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We give characterizations of closed, quasi-perfect, *d*-, *Z*-, *WZ*-, *W-open, *N*-, *WN*-, W_rN- and other maps using closed or open ultrafilters and investigate relations between these maps and various properties as generalizations of realcompactness, i.e., almost-, *a*-, *c*- and *wa*-real compactness, *cb**-ness and weak *cb**-ness. Finally we establish several theorems about the perfect *W**-open image of a weak *cb** space and its application to the absolute $E(X)$ of a given space X.**

We characterize closed, Z-, WZ-, N- and WN-maps by closed ultrafilters in §1 and show that φ is W^*-open iff $\varphi^{\#}\mathfrak{U}$ is an open ultrafilter for each open ultrafilter $\mathfrak{U}$ in §2. In §3, introducing the notion of *-open map, we show that $\beta\varphi$ is open iff φ is a *-open W_rN-map iff there is $\mathfrak{U}^p$ with $\varphi^{\#}\mathfrak{U}^p = \mathfrak{V}^q$ for each $q \in \beta Y$, each $\mathfrak{V}^q$ and each $p \in (\beta\varphi)^{-1}q$. In §4, we discuss invariance concerning CIP of closed or open ultrafilters under various maps and establish invariances and inverse invariance of various properties as a generalization of realcompactness under suitable maps in §5. In §6, we give several theorems about the perfect W^*-open image of weak cb^* spaces which contain, as corollaries, known results concerning the absolute $E(X)$ of X.

Throughout this paper, by a space we mean a completely regular Hausdorff space and assume familiarity with [3] whose notion and terminology will be used throughout. We denote by $\varphi\colon X \to Y$ a continuous onto map and by $\beta X(\upsilon X)$ the Stone-Čech compactification (realcompactification) of X and by $\beta\varphi$ the Stone extension over βX of φ. In the sequel, we use the following notation and abbreviation. $N =$ the set of positive integers, CIP $=$ countable intersection property, nbd $=$ neighborhood, $\mathfrak{F}^p =$ a closed ultrafilter converging to p. We denote by $\mathfrak{F}(\mathfrak{U})$ a closed (open) ultrafilter on X and by $\mathfrak{G}(\mathfrak{V})$ a closed (open) ultrafilter on Y. $\varphi^{\#}\mathfrak{F} = \{E \subset Y;\ \varphi^{-1}E \in \mathfrak{F}$ and E is closed in $Y\}$. Similarly define $\varphi^{\#}\mathfrak{U}$.

1. Closed ultrafilters.

1.1. In the sequel, we use frequently the following results.

(1) *If* $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^q = \bigcap\{\mathrm{cl}_{\beta X}\varphi^{-1}E;\ E \in \mathfrak{G}^q\}$, *then there is* $\mathfrak{F}^p$ *with* $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^q$. For, $\mathfrak{A} = \{\varphi^{-1}E \cap F;\ E \in \mathfrak{G}^q,\ F \in N(p)\}$ is a closed filter base where $N(p)$ is a closed nbd base of p in βX. Obviously $\mathfrak{A} \to p$. Thus any $\mathfrak{F}^p$ containing $\mathfrak{A}$ has the property $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^q$. It is easily seen that the same method above can be applied to open ultrafilter and

Z-ultrafilter respectively i.e., if $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{V}^q(\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{Z}^q)$, there is $\mathcal{U}^p(\mathcal{Z}^p)$ with $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q(\varphi^{\#}\mathcal{Z}^p = \mathcal{Z}^q)$.

(2) For $x \in X$, a closed ultrafilter $\mathcal{F}$ converging to x is unique and $\mathcal{F} = \{F;\ x \in F$ and F is closed$\}$. Obviously $\{x\} \in \mathcal{F}$. It is easy to see that X is normal iff for each $p \in \beta X$, a closed ultrafilter $\mathcal{F}$ converging to p is unique and $\mathcal{F} = \{F;\ p \in \mathrm{cl}_{\beta X}F$ and F is closed$\}$.

(3) For $p \in \beta X$, a Z-ultrafilter $\mathcal{Z}^p$ is unique and $\mathcal{Z}^p = \{Z;\ Z$ is a zero set and $p \in \mathrm{cl}_{\beta X}Z\}$.

1.2. *Let* $\varphi\colon X \to Y$, $(\beta\varphi)p = q$, $p \in \beta X$ *and* $q \in \beta Y$.

(1) $\bigcap \mathrm{cl}_{\beta Y}\varphi^{\#}\mathcal{F}^p = \{q\}$.

(2) $\varphi^{-1}\mathcal{E}^q \subset \mathcal{F}^p \Leftrightarrow \varphi^{\#}\mathcal{F}^p = \mathcal{E}^q$.

(3) $\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{E}^q \subset (\beta\varphi)^{-1}q$.

(4) $\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{E}^y = \mathrm{cl}_{\beta X}\varphi^{-1}y$ *for* $y \in Y$.

(5) $\varphi^{\#}\mathcal{F}^p \subset \mathcal{E}^q \Leftrightarrow \mathrm{cl}(\varphi F) \cap E \neq \varnothing$ *for* $F \in \mathcal{F}^p$ *and* $E \in \mathcal{E}^q$.

(6) *There is* $\mathcal{F}^p$ *such that* $\varphi^{\#}\mathcal{F}^p$ *is a closed ultrafilter iff there is* $\mathcal{E}^q$ *with* $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{E}^q$.

Proof. (1) It suffices to show that $\bigcap \mathrm{cl}_{\beta Y}\varphi^{\#}\mathcal{F}^p$ consists of only one point. Let $q_i \in \bigcap \mathrm{cl}_{\beta Y}\varphi^{\#}\mathcal{F}^p$ $(i = 1, 2)$. Then there are disjoint closed nbd's V_1 and V_2 of q_1 and q_2 in βY respectively, so $X \cap (\beta\varphi)^{-1}V_i \in \mathcal{F}^p$ $(i = 1, 2)$, a contradiction.

(2) Obvious.

(3) If $r \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{E}^q - (\beta\varphi)^{-1}q$, there is $\mathcal{F}^r$ with $\varphi^{-1}\mathcal{E}^q \subset \mathcal{F}^r$ by 1.1(1) and (2) above. This shows $(\beta\varphi)^{-1}q \ni r$, a contradiction.

(4) From $\{y\} \in \mathcal{E}^y$.

(5) $\Rightarrow$). From $\mathrm{cl}(\varphi F) \in \varphi^{\#}F^p$ for $F \in \mathcal{F}^p$. $\Leftarrow$). Let $K \in \varphi^{\#}\mathcal{F}^p - \mathcal{E}^q$. Then $\mathcal{F} = \varphi^{-1}K \in \mathcal{F}^p$. Since $K \notin \mathcal{E}^q$, there is $E \in \mathcal{E}^q$ with $K \cap E = \varnothing$, i.e., $\mathrm{cl}(\varphi F) \cap E = \varnothing$, a contradiciton.

(6) $\Rightarrow$). Let $\mathcal{E}^q = \varphi^{\#}\mathcal{F}^p$. Then $\varphi^{-1}\mathcal{E}^q \subset \mathcal{F}^p$, so $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{E}^q$. $\Leftarrow$). From 1.1(1).

1.3. Definition. We recall that $\varphi\colon X \to Y$ is a *Z-map* if φZ is closed for every zero set Z and φ is a *WZ-map* if $(\beta\varphi)^{-1}y = \mathrm{cl}_{\beta X}\varphi^{-1}y$ for each $y \in Y$. It is known that a closed map is a Z-map and a Z-map is WZ [**12**]. Woods [**21**] introduced the notions of N- and WN-map. φ is an *$N(WN)$-map* if $(\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}R = \mathrm{cl}_{\beta X}\varphi^{-1}R$ for every closed set (zero set) R of Y. An N-map is WN and WZ. In the following, we characterize maps mentioned above by closed ultrafilters.

Theorem 1.4. *Let* $\varphi\colon X \to Y$.

(1) φ *is* WZ *iff there is* $\mathcal{F}^p$ *with* $\varphi^{\#}\mathcal{F}^p = \mathcal{E}^y$ *for each* $y \in Y$ *and each* $p \in (\beta\varphi)^{-1}y$.

(2) φ *is a* Z*-map iff there is* $\mathcal{F}^p$ *such that* $Z \in \mathcal{F}^p$ *and* $\varphi^{\#} = \mathcal{E}^y$ *for each* $y \in Y$, *each* $p \in (\beta\varphi)^{-1}y$ *and each zero set* Z *with* $p \in \mathrm{cl}_{\beta X}Z$.

(3) *The following are equivalent*:

(i) φ *is closed.*

(ii) $\varphi^{\#}\mathfrak{F}$ *is a closed ultrafilter for any* $\mathfrak{F}$.

(iii) *There is* $\mathfrak{F}^p$ *such that* $F \in \mathfrak{F}^p$ *and* $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^y$ *for each* $y \in Y$, *each* $p \in (\beta\varphi)^{-1}$ *and each closed set* F *with* $p \in \mathrm{cl}_{\beta X} F$.

(4) *The following are equivalent*:

(i) φ *is an* N*-map.*

(ii) $(\beta\varphi)^{-1}q = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^q$ *for each* $q \in \beta Y$ *and each* $\mathfrak{G}^q$.

(iii) *There is* $\mathfrak{F}^p$ *with* $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^q$ *for each* $q \in \beta Y$, *each* $\mathfrak{G}^q$ *and each* $p \in (\beta\varphi)^{-1}q$.

(5) *The following are equivalent*:

(i) φ *is a* WN*-map.*

(ii) $\mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{Z}^q = (\beta\varphi)^{-1}q$ *for each* $q \in \beta Y$.

(iii) $\varphi^{\#}\mathfrak{Z}^p = \mathfrak{Z}^q$ *for each* $q \in \beta Y$ *and each* $p \in (\beta\varphi)^{-1}q$.

Proof. (1) $\Rightarrow$). Since φ is WZ, we have $(\beta\varphi)^{-1}y = \mathrm{cl}_{\beta X}\varphi^{-1}y$ and $(\beta\varphi)^{-1}y = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^y$ by 1.2(4). Thus there is $\mathfrak{F}^p$ with $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^y$ by 1.1(1) $\Leftarrow$). For each $p \in (\beta\varphi)^{-1}y$, we have $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^y$ by 1.2(6). Since $\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^y = \mathrm{cl}_{\beta X}\varphi^{-1}y$ by 1.2(4), $(\beta\varphi)^{-1}y \subset \mathrm{cl}_{\beta X}\varphi^{-1}y$, so $(\beta\varphi)^{-1}y = \mathrm{cl}_{\beta X}\varphi^{-1}y$ which shows that φ is WZ.

(2) $\Rightarrow$). Let $p \in (\beta\varphi)^{-1}y$ and Z a zero set with $p \in \mathrm{cl}_{\beta X}Z$. Since φ is a Z-map, φZ is closed, so $y \in \varphi Z$. On the other hand, $\varphi^{-1}y = X \cap (\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^y)$ by 1.2(4). If $p \in X$, then there is $\mathfrak{F}^p$ with $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^y$ by 1.2(6) and since $p \in X$, $p \in Z$ so $Z \in \mathfrak{F}^p$. Now suppose $p \notin X$. Since $y \in E$ for $E \in \mathfrak{G}^y$ and $\varphi Z \ni y$, we have $Z \cap \varphi^{-1}E \neq \varnothing$. We shall show $p \in \bigcap \mathrm{cl}_{\beta X}(Z \cap \varphi^{-1}E)$ for $E \in \mathfrak{G}^y$. Suppose contrary. There is a zero set K of βX such that $p \in \mathrm{int}_{\beta X}K$ and $K \cap \mathrm{cl}_{\beta X}(Z \cap \varphi^{-1}E) = \varnothing$. $Z' = K \cap Z \neq \varnothing$ and $p \in \mathrm{cl}_{\beta X}Z'$, but $y \notin \varphi Z'$, a contradiction. Thus there is $\mathfrak{F}^p \supset \{Z \cap \varphi^{-1}E;\ E \in \mathfrak{G}^y\}$ by 1.1(1). Obviously $\varphi^{-1}\mathfrak{G}^y \subset \mathfrak{F}^p$, so $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^y$ and $Z \in \mathfrak{F}^p$. $\Leftarrow$). Let Z be a zero set and $y \in \mathrm{cl}\,\varphi Z - \varphi Z$. Then we have $p \in \mathrm{cl}_{\beta X}Z \cap (\beta\varphi)^{-1}y$, so there is $\mathfrak{F}^p$ with $Z \in \mathfrak{F}^p$ and $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^y$. Since $\{y\} \in \mathfrak{G}^y$, $\varphi^{-1}y \in \mathfrak{F}^p$, but $Z \cap \varphi^{-1}y = \varnothing$, a contradiction.

(3) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Evident. (iii) $\Rightarrow$ (i). Suppose that there is a closed set F of X with $y \in \mathrm{cl}(\varphi F) - \varphi F$. Then $K = \mathrm{cl}_{\beta X}F \cap (\beta\varphi)^{-1}y \neq \varnothing$. Let $p \in K$. By (iii), there is $F \in \mathfrak{F}^p$ with $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^y$. Since $\{y\} \in \mathfrak{G}^y$ and $F \in \mathfrak{F}^p$, we have $F \cap \varphi^{-1}y \neq \varnothing$ which is a contradiction.

(4) (i) $\Rightarrow$ (ii). Since φ is an N-map and $q \in \mathrm{cl}_{\beta Y}E$ for each $E \in \mathfrak{G}^q$, we have $(\beta\varphi)^{-1}q \subset \bigcap(\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}\mathfrak{G}^q = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^q$, and hence $(\beta\varphi)^{-1}q = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{G}^q$ by 1.2(3). (ii) $\Rightarrow$ (iii). From (ii) and 1.2(6). (iii) $\Rightarrow$ (i). Suppose that there is a closed set E of Y with $K = (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}E - \mathrm{cl}_{\beta X}\varphi^{-1}E \neq \varnothing$. Let $p \in K$ and $(\beta\varphi)p = q$. Then $q \in \mathrm{cl}_{\beta Y}E$. Let $E \in \mathfrak{G}^q$. Take $\mathfrak{F}^p$ with $\varphi^{\#}\mathfrak{F}^p = \mathfrak{G}^q$. Since $p \notin \mathrm{cl}_{\beta X}\varphi^{-1}E$, we have $\varphi^{-1}E \notin \mathfrak{F}^p$, a contradiction.

(5) This is proven by the analogous method used in (4) above.

2. Open ultrafilters.

2.1. *Let* $g\colon X \to Y$ *and* $(\beta\varphi)p = q, p \in \beta X, q \in \beta Y$.

(1) $\bigcap \mathrm{cl}_{\beta Y}\varphi^{\#}\mathfrak{U}^p = \bigcap \mathrm{cl}_{\beta Y}\varphi\mathfrak{U}^p = \{q\}$.

(2) $\varphi^{-1}\mathfrak{V}^q \subset \mathfrak{U}^p \Leftrightarrow \varphi^{\#}\mathfrak{U}^p = \mathfrak{V}^q$.

(3) $\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{V}^q \subset \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}(\mathrm{cl}\,\mathfrak{V}^q) \subset (\beta\varphi)^{-1}q$.

(4) $\varphi^{\#}\mathfrak{U}^p \subset \mathfrak{V}^q \Leftrightarrow \varphi U \cap \mathrm{cl}\,V \neq \varnothing$ *for* $U \in \mathfrak{U}^p$ *and* $V \in \mathfrak{V}^q$.

(5) *There is* $\mathfrak{U}^p$ *such that* $\varphi^{\#}\mathfrak{U}^p$ *is an open ultrafilter iff there is* $\mathfrak{V}^q$ *with* $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathfrak{V}^q$.

The proof of 2.1 is obtained by the same method used in 1.2. By 1.1(1), "if part" of 2.1(5) implies that there is $\mathfrak{U}^p$ with $\varphi^{\#}\mathfrak{U}^p = \mathfrak{V}^q$. As is shown by 2.2 below, it is not necessarily true that if there is $\mathfrak{V}^q$ with $p \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}(\mathrm{cl}\,\mathfrak{V}^q)$, then there is $\mathfrak{U}^p$ with $\varphi^{\#}\mathfrak{U}^p = \mathfrak{V}^q$.

EXAMPLE 2.2. Let $X = [0, 1) \oplus [1, 2]$ and $Y = [0, 2]$. Define $\varphi\colon X \to Y$ by $\varphi(x) = x$ for $x \in X$. Let $\mathfrak{V}^q \ni [0, 1)$, $q = 1 \in Y$. Then $p = 1 \in \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}(\mathrm{cl}\,\mathfrak{V}^q)$ and any $\mathfrak{U}^p$ contain $(1, 2]$ and hence $\varphi^{\#}\mathfrak{U}^p \neq \mathfrak{V}^q$ (cf. 3.1 below).

LEMMA 2.3. *Let* $\varphi^{\#}\mathfrak{U}^p \subset \mathfrak{V}^q$, $U \in \mathfrak{U} = \mathfrak{U}^p$, $V \in \mathfrak{V}^q = \mathfrak{V}$ *and let us put* $B(U, V) = U \cap \varphi^{-1}(\mathrm{cl}\,V)$. *Then we have*

(1) Int $B(U, V) \in \mathfrak{U}$.

(2) *If* $\varphi^{\#}\mathfrak{U} \subsetneqq \mathfrak{V}$ *and* $V \cap \varphi U = \varnothing$, *then* $\mathrm{int\,cl}(\mathrm{cl}\,V \cap \varphi U) = \varnothing$.

(3) *If* $\varphi^{\#}\mathfrak{U} = \mathfrak{V}$, *then* $\mathrm{int\,cl}(\varphi U) \in \mathfrak{V}$.

Proof. (1). By 2.1(4), $B(U, V) \neq \varnothing$. Suppose $S = \mathrm{int}\,B(U, V) = \varnothing$. $U - B(U, V)$ is open in U, so in X. Since $(X - \mathrm{cl}\,U) \cup (U - B(U, V))$ is dense in X and $\mathfrak{U}$ is prime, we have $U - B(U, V) \in \mathfrak{U}$. But $\varphi^{-1}\,\mathrm{cl}\,V \cap (U - B(U, V)) = \varnothing$, and hence $\mathrm{cl}\,V \cap \varphi(U - B(U, V)) = \varnothing$, a contradiction by 2.1(4). Thus $S \neq \varnothing$. If $S \notin \mathfrak{U}$, there is $W \in \mathfrak{U}$ with $W \cap S = \varnothing$. This implies $S \cap W = \mathrm{int}(U \cap \varphi^{-1}(\mathrm{cl}\,V) \cap W) = \mathrm{int}(U \cap W \cap \varphi^{-1}(\mathrm{cl}\,V)) = \mathrm{int}\,B(U \cap W, V) = \varnothing$, a contradiction.

(2) Since $V \cap \varphi U = \varnothing$ implies $V \cap \mathrm{cl}(\varphi U) = \varnothing$, we have

$$\mathrm{cl}(\varphi U \cap \mathrm{cl}\,V) \subset \mathrm{cl}\,\varphi U \cap \mathrm{cl}\,V \subset \mathrm{cl}(\varphi U) \cap (\mathrm{cl}\,V - V),$$

so $\mathrm{int\,cl}(\varphi U \cap \mathrm{cl}\,V) = \varnothing$.

(3) If $\mathrm{int\,cl}\,\varphi U \notin \mathfrak{V}$, we have $Y - \mathrm{cl}\,\varphi U \in \mathfrak{V}$, so $X - \varphi^{-1}\,\mathrm{cl}(\varphi U) \in \mathfrak{U}$, a contradiction.

THEOREM 2.4. $\varphi^{\#}\mathfrak{U}^p$ *is an open ultrafilter iff* $\mathrm{int\,cl}(\varphi U) \neq \varnothing$ *for* $U \in \mathfrak{U}^p$.

Proof. $\Rightarrow$) Let $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$. Then this follows from 2.3(3). $\leftarrow$). Suppose $\varphi^{\#}\mathcal{U}^p \subsetneq \mathcal{V}^q$ for some $q \in \beta Y$. Put $\mathcal{U} = \mathcal{U}^p$ and $\mathcal{V} = \mathcal{V}^q$. There is $V \in \mathcal{V} - \varphi^{\#}\mathcal{U}$ with $V \cap \varphi U = \varnothing$ for some $U \in \mathcal{U}$. By 2.3(1), $W = \operatorname{int} B(U, V) \in \mathcal{U}$ and $\varphi W \cap V = \varnothing$, so $\operatorname{int}\operatorname{cl}(\varphi W) = \varnothing$ by 2.3(2), a contradiction.

2.5. Definition. φ: $X \to Y$ is said to be a *W^*-open* map if $\operatorname{cl}\varphi U$ is regular closed (i.e., $\operatorname{cl}(\operatorname{int}(\operatorname{cl}\varphi U)) = \operatorname{cl}\varphi U$) whenever U is open [**8**]. This is a generalization of an open map. We use this notion in the following.

Theorem 2.6. *Let φ: $X \to Y$. The following are equivalent:*

(1) *φ is W^*-open.*

(1′) Cl φU *is regular closed whenever U is a basic open set of X.*

(2) Int(cl φU) $\neq \varnothing$ *for each non-empty open set U of X.*

(2′) Int(cl φU) $\neq \varnothing$ *for each non-empty basic open set U of X.*

(3) $\operatorname{Int}(\operatorname{cl}\varphi^{-1}V) = \operatorname{int}\varphi^{-1}(\operatorname{cl}V)$ *for each open set V of Y.*

(4) $\varphi^{\#}\mathcal{U}$ *is an open ultrafilter for any $\mathcal{U}$.*

(5) *There is $\mathcal{U}^p$ such that $\varphi^{\#}\mathcal{U}^p$ is an open ultrafilter for each $q \in \beta Y$ and each $p \in (\beta\varphi)^{-1}q$.*

(6) $(\beta\varphi)^{-1}q = \bigcup\{\bigcap \operatorname{cl}_{\beta X}\varphi^{-1}\mathcal{V};\ \mathcal{V}$ *is an open ultrafilter converging to $q\}$ for each $q \in \beta Y$.*

Proof. (1) $\Rightarrow$ (1′) $\Rightarrow$ (2′) $\Leftrightarrow$ (2) and (4) $\Rightarrow$ (5) are evident. (2) $\Leftrightarrow$ (4). From 2.4 (5) $\Leftrightarrow$ (6). From 2.1(3, 5).

(2) $\Rightarrow$ (3). It suffices to show $\operatorname{int}\varphi^{-1}\operatorname{cl}V \subset \operatorname{cl}(\varphi^{-1}V)$. Suppose $x \in \operatorname{int}\varphi^{-1}(\operatorname{cl}V) - \operatorname{cl}(\varphi^{-1}V)$. There is an open set $W \ni x$ such that $W \cap \operatorname{cl}(\varphi^{-1}V) = \varnothing$ and $W \subset \operatorname{int}\varphi^{-1}(\operatorname{cl}V)$. Then $V \cap \varphi W = \varnothing$, so $V \cap \operatorname{cl}\varphi W = \varnothing$. On the other hand, $\varphi W \subset \operatorname{cl}V$, so $\operatorname{int}(\operatorname{cl}\varphi W) \subset \operatorname{cl}V - V$ and hence $\operatorname{int}\operatorname{cl}(\varphi W) = \varnothing$, a contradiction.

(5) $\Rightarrow$ (2). Let $U \subset X$ be open and $x \in U$. Then any open ultrafilter $\mathcal{U}$ converging to x contains U. There is $\mathcal{U}^x$ such that $\varphi^{\#}\mathcal{U}^x$ is an open ultrafilter by (5). Thus $\operatorname{int}\operatorname{cl}\varphi U \neq \varnothing$ by 2.4.

(3) $\Rightarrow$ (2). Suppose that there is an open set U with $\operatorname{int}\operatorname{cl}\varphi U = \varnothing$. Let us put $V = Y - \operatorname{cl}\varphi U$. Then $\operatorname{cl}V = Y$ and $\operatorname{int}\varphi^{-1}(\operatorname{cl}V) = X$. But $\operatorname{int}(\operatorname{cl}\varphi^{-1}V) \cap U = \varnothing$, a contradiction.

(2) $\Rightarrow$ (1). Let U be open and put $K = \operatorname{cl}\operatorname{int}(\operatorname{cl}\varphi U)$. Suppose $y \in \varphi U - K$. Then there is an open set $W \ni y$ with $K \cap \operatorname{cl}W = \varnothing$. Since $T = U \cap \varphi^{-1}W \neq \varnothing$ and $\operatorname{cl}\varphi T \subset \operatorname{cl}W \cap \operatorname{cl}\varphi U$, $\operatorname{int}\operatorname{cl}(\varphi T) \subset \operatorname{int}(\operatorname{cl}W) \cap \operatorname{int}(\operatorname{cl}\varphi U) = \varnothing$, a contradiction. Thus $\varphi U \subset K$ and hence $\operatorname{cl}\varphi U \subset K$, i.e., $\operatorname{cl}\varphi U = K$.

3. *-open mappings.

3.1. Definition. $\varphi: X \to Y$ is said to be **-open* if $\operatorname{int}(\operatorname{cl} \varphi U) \supset \varphi U$ for each open set U of X. An open map is *-open but a *-open map is not necessarily open by 3.2 below. A *-open map is W^*-open by 2.6 but a W^*-open map is not necessarily *-open by 2.2 in which it is easy to see that φ is W^*-open. Let $U = [1,2] \subset X$. Then U is open in X and $\operatorname{int}(\operatorname{cl} \varphi U) = (1,2] \not\supset \varphi U = [1,2]$, so φ is not *-open (cf. 5.6 below). We say that φ is a *W_rN-map* if $\operatorname{cl}_{\beta X} \varphi^{-1} R = (\beta\varphi)^{-1} \operatorname{cl}_{\beta Y} R$ for every regular closed set R of Y **[10]**. X is *almost normal* **[17]** (*κ-normal* **[16]**) if each regular closed set is completely separated from each closed (regular closed) set disjoint from it.

Example 3.2. Let P be the set of rational numbers in $[0,1]$, $X = [0,1] \oplus P$, $Y = [0,1]$ and $\varphi(x) = x \in Y$ for each $x \in X$. Then φ is not open. To show that φ is *-open, it suffices to prove that $\operatorname{int}(\operatorname{cl} \varphi U) \supset \varphi U$ for each open set U of P. Let $U \subset P$ be open. There is an open set $W \subset [0,1]$ with $P \cap W = U$. W is the union of countably many disjoint open interval $W_n = (a_n, b_n)$. Put $P_n = P \cap W_n$. Obviously $\operatorname{cl} \varphi P_n = [a_n, b_n]$ and $\operatorname{int}(\operatorname{cl} \varphi P_n) \supset P_n$, so $\operatorname{int}(\operatorname{cl} \varphi U) \supset \varphi U$, i.e., φ is *-open.

Theorem 3.3. *Let $\varphi: X \to Y$. The following are equivalent*:

(1) *φ is *-open.*

(2) $\operatorname{Cl} \varphi^{-1} V = \varphi^{-1} \operatorname{cl} V$ *for each open set V of Y.*

(3) $\bigcap \operatorname{cl}_{\beta X} \varphi^{-1} \mathcal{V}^y \supset \operatorname{cl}_{\beta X} \varphi^{-1} y$ *for each $y \in Y$ and each $\mathcal{V}^y$.*

(4) *There is $\mathcal{U}^p$ with $\varphi^{\#} \mathcal{U}^p = \mathcal{V}^y$ for each $y \in Y$, each $p \in \operatorname{cl}_{\beta X} \varphi^{-1} y$ and each $\mathcal{V}^y$.*

Proof. (1) $\Rightarrow$ (2). Suppose that there is an open set V of Y with $x \in \varphi^{-1} \operatorname{cl} V - \operatorname{cl} \varphi^{-1} V$. Take an open set $W \ni x$ disjoint from $\operatorname{cl} \varphi^{-1} V$. Since $V \cap \operatorname{cl} \varphi W = \varnothing$ and φ is *-open, we have $\operatorname{int}(\operatorname{cl} \varphi W) \cap \operatorname{cl} V = \varnothing$ and $\operatorname{int}(\operatorname{cl} \varphi W) \supset \varphi W \ni \varphi(x)$, a contradiction.

(2) $\Rightarrow$ (3). Take $\mathcal{V}^y$. Since $\operatorname{cl}_{\beta X} \varphi^{-1} V = \operatorname{cl}_{\beta X} \varphi^{-1}(\operatorname{cl} V)$ and $y \in \operatorname{cl} V$ for $V \in \mathcal{V}^y$, we have $\varphi^{-1} y \subset \bigcap \operatorname{cl}_{\beta X} \varphi^{-1} \mathcal{V}^y$, so $\operatorname{cl}_{\beta X} \varphi^{-1} y \subset \bigcap \operatorname{cl}_{\beta X} \varphi^{-1} \mathcal{V}^y$.

(3) $\Rightarrow$ (4). From 2.1(5).

(4) $\Rightarrow$ (1). Suppose that there is an open set U with $x \in U$ and $y = \varphi(x) \in \varphi U - \operatorname{int}(\operatorname{cl} \varphi U)$. Let $W \ni y$ be open. Then $V = W \cap (Y - \operatorname{cl} \varphi U) \neq \varnothing$, $y \notin V$ and $y \in \operatorname{cl} V$. Take $\mathcal{V}^y \ni V$. Any $\mathcal{U}^x$ contains U. If $\varphi^{\#} \mathcal{U}^x = \mathcal{V}^y$ for some $\mathcal{U}^x$, then $\varphi^{-1} V \in \mathcal{U}^x$, but $\varphi^{-1} V \cap U = \varnothing$, a contradiction.

In general the equality in 3.3(3) does not hold by 3.8 below. From the definition of a WZ-map, 2.1(3) and 3.3(3) we have

COROLLARY 3.4. *If* φ: $X \to Y$ *is* *-*open* WZ, *then* $(\beta\varphi)^{-1}y = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{V}^y$ *for each* $y \in Y$ *and each* $\mathcal{V}^y$.

EXAMPLE 3.5. We give an example which shows that the converse of 3.4 is not necessarily true. Let $X = [0, \omega_1] \oplus [0, \omega_1)$, $Y = [0, \omega_1]$ and $\varphi(x) = x \in Y$ for each $x \in X$ where ω_1 is the first uncountable ordinal. Then φ is open but not WZ. It is easy to see $(\beta\varphi)^{-1}y = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{V}^y$ for each $y \in Y$ and each $\mathcal{V}^y$.

THEOREM 3.6. φ: $X \to Y$ *is* W_rN *iff* $(\beta\varphi)^{-1}q = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,\mathcal{V}^q$ *for each* $q \in \beta Y$ *and each* $\mathcal{V}^q$.

Proof. $\Rightarrow$). Since $\mathrm{cl}_{\beta X}(\varphi^{-1}\,\mathrm{cl}\,V) = (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}V$ for $V \in \mathcal{V}^q$, $(\beta\varphi)^{-1}q \subset \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,\mathcal{V}^q$, so we have the equality by 2.1(3). $\leftarrow$). Let $p \in (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}V - \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,V$ for some open set V of Y. Then $p \in (\beta\varphi)^{-1}q$ for some $q \in \mathrm{cl}_{\beta Y}V$. Take $\mathcal{V}^q$ with $V \in \mathcal{V}^q$. Then $\mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,V \not\supset (\beta\varphi)^{-1}q$, a contradiction.

THEOREM 3.7. (1) *The following are equivalent* ([**10**], *Theorems* 1 *and* 6):

(i) *Y is almost normal.*
(ii) *Any WZ-map onto Y is W_rN.*
(iii) *Any perfect map onto Y is W_rN.*

(2) *The following are equivalent*:

(i) *Y is κ-normal.*
(ii) *Any W^*-open WZ-map onto Y is W_rN.*
(iii) *Any W^*-open perfect map onto Y is W_rN.*

Proof. (2) (i) $\Rightarrow$ (ii). Let φ: $X \to Y$ be W^*-open and WZ. Suppose $p \in (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}V - \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,V$ for some open set V of Y. Then $(\beta\varphi)p = q \in \mathrm{cl}_{\beta Y}V$ and take an open set W of βX such that $p \in W$ and $\mathrm{cl}_{\beta X}W \cap \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,V = \varnothing$. Since φ is W^*-open and WZ, we have that $(\beta\varphi)\,\mathrm{cl}_{\beta X}W \cap \mathrm{cl}\,V = \varnothing$ and $\mathrm{cl}\,\varphi(X \cap W)$ is regular closed. Thus $\mathrm{cl}\,\varphi(X \cap W) \cap \mathrm{cl}\,V = \varnothing$, and hence $\mathrm{cl}_{\beta Y}\varphi(X \cap W) \cap \mathrm{cl}_{\beta Y}V = \varnothing$ because Y is κ-normal. On the other hand, $\mathrm{cl}_{\beta X}(X \cap W) = \mathrm{cl}_{\beta X}W \ni p$, so $q \in \mathrm{cl}_{\beta X}\varphi(X \cap W) \cap \mathrm{cl}_{\beta Y}V$, a contradiction. (ii) $\Rightarrow$ (iii). Evident.

(iii) $\Rightarrow$ (i). This follows from the same method used in 1.5 of [**21**]. Suppose that there are disjoint regular closed sets E and K such that $\mathrm{cl}_{\beta Y}E \cap \mathrm{cl}_{\beta Y}K \ni q$. Let $X = Y \oplus E$. Define φ: $X \to Y$ by $\varphi(x) = x$ for $x \in X$. It is evident that φ is W^*-open perfect. On the other hand, $\mathrm{cl}_{\beta X}\varphi^{-1}K = \mathrm{cl}_{\beta Y}K$ and $(\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}K \cap \beta E \neq \varnothing$, so $(\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y}K \neq \mathrm{cl}_{\beta X}\varphi^{-1}K$ which shows that φ is not W_rN.

EXAMPLE 3.8. In 3.7(2, ii), "WZ-ness of φ" is necessary as shown by the following. Let $Y = [0,3]$, $X = [0,2) \oplus (1,3]$ and $\varphi(x) = x$ for $x \in X$. Then φ is open and Y is metrizable. $\varphi^{-1}(1) = 1$ and $(\beta\varphi)^{-1}1 \neq \mathrm{cl}_{\beta X}\varphi^{-1}(1) = 1$ and hence φ is not WZ. Let $Y \ni y = 1$ and $\mathcal{V}^y \ni [0,1)$. Then it is obvious $\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\,\mathcal{V}^y = \{1\} \subsetneq (\beta\varphi)^{-1}y$. Thus φ is not W_rN by 3.6 and hence $\beta\varphi$ is not open by 3.10 below. But $\beta\varphi$ is W^*-open by Theorem 4 of [7]. Let $Y \ni z = 2$ and $\mathcal{V}^z \ni [0,2)$. Then it is easy to see that $\bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{V}^z \supsetneq \mathrm{cl}_{\beta X}\varphi^{-1}z = \{2\}$ (cf. Remark of 3.3).

THEOREM 3.9. *If φ: $X \to Y$ is a *-open Z-map, then it is open.*

Proof. Let U be open in X and $x \in U$. Then there is a zero set Z with $x \in \mathrm{int}\, Z = W \subset Z \subset U$ and $\varphi U \supset \varphi Z = \mathrm{cl}\,\varphi Z \supset \mathrm{cl}\,\varphi(\mathrm{int}\, Z) \supset \mathrm{int}(\mathrm{cl}\,\varphi(\mathrm{int}\, Z)) \supset \varphi W \ni \varphi(x)$, and hence φ is open.

THEOREM 3.10. *Let φ: $X \to Y$. Then the following are equivalent*:

(1) *$\beta\varphi$ is open.*

(2) *φ is *-open and W_rN.*

(3) *$\mathrm{Cl}_{\beta X}\varphi^{-1}V = (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y} V$ for each open set V of Y.*

(4) *$(\beta\varphi)^{-1}q = \bigcap \mathrm{cl}_{\beta X}\varphi^{-1}\mathcal{V}^q$ for each $q \in \beta Y$ and each $\mathcal{V}^q$.*

(5) *There is $\mathcal{U}^p$ with $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$ for each $q \in \beta Y$, each $\mathcal{V}^q$ and each $p \in (\beta\varphi)^{-1}q$.*

Proof. (1) $\Rightarrow$ (2). Let U be open in X and put $W = \beta X - \mathrm{cl}_{\beta X}(X - U)$. Then $U = W \cap X$ and $\mathrm{cl}_{\beta X} W = \mathrm{cl}_{\beta X} U$. Since $\beta\varphi$ is closed, we have $(\beta\varphi)\,\mathrm{cl}_{\beta X} W = \mathrm{cl}_{\beta Y}(\beta\varphi)U = \mathrm{cl}_{\beta Y}\varphi U \supset (\beta\varphi)W \supset \varphi U$ and $\mathrm{cl}\,\varphi U = Y \cap \mathrm{cl}_{\beta Y}\varphi U \supset Y \cap (\beta\varphi)W \supset \varphi U$. Since $\beta\varphi$ is open, $\mathrm{int}(\mathrm{cl}\,\varphi U) \supset \varphi U$, i.e., φ is *-open. We shall show that φ is W_rN. Let V be open in Y. $T = \beta Y - \mathrm{cl}_{\beta Y}(Y - V)$ is open and $V = Y \cap T$. Since $\mathrm{cl}_{\beta Y} T = \mathrm{cl}_{\beta Y} V$ and $\beta\varphi$ is *-open, $\mathrm{cl}_{\beta X}(\beta\varphi)^{-1}T = (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y} T = (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y} V$. Thus it suffices to show $\mathrm{cl}_{\beta X}(\beta\varphi)^{-1}T = \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\, V$. Suppose $p \in (\beta\varphi)^{-1}T - \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\, V$. Let $q \in T$ and $(\beta\varphi)p = q$. Take an open set S of βX such that $S \ni p$ and $\mathrm{cl}_{\beta X}S \cap \mathrm{cl}_{\beta X}\varphi^{-1}\,\mathrm{cl}\, V = \varnothing$. Let us put $K = \mathrm{int}_{\beta Y}((\beta\varphi)\,\mathrm{cl}_{\beta X}S)$. Then $K = \mathrm{int}_{\beta Y}(\mathrm{cl}_{\beta Y}(\beta\varphi)S) \supset (\beta\varphi)S \ni q$ and $K \cap V = \varnothing$, so $K \cap \mathrm{cl}_{\beta Y} V = \varnothing$. This is a contradiction because $q \in \mathrm{cl}_{\beta Y} V$. (2) $\Rightarrow$ (3). From 3.3(2). (3) $\Rightarrow$ (4). From 2.1(3) and the fact that $q \in \mathrm{cl}_{\beta Y} V$ for each $V \in \mathcal{V}^q$. (4) $\Rightarrow$ (5). From 2.1(5).

(5) $\Rightarrow$ (1). We first show that $\beta\varphi$ is *-open. Let $p \in (\beta\varphi)^{-1}\,\mathrm{cl}_{\beta Y} W - \mathrm{cl}_{\beta X}(\beta\varphi)^{-1}W$ for some open set W of βY. Then there is an open set U of βX with $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} U$ and $\mathrm{cl}_{\beta X} U \cap \mathrm{cl}_{\beta X}(\beta\varphi)^{-1}W = \varnothing$. Let $(\beta\varphi)p = q$ and take $\mathcal{V}^q$ with $W \in \mathcal{V}^q$. Then any $\mathcal{U}^p$ contains U. If $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$ for some $\mathcal{U}^p$, then $\varphi^{-1}W \in \mathcal{U}^p$, but $U \cap \varphi^{-1}V = \varnothing$, a contradiction. Thus $\beta\varphi$ is *-open by 3.3, so open by 3.9.

If $\varphi: X \to Y$ is open WZ, then $\beta\varphi$ is open by Theorem 4.4(1) of **[12]**. Let $X \subset Z \subset \beta X$ and $\zeta = (\beta\varphi) \mid Z$. Then $\zeta: Z \to \zeta Z$ has the Stone extension $\beta\zeta = \beta\varphi$, so $\beta\zeta$ is open, and hence ζ is *-open $W_r N$ by 3.10. Thus we have

THEOREM 3.11. *Let $\varphi: X \to Y$ be open WZ. Then for any space Z, $X \subset Z \subset \beta X$, $\zeta: Z \to \zeta Z \subset \beta Y$ is *-open $W_r N$ where $\zeta = (\beta\varphi) \mid Z$.*

4. Countable intersection property.

4.1. DEFINITION. We denote by $\{F_n\}_{\mathrm{cl}} \downarrow \varnothing$ ($\{F_n\}_{ze} \downarrow \varnothing$ or $\{F_n\}_{re} \downarrow \varnothing$ resp.) a decreasing sequence of closed sets (zero sets or regular closed sets resp.) with empty intersection. $\varphi: X \to Y$ is said to be a d (d' or d^* resp.)-*map* if $\bigcap \mathrm{cl}\, \varphi F_n = \varnothing$ for each $\{F_n\}_{\mathrm{cl}} \downarrow \varnothing$ ($\{F_n\}_{re} \downarrow \varnothing$ or $\{F_n\}_{ze} \downarrow \varnothing$ resp.) **[5, 8, 11]**. Obviously a d-map is d' and a d'-map is d^* (**[8]**, Theorem 7). We say that φ is *hyper-real* if $(\beta\varphi)(\beta X - vX) \subset \beta Y - vY$. A hyper-real map is d^* **[11]** (cf. the diagram of 5.4 below). Let us put $X^* = \beta X - X$.

$F(X; 0) = \{p \in X^*;$ any $\mathfrak{F}^p$ has CIP$\}$.

$F(X; 0, \Delta) = \{p \in X^*;$ there is $\mathfrak{F}_1^p$ with CIP and $\mathfrak{F}_2^p$ without CIP$\}$.

$F(X, \Delta) = \{p \in X^*;$ any $\mathfrak{F}^p$ does not have CIP$\}$.

$F(X; v, \Delta) = (vX - X) \cap F(X; \Delta)$.

Similarly we define $U(X; 0)$, $U(X; 0, \Delta)$, $U(X; \Delta)$ and $U(X; v, \Delta)$ using free open ultrafilters. It is known that $\beta X - vX \subset U(X; \Delta)$, $U(X; \Delta) \subset F(X; \Delta)$ and $F(X; 0) \subset U(X; 0)$ **[13]**. Concerning invariance of CIP under a map, we note the following. Let $\varphi: X \to Y$.

(1) If $\mathfrak{U}$ has CIP, then any $\mathfrak{V} \supset \varphi^{\#}\mathfrak{U}$ has CIP by 2.3(1) where "$\mathfrak{U}$ has CIP" means "$\bigcap \mathrm{cl}\, U_n \neq \varnothing$ for $U_n \in \mathfrak{U}$". Thus, in general, for $\varphi: X \to Y$, we have $U(Y; \Delta) \cap (\beta\varphi)(U(X; 0) \cup U(X; 0, \Delta)) = \varnothing$ and hence $(\beta\varphi)^{-1}U(Y; \Delta) \subset U(X; \Delta)$.

(2) If $\mathfrak{F}$ has CIP and $\varphi^{\#}\mathfrak{F} = \mathfrak{E}$, then $\mathfrak{E}$ has CIP. This follows from $\varphi^{-1}E \in \mathfrak{F}$ for $E \in \mathfrak{E}$.

(3) The following (a) and (b) are not necessarily true as is shown by 4.2 below.

(a) $\varphi^{\#}\mathfrak{U} = \mathfrak{V}$ does not have CIP for $\mathfrak{U}$ without CIP.

(b) $\varphi^{\#}\mathfrak{F} = \mathfrak{E}$ does not have CIP for $\mathfrak{F}$ without CIP.

Problem. Does $\mathfrak{E} \supset \varphi^{\#}\mathfrak{F}$ have CIP whenever $\mathfrak{F}$ has CIP?

4.2. EXAMPLE. Let $Y = \{y\}$. In (1) and (2) below, define $\varphi(x) = y$. Then φ is open, closed, RC-preserving, Z-preserving and an N-map where φ is $RC(Z)$-*preserving* if φE is regular closed (a zero) set whenever E is a regular closed set (a zero set).

(1) Let X be pseudocompact but not countably compact. Then φ is a d'-map but not a d-map. Evidently there is $\mathfrak{F}$ without CIP but $\varphi^{\#}\mathfrak{F} = \{y\}$ has CIP.

(2) Let X be a non-pseudocompact space. Then φ is not a d^*-map. Evidently there is $\mathfrak{U}$ without CIP but $\varphi^{\#}\mathfrak{U} = \{y\}$ has CIP. It is easy to construct an N-map which is not a d^*-map by taking a suitable space X.

THEOREM 4.3. *Let* $\varphi: X \to Y$. *The following are equivalent*:

(1) φ *is a d-map.*

(2) *If* $\mathfrak{F}$ *does not have* CIP, *so neither does any* $\mathfrak{G} \supset \varphi^{\#}\mathfrak{F}$.

(3) $(\beta\varphi)^{-1}(Y \cup F(Y; 0)) \subset X \cup F(X, 0)$.

(4) $(\beta\varphi)^{-1}Y \subset X \cup F(X; 0)$.

Proof (1) $\Rightarrow$ (2). From the fact that $\bigcap \operatorname{cl} \varphi F_n = \varnothing$ for $\{F_n \in \mathfrak{F}\} \downarrow \varnothing$ and $\operatorname{cl} \varphi F_n \in \mathfrak{G}$.

(2) $\Rightarrow$ (3). There is $\mathfrak{F}^p$ without CIP for $p \in F(X; \Delta) \cup F(X; 0, \Delta)$, so every $\mathfrak{G} \supset \varphi^{\#}\mathfrak{F}^p$ does not have CIP by (2) and hence $(\beta\varphi)p \notin Y \cup F(Y, 0)$, so $(\beta\varphi)^{-1}(Y \cup F(Y; 0)) \subset X \cup F(X; 0)$.

(3) $\Rightarrow$ (4). Evident.

(4) $\Rightarrow$ (1). Let $\{F_n\}_{\mathrm{cl}} \downarrow \varnothing$ and $y \in \bigcap \operatorname{cl} \varphi F_n$. Then $\operatorname{cl}_{\beta X} F_n \cap (\beta\varphi)^{-1}y \neq \varnothing$ for $n \in N$. Take $p \in (\bigcap \operatorname{cl}_{\beta X} F_n) \cap (\beta\varphi)^{-1}y$ and $\mathfrak{F}^p$ with $F_n \in \mathfrak{F}^p$, $n \in N$. Then $p \in F(X; 0)$ by (4) but $\mathfrak{F}^p$ does not have CIP, a contradiction.

REMARK. In general, the equality of 4.3(3) does not hold as shown by 5.6 below. An analogous theorem concerning a d^*- and d'-map was obtained respectively (see, 4.4(2, 3) below). A closed d-map is precisely quasi-perfect (= closed and each fiber is countably compact), so we have the following 4.4(1) using 1.4(3) and 4.3.

4.4. *Let* $\varphi: X \to Y$. (1) φ *is quasi-perfect iff* $\varphi^{\#}\mathfrak{F}$ *is a closed ultrafilter for each* $\mathfrak{F}$ *and* $\varphi^{\#}\mathfrak{F}$ *does not have* CIP *for each* $\mathfrak{F}$ *without* CIP.

(2) φ *is a* d^**-map iff* $(\beta\varphi)^{-1}Y \subset \mathfrak{U}X$ **[11]**.

(3) φ *is a* d'*-map iff* $(\beta\varphi)^{-1}Y \subset X \cup U(X; 0)$ **[5]**.

4.5. *Let* $\varphi: X \to Y$.

(1) *Let* φ *be a* d'*-map and* $\varphi^{\#}\mathfrak{U} = \mathfrak{V}$. *If* $\mathfrak{U}$ *does not have* CIP, *then neither does* $\mathfrak{V}$.

(2) *If* φ *is not a* d'*-map, there is* $\mathfrak{U}$ *without* CIP *such that every* $\mathfrak{V} \supset \varphi^{\#}\mathfrak{U}$ *has* CIP.

(3) *If* φ *is* W^**-open, then* φ *is a* d'*-map iff* $\varphi^{\#}\mathfrak{U}$ *does not have* CIP *for each* $\mathfrak{U}$ *without* CIP (cf., 4.6(2)).

Proof. (1) Since $\mathcal{U}$ does not have CIP, there is $\{U_n \in \mathcal{U}\}\downarrow$ with $\bigcap \mathrm{cl}\, U_n = \varnothing$. If $\mathcal{V}$ has CIP, $Y - \mathrm{cl}\, \varphi U_n \in \mathcal{V}$ for some n. $\varphi^{\#}\mathcal{U} = \mathcal{V}$ implies $\varphi^{-1}(Y - \mathrm{cl}\, \varphi U_n) = X - \varphi^{-1}(\mathrm{cl}\, \varphi U_n) \in \mathcal{U}$, a contradiction.

(2) Since φ is not d', there is $\{U_n\}_{\mathrm{open}} \downarrow \varnothing$ with $y \in \bigcap \mathrm{cl}\, \varphi U_n$ for some $y \in Y$. This implies $(\beta\varphi)^{-1}y \cap \mathrm{cl}_{\beta X} U_n \neq \varnothing$ for $n \in N$. By 1.1(2), there is $\mathcal{U}^p$ without CIP and $U_n \in \mathcal{U}^p$ where $p \in (\bigcap \mathrm{cl}_{\beta X} U_n) \cap (\beta\varphi)^{-1}y$. Obviously any $\mathcal{V} \supset \varphi^{\#}\mathcal{U}^p$ converges to y, i.e., $\mathcal{V}$ has CIP.

(3) $\Rightarrow$). From (1) and 2.6 $\Leftarrow$). From (2) and 2.6.

4.6. Definitions and some properties. Let φ: $X \to Y$. φ is said to be an *sd-map* if $\mathcal{F}$ does not have CIP iff no $\mathcal{G} \supset \varphi^{\#}\mathcal{F}$ has CIP. We say that φ is an *sd'-map* if some $\mathcal{V} \supset \varphi^{\#}\mathcal{U}$ does not have CIP for $\mathcal{U}$ without CIP.

(1) A quasi-perfect map is *sd* by 4.4 and an *sd*-map is *d* by 4.3.

(2) Any W^*-open d'-map is sd' by 4.5(3) and an sd'-map is d' by 4.5(2).

(3) If φ is *sd*, then we have that $(\beta\varphi)^{-1}(Y \cup F(Y; 0)) \subset X \cup F(X; 0)$, $(\beta\varphi)F(X; 0, \Delta) \subset F(Y; 0, \Delta)$ and $(\beta\varphi)F(X; \Delta) \subset F(Y; \Delta) \cup F(Y; 0, \Delta)$.

(4) If φ is sd', then we have that $(\beta\varphi)^{-1}(Y \cup U(Y; 0)) \subset X \cup U(X; 0)$, $(\beta\varphi)U(X; 0, \Delta) \subset U(Y; 0, \Delta)$ and $(\beta\varphi)U(X; \Delta) \subset U(Y; \Delta) \cup U(Y; 0, \Delta)$.

(5) If φ is $*$-open W_rN, then $(\beta\varphi)^{-1}U(Y; 0, \Delta) \subset (X; 0, \Delta)$, $(\beta\varphi)^{-1}U(Y; \Delta) \subset U(X, \Delta)$ and $(\beta\varphi)U(X; 0) \subset Y \cup U(Y; 0)$ by 3.10 and 4.1(1).

(6) If φ is a $*$-open W_rN d'-map, then $(\beta\varphi)^{-1}U(Y; \Delta) = U(X; \Delta)$ by 3.10. $(\beta\varphi)^{-1}U(Y; 0, \Delta) = U(X; 0, \Delta)$ and $(\beta\varphi)^{-1}(Y \cup U(Y; 0) = X \cup U(X; 0)$.

(7) If φ is closed, then $(\beta\varphi)(F(X; 0) \cup F(X; 0, \Delta)) \cap F(Y; \Delta) = \varnothing$ by 1.4(3) and 4.1(2).

(8) If φ is an N-map, then we have $(\beta\varphi)F(X; 0) \cap (F(Y; 0, \Delta) \cup F(Y; \Delta)) = \varnothing$ by 1.1(1) and 1.4(4).

It is not necessarily true that a perfect map is sd' as shown by 4.7 below. X is said to be $nd - cp$ if for a decreasing sequence $\{F_n\}$ of nowhere dense closed sets with $\bigcap F_n = \varnothing$, there is $\{U_n\}_{\mathrm{open}} \downarrow$ with $F_n \subset U_n$ and $\bigcap \mathrm{cl}\, U_n = \varnothing$. It is easy to see the following

(9) If X is countably paracompact, then X is $nd - cp$.

(10) If X is pseudocompact, then X is countably compact iff X is $nd - cp$.

4.7. *If Y is pseudocompact but not countably compact, then there is a space X and a perfect map φ: $X \to Y$ which is neither sd' nor W^*-open.*

Proof. Let $A = \{a_n;\ n \in N\}$ be a discrete closed set of Y and put $X = Y \oplus A$. Define $\varphi(x) = x$. Obviously φ is perfect but not W^*-open. Let us put $U_n = \{a_m;\ m \geq n\} \subset A \subset X$ and take $\mathfrak{U}$ with $U_n \in \mathfrak{U}$, $n \in N$. Then $\mathfrak{U}$ does not have CIP but any $\mathfrak{V} \supset \varphi^{\#}\mathfrak{U}$ has CIP because Y is pesudocompact.

THEOREM 4.8. *Let* φ: $X \to Y$.
(1) *If* Y *is countably compact, then* X *is countably compact iff* φ *is sd.*
(2) *If* Y *is pseudocompact, then* X *is pseudocompact iff* φ *is sd'.*

4.8(2) is a generalization of 4.3 of **[12]** and Theorem 12 of **[8]**.

Proof. (1) $\Rightarrow$). Evident. $\Leftarrow$). If X is not countably compact, there is $\{F_n\}_{\mathrm{cl}} \downarrow \varnothing$. Take $\mathfrak{F} \ni F_n$ for each n. Then $\mathfrak{F}$ does not have CIP and hence there is $\mathfrak{G}$ without CIP containing $\varphi^{\#}\mathfrak{F}$ because φ is *sd*. But this is a contradiction because Y is countably compact.

(2) is obtained by the same method used in the proof of (1).

THEOREM 4.9. *Let* φ: $X \to Y$ *and* Y *be* $nd - cp$.
(1) *If* φ *is* d', *then* φ *is* sd'.
(2) *If* φ *is* d, *then* φ *is sd.*

Proof. (1). Suppose that there is $\mathfrak{U}$ without CIP such that each $\mathfrak{V} \supset \varphi^{\#}\mathfrak{U}$ has CIP. If $\varphi^{\#}\mathfrak{U} = \mathfrak{V}$, then $\mathfrak{V}$ does not have CIP by 4.5(1), and hence we may assume that $\varphi^{\#}\mathfrak{U} \neq \mathfrak{V}$ for each $\mathfrak{V} \supset \varphi^{\#}\mathfrak{U}$. Since $\mathfrak{U}$ does not have CIP, there is $\{U_n \in \mathfrak{U}\} \downarrow \varnothing$ with $\bigcap \mathrm{cl}\, U_n = \varnothing$. φ being d', $\bigcap \mathrm{cl}\, \varphi U_n = \varnothing$. Let $V \in \mathfrak{V} - \varphi^{\#}\mathfrak{U}$. Then there is $U \in \mathfrak{U}$ with $U \cap \varphi^{-1}V = \varnothing$ and hence we may assume $U_n \subset U$ for each n. Now $\varphi B(U_n, V) \subset \varphi U_n \cap \mathrm{cl}\, V$, so by 2.3(2) $K_n = \mathrm{cl}\, \varphi(\mathrm{int}\, B(U_n, V))$ is nowhere dense and $\bigcap K_n = \varnothing$. Since Y is $nd - cp$, there is $\{V_n\}_{\mathrm{open}} \downarrow \varnothing$ such that $K_n \subset V_n$ and $\bigcap \mathrm{cl}\, V_n = \varnothing$. Obviously $\varphi^{-1}V_n \supset \mathrm{int}\, B(U_n, V)$, so $V_n \in \mathfrak{V}$ by 2.3(1) which shows that $\mathfrak{V}$ does not CIP, a contradiction.

(2) By 4.3, it suffices to show that if $\mathfrak{F}$ has CIP, then any $\mathfrak{G} \supset \varphi^{\#}\mathfrak{F}$ has CIP. Suppose that $\mathfrak{F}$ has CIP and some $\mathfrak{G} \supset \varphi^{\#}\mathfrak{F}$ does not have CIP. We may assume $\mathfrak{G} \neq \varphi^{\#}\mathfrak{F}$. There is $\{E_n \in \mathfrak{G} - \varphi^{\#}\mathfrak{F}\} \downarrow \varnothing$. Then there is $F \in \mathfrak{F}$ with $E_1 \cap \varphi F = \varnothing$, and hence $E_n \cap \varphi F = \varnothing$ for each n. Since $\mathfrak{G} \ni K_n = E_n \cap \mathrm{cl}\, \varphi F \neq \varnothing$ and K_n is nowhere dense, there is $\{V_n\}_{\mathrm{open}} \downarrow \varnothing$ such that $K_n \subset V_n$ and $\bigcap \mathrm{cl}\, V_n = \varnothing$. If $\mathrm{cl}\, V_n \notin \varphi^{\#}\mathfrak{F}$, then there is $D \in \mathfrak{F}$ with $\mathrm{cl}\, V_n \cap \varphi D = \varnothing$. V_n being open, $V_n \cap \mathrm{cl}\, \varphi D = \varnothing$ and hence $K_n \cap \mathrm{cl}\, \varphi D = \varnothing$ which contradicts $\mathfrak{G} \supset \varphi^{\#}\mathfrak{F}$. This shows $\mathrm{cl}\, V_n \subset \varphi^{\#}\mathfrak{F}$ for each n, so F does not have CIP, a contradiction.

5. Spaces and mappings.

5.1. We recall the following [**13**].

(1) X is almost realcompact iff $U(X; 0) \cup U(X; 0, \Delta) = \varnothing$.

(2) X is c-realcompact iff $U(X; 0) = \varnothing$.

(3) X is a-realcompact iff $F(X; 0) \cup F(X; 0, \Delta) = \varnothing$.

(4) X is wa-realcompact iff $F(X; 0) = \varnothing$.

(5) X is weak cb^* iff $U(X; v, \Delta) \cup U(X; 0, \Delta) = \varnothing$.

(6) X is pseudocompact iff $U(X; \Delta) \cup U(X; 0, \Delta) = \varnothing$.

(7) X is cb^* iff $F(X; v, \Delta) \cup F(X; 0, \Delta) = \varnothing$.

(8) X is countably compact iff $F(X; \Delta) \cup F(X; 0, \Delta) = \varnothing$.

Dykes and Frolík proved the following respectively.

(9) Let $\varphi: X \to Y$ be perfect. Then

(i) X is almost realcompact iff Y is almost realcompact [**2**].

(ii) X is a-realcompact iff Y is a-realcompact [**1**].

From (1) $\sim$ (8), we have the following diagram.

$$\begin{array}{ccccc}
 & & \textit{countably compact} & \Rightarrow & \textit{pseudocompact}\\
 & & \Downarrow & & \Downarrow\\
\textit{realcompact} & \Rightarrow & cb^* & \Rightarrow & \textit{weak } cb^*\\
\Downarrow & & & & \\
\textit{almost realcompact} & \Rightarrow & a\textit{-realcompact} & & \\
\Downarrow & & \Downarrow & & \\
c\textit{-realcompact} & \Rightarrow & wa\textit{-realcompact} & &
\end{array}$$

5.2. Let $p \in X^*$, $Z = X \cup \{p\} \subset \beta X$ and Y the space obtained from Z by identifying p and a fixed point x_0 of X. It is easy to see that the identifying map φ is W^*-open but not $*$-open. In this case we have

(1) *If* $p \in \mathcal{V}X - X$, *then* φ *is* d^* [**11**].

(2) *If* $p \in U(X; 0)$, *then* φ *is* d' [**5**].

THEOREM 5.3. (1) *The following are equivalent*:

(i) *X is wa-realcompact.*

(ii) *Any d-map defined on X is perfect.*

(iii) *Any W^*-open sd-map defined on X is perfect.*

(2) *The following are equivalent* ([**5**], *Theorem* 1 *and* [**8**], *Theorem* 13):

(i) *X is c-realcompact.*

(ii) *Any d'-map defined on X is perfect.*

(iii) *Any W^*-open d'-map defined on X is perfect.*

(3) *The following are equivalent* ([**11**], *Theorem* 6.3):

(i) *Y is cb^*.*

(ii) *Any d^*-map onto Y is hyper-real.*

(iii) *Any perfect map onto Y is hyper-real.*

(4) *The following are equivalent*:
(i) *Y is weak cb**.
(ii) *Any sd'-map onto Y is hyper-real.*
(iii) *Any W^*-open d'-map onto Y is hyper-real.*
(iv) *Any W^*-open perfect map onto Y is hyper-real.*

Proof. (1) (i) $\Rightarrow$ (ii). From 4.3(2, 3) and *wa*-realcompactness. (ii) $\Rightarrow$ (iii). Evident. (iii) $\Rightarrow$ (i). If X is not *wa*-realcompact, take $p \in F(X;\ 0)$ in 5.2. Obviously φ is W^*-open *sd*-map but $\varphi^{-1}(x_0) = x_0$ and $(\beta X)^{-1}x_0 \ni p$, so φ is not perfect.

(4) (i) $\Rightarrow$ (ii). Since φ is *sd'*, $(\beta\varphi)(\beta X - vX) \subset (\beta\varphi)U(X;\ \Delta) \cup U(Y;\ \Delta) \cup U(Y;\ 0, \Delta) = \beta Y - vY$ because Y is weak *cb**, i.e., φ is hyper-real. (ii) $\Rightarrow$ (iii). From 4.6(2). (iii) $\Rightarrow$ (iv). Evident. (iv) $\Rightarrow$ (i). Suppose that there is $\mathfrak{U}^p$ without CIP and $p \in vY - Y$. There is $\{U_n \in \mathfrak{U}^p\} \downarrow \varnothing$ with $\bigcap \operatorname{cl} U_n = \varnothing$. Let us put $X = Y \oplus \sum \oplus \operatorname{cl} U_n$ and define $\varphi(x) = x$. Obviously φ is W^*-open perfect. On the other hand, $vX = vY \oplus \sum \oplus v(\operatorname{cl} U_n)$ and $v\varphi$ is onto vY, but $(v\varphi)^{-1}p$ $(p \in vY)$ is not compact where $v\varphi = (\beta\varphi) \mid (vX)$, and hence φ is not hyper-real.

5.4. Note and Problem. We define that φ: $X \to Y$ is a *$d_1(d_2)$-map* if $(\beta\varphi)^{-1}Y \subset X \cup U(X;\ 0) \cup U(X;\ 0, \Delta)(\subset X \cup F(X;\ 0) \cup F(X;\ 0, \Delta))$. Then we have the following:

(1) *X is almost realcompact iff any d_1-map defined on X is perfect.*

(2) *X is a-realcompact iff any d_2-map defined on X is perfect.*

"only if" part of (1) and (2) are obvious and "if" part of (1) and (2) are obtained by the method used in 5.2 taking $p \in U(X;\ 0, \Delta) \cup U(X;\ 0)$ and $p \in F(X;\ 0, \Delta) \cup F(X;\ 0)$ respectively. But these definitions of d_1- and d_2-map are affected.

Problem. What is the intrinsic definition of a d_1 (or d_2)-map? Concerning various maps in this paper, we have the following:

$$
\begin{array}{lcccccccc}
\textit{open} \Rightarrow \textit{*-open} & \Rightarrow & W^*\textit{-open} & \Leftarrow & & W^*\textit{-open and } d' & & & \\
 & & & & & \Downarrow & & & \\
\textit{perfect} \quad \textit{open } WZ & \Rightarrow & W_rN & \Leftarrow & N \Rightarrow WN & sd' & & & \textit{hyper-real} \\
\Downarrow & & & & \Downarrow & \Downarrow & & & \Downarrow \\
\textit{quasi-perfect} \Rightarrow \textit{closed} & \Rightarrow & Z & & \Rightarrow WZ & d' & & & d^* \\
\Updownarrow & & & & \Nearrow & & \Searrow & & \Uparrow \\
\textit{closed and } d & \Rightarrow & sd & \Rightarrow & d & \Rightarrow & d_2 & \Rightarrow & d_1.
\end{array}
$$

Theorem 5.5. *Let φ: $X \to Y$.*

(1) *Suppose that φ is a d-map. Then we have*

(i) *If X is wa-realcompact, so is Y.*

(ii) *If X is a-realcompact, so is Y.*

(2) *Let φ be an sd'-map. Then if X is c-realcompact, so is Y (this is a generalization of Theorem* 1.3 *of* [7] *by* 4.6(2)).

(3) *Let φ be a d'-map. Then if X is almost realcompact, so is Y.*

(4) *Let φ be hyper-real. Then if X is weak cb*, so is Y.*

(5) *Let φ be hyper-real. Then if X is cb*, so is Y* ([**11**], *Theorem* 5.7(2)).

Proof. (1) (i). From 5.1(4), 5.3(1) and 4.3(3) (note that a perfect map is *sd*). (ii). From the diagram of 5.1, 5.3, (i) above and 5.1(9(ii)).

(2) From $U(Y;\, 0) = \varnothing$ by 4.6(4) and $U(X;\, 0) = \varnothing$, or from 4.6(4), Theorem 2 of [**4**] and the fact that $uX = X \cup U(X;\, 0)$.

(3) From the diagram of 5.1, 5.3(2) and 5.1(9(i)).

(4) Suppose that there is $\mathcal{V}^q$ without CIP for $q \in vY - Y$. Then $(\beta\varphi)^{-1}q \subset U(X;\, 0)$. Take $p \in (\beta\varphi)^{-1}q$ and $\mathcal{U}^p \supset \varphi^{-1}\mathcal{V}^q$. Since $\mathcal{U}^p$ has CIP, so does $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$, a contradiction. Thus $U(Y;\, v, \Delta) \cup U(Y;\, 0, \Delta) = \varnothing$, so Y is weak *cb**.

Since a compact space is realcompact, by 4.2(1, 2), it is easily seen that almost-, *c*-, *a*- and *wa*-realcompactness, *cb**-ness and weak *cb**-ness are not inverse invariant under an open, closed, *Z*-preserving, *N*-map. Moreover, by the following Example 5.6, we have that (1) *c*-realcompactness is not inverse invariant under a W^*-open perfect map and (2) *cb**-ness and weak *cb**-ness are not invariant under a W^*-open perfect map.

5.6. EXAMPLE. K. Morita [**15**] constructed an *M*-space, non *c*-realcompact space X and a perfect map φ such that the perfect image Y [**14**] of X by φ is not an M space. It is easy to see that φ is W^*-open but not *-open. An *M*-space is *cb** and hence weak *cb**. On the other hand, Y is *c*-realcompact [**6**] but neither *a*-realcompact [**22**] nor weak *cb** [**11**] and $vY - Y = U(Y;\, 0, \Delta) = F(Y;\, 0, \Delta)$ consists of only one point (see [**12**, **15**]). We note that $(\beta\varphi)^{-1}(Y \cup F(Y;\, 0)) = (\beta\varphi)^{-1}Y \neq X \cup F(X;\, 0)$ (cf. Remark of 4.3 and Remark 6.4 below).

THEOREM 5.7. *Let* $\varphi\colon X \to Y$.

(1) *Let φ be an sd'-map. Then if Y is weak cb*, so is X.*

(2) *Let φ be a d-map. Then if Y is cb*, so is X* ([**11**], *Theorem* 5.5).

(3) *Let φ be a d'-map and Y almost realcompact. Then we have*

(i) $U(X;\, 0, \Delta) = \varnothing$.

(ii) *If X is c-realcompact, then X is almost realcompact.*

(iii) *If φ is perfect, then X is almost realcompact* (5.1(9)).

(4) *Let φ be an sd-map and Y a-realcompact. Then we have*

(i) $F(X;\, 0, \Delta) = \varnothing$.

(ii) *If X is wa-realcompact, then X is a-realcompact.*

(iii) *If φ is perfect, then X is a-realcompact* (5.1(9)).

(5) *Let* φ *be a perfect open map. If* Y *is a c-realcompact, so is* X ([**5**], *Theorem* 4).

(6) *Let* φ *be a perfect N-map. Then if* Y *is wa-realcompact, so is* X.

Proof. (1) φ being hyper-real, by 5.3(4)$\beta X - \upsilon X = (\beta\varphi)^{-1}(\beta Y - \upsilon Y)$ and $U(X;\, \upsilon, \Delta) \cup U(X;\, 0, \Delta) = \varnothing$ by 4.6(4) and 5.1(5), and hence X is weak cb^*.

(3) (i). By 4.1(1) and 4.4(3), $(\beta\varphi)U(X;\, 0, \Delta) \subset U(Y;\, 0, \Delta)$ and hence we have $U(X;\, 0, \Delta) = \varnothing$ because Y is almost realcompact. (ii). From (i) and 5.1(1, 2). (iii). (New proof) Let $p \in U(X;\, 0)$. Then any $\mathcal{V} \supset \varphi^{\#}\mathcal{U}^p$ has CIP and converges to a point $q \in \upsilon Y - Y$ by 4.1(1) and $X = (\beta\varphi)^{-1}Y$. Since Y is almost realcompact, $\upsilon Y - Y = U(Y;\, \upsilon, \Delta)$, a contradiction. Our assertion follows from (i) and 5.1(1).

(4) (i). By 4.6(3), $(\beta\varphi)F(X;\, 0, \Delta) \subset F(Y;\, 0, \Delta)$, so $F(X;\, 0, \Delta) = \varnothing$ and hence X is a-realcompact because Y is a-realcompact. (ii). From (i) and 5.1(3, 4). (iii). (New proof) Let $p \in F(X;\, 0)$. Since φ is *sd*, some $\mathcal{G} \supset \varphi^{\#}\mathcal{F}$ has CIP and converges to a point $q \in \upsilon Y - Y$ by $X = (\beta\varphi)^{-1}Y$. Since Y is c-realcompact, $\upsilon Y - Y = F(Y;\, \upsilon, \Delta)$, a contradiction. Our assertion follows from (i) and 5.1(3).

(5) (New proof) From 4.6(6) and $X = (\beta\varphi)^{-1}Y$.

(6) Since φ is $N(\beta\varphi)F(X;\, 0) \subset Y \cup F(Y;\, 0) = Y$ by 4.6(8), and since φ is perfect $(\beta\varphi)^{-1}Y = X$ and $F(Y;\, 0) = \varnothing$ because Y is *wa*-realcompact and hence X is *wa*-realcompact.

6. Weak ***cb******-ness and absolute.** Using preceding results we give new proofs of several theorems concerning the absolute $E(X)$ of X which are obtained as corollaries of theorems about perfect W^*-open images of weak cb^* spaces.

THEOREM 6.1. *Let* φ *be a perfect* W^**-open map of a weak cb* space* X *onto* Y. *Then we have*

(1) φ *is hyper-real iff* Y *is weak cb**.

(2) $(\beta\varphi)\upsilon X = Y \cup U(Y;\, 0) \cup U(Y;\, 0, \Delta)$.

(3) X *is realcompact iff* Y *is almost realcompact.*

(4) $\upsilon X = (\beta\varphi)^{-1}T$ *for some* T *with* $Y \subset T \subset \beta Y$ *iff* $T = Y \cup U(Y;\, 0)$ *and* $U(Y;\, 0, \Delta) = \varnothing$.

Proof. (1) From 5.3(4) and 5.5(4).

(2) Suppose $(\beta\varphi)^{-1}q \subset \beta X - \upsilon X$ for some point $q \in U(Y;\, 0) \cup U(Y;\, 0, \Delta)$. Then there is $\mathcal{V}^q$ with CIP and $\mathcal{U}^p$ with $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$ for $p \in (\beta\varphi)^{-1}q$. Since $\mathcal{U}^p$ does not have CIP and φ is sd', $\mathcal{V}^q$ does not have CIP, a contradiction.

(3) $\Rightarrow$). Since φ is perfect and $X = vX$, we have $U(Y, 0) \cup U(Y; 0, \Delta) = \varnothing$ by (2), so Y is almost realcompact $\Leftarrow$). Since Y is almost realcompact $(\beta\varphi)vX = Y$ by (2). On the other hand, $(\beta\varphi)^{-1}Y = X$, and hence $vX = X$, i.e., X is realcompact.

(4) $\Rightarrow$). By (2), we have $(\beta\varphi)vX = T = Y \cup U(Y; 0) \cup U(Y; 0, \Delta)$. Since φ is perfect and W^*-open, φ is *sd'* and $(\beta\varphi)^{-1}(Y \cup U(Y; 0)) \subset X \cup U(X; 0) = vX$ by 4.6(4). We shall show $U(Y; 0, \Delta) = \varnothing$. Let $q \in U(Y; 0, \Delta)$. Then $(\beta\varphi)^{-1}q \subset U(X; 0)$ and there is $\mathcal{V}^q$ without CIP but any $\mathcal{U}^p$ has CIP for each $p \in (\beta\varphi)^{-1}q$. Since φ is W^*-open, $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$ for some $p \in (\beta\varphi)^{-1}q$ and some $\mathcal{U}^p$ and hence $\mathcal{V}^q$ has CIP by 4.1(1), a contradiction $\Leftarrow$). By (2), $(\beta\varphi) \cup X = Y \cup U(Y; 0) \cup U(Y; 0, \Delta) = Y \cup U(Y; 0)$. Since φ is *sd'*, $(\beta\varphi)U(X; \Delta) \subset U(Y; \Delta) \cup U(Y; 0, \Delta) = U(Y, \Delta)$ by 4.6(4). Thus $(\beta\varphi)^{-1}T = vX$ where $T = Y \cup U(Y; 0)$.

Let $E(X)$ be the set of all fixed open ultrafilters on X topologized by using $\{U^0;\ U \text{ is open in } X\}$ as a basis where $U^0 = \{\mathcal{U};\ U \in \mathcal{U}\}$. $E(X)$ is called the *absolute of* X and it is a Hausdorff extremally disconnected space. Define η: $\eta\mathcal{U} = \bigcap \operatorname{cl} \mathcal{U}$. Then it is known that η is a perfect irreducible map and $\beta E(X) = E(\beta X)$. Since $\eta U^0 = \operatorname{cl} U$ **[18]**, η is W^*-open by 2.6(2). We note that an extremally disconnected space is weak *cb**.

Corollary 6.2. (1) $vE(X) = (\beta\eta)^{-1}vX\ (= E(vX))$ *iff* $uX = vX$ ([7], *Theorem* 2.4 *and* **[8]**, *Theorem* 4.2) *iff X is weak cb**.

(2) $(\beta\eta)vE(X) = a_1X$ (**[22]**, *Lemma* 2.1).

(3) $E(X)$ *is realcompact iff X is almost realcompact* **[1]**.

(4) $vE(X) = (\beta\eta)^{-1}T$ *for some* T *with* $X \subset T \subset \beta X$ *iff* $T = X \cup U(X; 0)$ *and* $U(X; 0, \Delta) = \varnothing$ (**[20]**, *p.* 330 *and* **[22]**, *Theorem* 3.3).

(5) $E(X)$ *is pseudocompact* iff X *is pseudocompact* (**[20]**, *Proposition* 2.5).

Proof. We note that $E(X)$ is weak *cb** and η is perfect W^*-open. (1) Since $uX = \{p \in \beta X;\ \text{each } \mathcal{U}^p \text{ has CIP}\}$ ([7], Lemma 2.5) and $uX = X \cup U(X; 0)$ by 4.4, we have that $vX = uX$ iff X is weak *cb**. Thus (1) follows from 6.1(1). (2) From 6.1(2) and $a_1X = X \cup U(X; 0) \cup U(X; 0, \Delta)$ (**[22]**, Theorem 2.3). (3) From 6.1(3). (4) From 6.1(4). (5) From 4.6(2) and 4.8(2).

Theorem 6.3. *Let φ be a perfect W^*-open map of a non-realcompact cb* space X onto Y. Then we have*

(1) *Y is cb* iff φ is hyper-real.*

(2) *If Y is weak cb* then Y is cb*.*

(3) *If $vY = Y \cup \{q\}$, then Y is not weak cb* iff Y is c-realcompact but not a-realcompact.*

Proof. (1) From 5.3(3) and 5.5(5). (2) Since Y is weak cb^*, φ is hyper-real by 5.3(4), so Y is cb^* by 5.5(5) because X is cb^*.

(3) $\Rightarrow$). By 5.1(5) and $vY = Y \cup \{q\}$, we have $U(Y; 0) = \varnothing$, so Y is c-realcompact by 5.1(2). On the other hand, $(\beta\varphi)F(X; 0) \subset F(Y; 0) \cup F(Y; 0, \Delta) = F(Y; 0, \Delta)$ because $F(Y; 0) \subset U(Y; 0) = \varnothing$. Thus Y is not a-realcompact $\Leftarrow$). From realcompactness = (weak cb^*-ness) + (c-realcompactness).

6.4. REMARK. The space X in Example 5.6 is not weak cb^* **[11]** and Y is a perfect W^*-open image of an M-space (we note that an M-space is cb^*). Thus Y is c-realcompact but not a-realcompact by 6.5(3). On the other hand, this assertion follows also from the following Corollary 6.7 since φ: $X \to Y$ in 5.6 is irreducible **[5]**.

COROLLARY 6.5. *Let φ be a perfect irreducible map of a non-realcompact cb^* space X onto Y with $vY = Y \cup \{q\}$. Then Y is not weak cb^* iff Y is c-realcompact but not a-realcompact.*

Proof. By Proposition 1.9 of **[19]**, X and Y are co-absolute, so $E(X)$ and $E(Y)$ are homeomorphic. Since X is cb^*, $E(X)$ is cb^* by 5.6(2), so $E(Y)$ is also. Since the canonical map: $E(Y) \to Y$ is perfect and W^*-open, we have our assertion by 6.3(3).

THEOREM 6.6. (1) *If V is an open set of Y with pseudocompact closure, then any $\mathcal{V}^q \ni V$ has* CIP.

(2) *Let φ: $X \to Y$ be W^*-open and d'. Then $S = \beta X - (\beta\varphi)^{-1}vY$ is dense in $\beta X - vX$ and $\beta Y - (\beta\varphi)\,\mathrm{cl}_{\beta X}S \subset Y \cup U(Y; 0)$ (this is a generalization of Theorem* 2.8 *of* **[20]**).

(3) *Let vY be locally compact. Then we have*

(i) *Y is weak cb^** **[4]**.

(ii) *If φ: $X \to Y$ is sd', then φ is hyper-real.*

(iii) $E(vY) = vE(Y)$ (**[20]**, *Proposition* 2.10).

Proof. (1) Suppose that there is $\{V_n \in \mathcal{V}^q\}\downarrow$ with $\bigcap \mathrm{cl}\, V_n = \varnothing$. Then we have $\{\mathrm{cl}(V \cap V_n)\}\,\varnothing$ which contradicts the pseudocompactness of $\mathrm{cl}\, V$.

(2) Suppose $p \in (\beta X - vX) - \mathrm{cl}_{\beta X}S$. Then any $\mathcal{U}^p$ does not have CIP, so $\varphi^{\#}\mathcal{U}^p = \mathcal{V}^q$ for some $\mathcal{V}^q$, $q \in vY - Y$ and hence $\mathcal{V}^q$ does not have CIP by 4.5(1). There is $U \in \mathcal{U}^p$ and an open set W of βX such that $W \cap X = U$ and $\mathrm{cl}_{\beta X} W \cap \mathrm{cl}_{\beta X}S = \varnothing$. By 2.3(3), $\mathrm{int}(\mathrm{cl}\, \varphi U) \in \mathcal{V}^q$. Since $(\beta Y - vY) \cap \mathrm{cl}_{\beta Y}(\beta\varphi)W = \varnothing$ and $\mathrm{cl}_{\beta Y}(\mathrm{int}(\mathrm{cl}\, \varphi U))$ is compact and contained in vY, $\mathrm{cl}\, \varphi U$ is a regular closed by 2.6 and pseudocompact **[4]**. Thus $\mathcal{V}^q$ has CIP by (1), a contradiction. Let us put $R = \beta Y - (\beta\varphi)\,\mathrm{cl}_{\beta X}S$. R is locally compact and $X \cap R \in \mathcal{V}^q$ for any point $q \in R$ and any $\mathcal{V}^q$.

Thus $\mathcal{V}^q$ has a member whose closure is pseudocompact, so has CIP by (1) and hence $R \subset Y \cup U(Y; 0)$.

(3) (i) From (1). (ii). From (i) and 5.3(4). (iii). From (i) and 6.2(1).

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