alpha_n}$ as being a subsurface of T_n . Let $V = \mathbf{C} \setminus \bigcup_{1 \leq n < \infty} \{ |z - 3n| \leq 1 \}$. We weld $W_{2\alpha_n}$ to V by identifying $\partial W_{2\alpha_n}$ with $\{ |z - 3n| = 1 \}$, $n = 1, 2, \dots$ and let

R be the resulting Riemann surface. We now view $W_{\alpha_n}, W_{2\alpha_n}$ as sub-surfaces of R and denote them simply by W_n, U_n . We regard $v_{\alpha_n}, v_{2\alpha_n}$ as being defined on W_n, U_n and denote them by v_n, u_n . Let Δ be the harmonic boundary of R . Since $\dim HBD(W_n; \partial W_n) = 1$, $\overline{W_n} \cap \Delta$ consists of a single point p_n^* . Set $\Delta_1 = \{p_1^*, p_2^*, \dots\}$. The fact that $u_n|_{\partial W_n} = \frac{1}{2}$, $n = 1, 2, \dots$, i.e. (14), implies $\overline{\Delta_1} = \Delta$ (cf. [3]). Let $W = \bigcup_{1 \leq n < \infty} W_n$ and define v on R by $v|_{W_n} = v_n, n = 1, 2, \dots$ and $v|_{R \setminus W} = 0$. Then by (11) we see that $v \in HBD(W; \partial W)$. Since $v|_{\Delta_1} = 1$, we must have $v|_{\Delta} = 1$ and consequently $\overline{W} \setminus \partial \overline{W}$ is a neighborhood of Δ in R^* .

Define a 2-form P on R by

$$P|_{W_n} = P_{\lambda_n}^{(\beta_n)}, n = 1, 2, \dots \text{ and } P|_{R \setminus W} = 0.$$

We view $w_{\lambda_n}^{(\beta_n)}$ as a function on W_n and use the simplified notation w_n for it. In this notation (12) and (13) are written as

$$(15) \quad \delta_n = D_{W_n}(w_n) < \frac{1}{4^n},$$

$$(16) \quad \varepsilon_n = E_{W_n}^P(w_n) = \frac{1}{2^n},$$

$n = 1, 2, \dots$. For $X = D, E$ define measures m^{PX} on Δ by setting $m^{PX}(\Delta \setminus \Delta_1) = 0$ and

$$m^{PD}(p_n^*) = \delta_n, m^{PE}(p_n^*) = \varepsilon_n,$$

$n = 1, 2, \dots$. We denote the bounded continuous functions on Δ by $B(\Delta)$.

LEMMA. For $X = D$ or E

- (i) $PBX(W; \partial W)|_{\Delta} = B(\Delta)$,
- (ii) $PX(W; \partial W)|_{\Delta} = L^2(\Delta, m^{PX})$.

Since (i) is an easy consequence of (ii) we proceed directly to the proof of (ii). We consider only the case $X = E$ as $X = D$ is analogous. Let $s \in PE(W; \partial W)$. Then $+\infty > E_W^P(s) = \sum_1^\infty E_{W_n}^P(s)$. Recall that $PE(W_n; \partial W_n)$ is spanned by w_n . Thus $s|_{W_n} = a_n w_n$ with $a_n = s(p_n^*)$. We see by (16) that $E_{W_n}^P(s) = a_n^2 \varepsilon_n$ and hence $\{a_n\} \in L^2(\Delta, m^{PE})$. Conversely, if $\{a_n\} \in L^2(\Delta, m^{PE})$, then by (16) the function $s = \sum_1^\infty a_n w_n$ is in $PE(W; \partial W)$ with $s|_{\Delta} = a_n, m^{PE}$ -a.e.

5. We arrive at our main result.

THEOREM. *The 2-form P and the Riemann surface R have the property that*

$$PBE(R) = PBD(R) \text{ and } PE(R) \neq PD(R).$$

Since $\overline{W} \setminus \partial \overline{W}$ is a neighborhood of Δ , we see that μ_{BD} is surjective (cf. [8]). By Theorem 1 we see that μ_{BD}^P and μ_{BE}^P are surjective as well. From Lemma 4(i) we deduce that $PBD(W; \partial W) = PBE(W; \partial W)$. Thus the mapping $\mu_{BD}^P \circ (\mu_{BE}^P)^{-1}: PBE(R) \rightarrow PBD(R)$ is a bijection and the first part of the assertion follows.

Let f be defined on Δ , by $f(p_n^*) = 2^{n/2}$, $n = 1, 2, \dots$. By (15) and (16) we see that $f \in L^2(\Delta, m^{PD})$ but $f \notin L^2(\Delta, m^{PE})$. According to Lemma 4(ii) there is a function $s \in PD(W; \partial W)$ such that $s|_{\Delta} = f$, m^{PD} -a.e. Set $u = \mu_D^P s \in PD(R)$ and $h = T_D u$. By Theorem 1 we have $h \in \mu_D(HD(W; \partial W))$. If u were in $PE(R)$, then in view of $h = T_E u$ Theorem 1 would imply that $u \in \mu_E^P(PE(W; \partial W))$. But since $u|_{\Delta} \notin L^2(\Delta, m^{PE})$, Lemma 4(ii) rules out the possibility of u being in $\mu_E^P(PE(W; \partial W))$ and the assertion $u \notin PE(R)$ follows.

It is clear that there is a neighborhood V^* of Δ with $\int_{V^* \cap R} P < +\infty$ but we have not been able to determine whether $\int_R P < +\infty$. Thus the relation between $\int_R P < +\infty$ and $PE(R) = PD(R)$ remains open.

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PENNSYLVANIA STATE UNIVERSITY
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