COMPACT OPERATORS AND DERIVATIONS INDUCED BY WEIGHTED SHIFTS

C. Ray Rosentrater
COMPACT OPERATORS AND DERIVATIONS
INDUCED BY WEIGHTED SHIFTS
C. Ray Rosentrater

In this paper we study the question: which compact operators are contained in \( l^s \), the norm closure of the range of the derivation \( \delta_S(X) = SX - XS \) induced by a weighted shift \( S \)? We find that \( l^s \) always contains the lower triangular (with respect to the basis \( (e_i) \) on which \( S \) is a shift) compact operators. Further, \( l^s \) contains the \( n \)-lower triangular (operators \( T \) satisfying \( (Te_i, e_j) = 0 \) for \( i - j > n \)) compact operators if and only if \( e_i \otimes e_{n+1} \in l^s \). We also find necessary and sufficient conditions on the weights of \( S \) in order that \( e_i \otimes e_{n+1} \in l^s \) and that \( K \), the algebra of compact operators, be contained in \( l^s \). These results completely answer the question: which essentially normal weighted shifts are \( d \)-symmetric?

Let \( T \in B(H) \), the algebra of bounded linear operators on a complex Hilbert space \( H \). The derivation induced by \( T \) is the map \( \delta_T(X) = TX - XT \) from \( B(H) \) to itself. Let \((e_n)_{n=1}^\infty \) (respectively \((e_n)_{n=-\infty}^\infty \)) be an orthonormal basis for \( H \) and let \( S \) be the unilateral (respectively bilateral) weighted shift \( Se_n = w_n e_{n+1}, \ n \in \mathbb{N} \) (respectively \( n \in \mathbb{Z} \)) with nonzero weights \( w_n \). By taking a unitarily equivalent weighted shift, we may assume that \( w_n = |w_n| > 0 \).

Recall that for \( f, g \in B(H) \), the operator \( f \otimes g \in B(H) \) is defined by \((f \otimes g)h = (h, g)f \) for \( h \in B(H) \). In particular, \((e_i \otimes e_j)e_n = e_i \) if \( n = j \) and \((e_i \otimes e_j)e_n = 0 \) otherwise. In Theorem 2 we show that \( e_i \otimes e_{n+1} \in l^s \) if and only if \( \sum w_k \cdot w_{k+1} \cdots \cdot w_{n+k-1} = 0 \). In Corollary 2, we find that this is also equivalent to \( l^s \) containing all the \( n \)-lower triangular compact operators.

The above results enable us to characterize those essentially normal weighted shifts that are \( d \)-symmetric (i.e., satisfy \( l^s = l^{s^*} \)). Namely, an essentially normal weighted shift is \( d \)-symmetric if and only if \( S \) satisfies the total products condition \( \sum w_k \cdot w_{k+1} \cdots \cdot w_{n+k} = \infty \) for all \( n \in \mathbb{N} \). This yields another proof of the fact proved in Corollary 4 of [8] that all hyponormal (and hence all subnormal) weighted shifts are all \( d \)-symmetric.

**Theorem 1.** Let \( S \) be the unilateral (bilateral) weighted shift \( Se_n = w_n e_{n+1}, \ n \in \mathbb{N} (\mathbb{Z}) \). Then \( e_i \otimes e_j \in l^s \) for all \( i, j \in \mathbb{N} (\mathbb{Z}) \) with \( i > j \).

**Proof.** Write \( i = j + n \) with \( n > 0 \). Let \( a_0 = 1/w_j, \ a_k = w_{j+n} \cdots \cdot w_{j+n+k-1}/w_j \cdots \cdot w_{j+k} \) for \( k \geq 1 \), and \( a_k = 0 \) for \( k < 0 \). Then
for $k > n$, cancellation is possible and

$$a_k = w_{j+k+1} \cdots w_{j+n+k-1} / w_j \cdots w_{j+n-1} \leq \|S\|^n / w_j \cdots w_{j+n-1}. $$

Thus the $a_k$'s are uniformly bounded by some constant $B_n$. Also note that $a_k w_{j+n+k} = a_{k+1} w_{j+k+1}$ for $k \neq 0$ so $w_{m+n-1} a_{m-j-1} = a_{m-j} w_m$ for $m - j - 1 \neq -1$.

Now define $T = \sum_{k=0}^{\infty} a_k e_{j+n+k} \otimes e_{j+k+1}$. Then $\|T\| = \sup_k a_k \leq B_n$ so $T \in \mathcal{B}(\mathcal{H})$. Further,

$$(ST - TS)(e_m) = Sa_{m-j} e_{j+n+(m-j-1)} \otimes e_{j+(m-j)+1}(e_m)$$

$$= a_{m-j} e_{j+n+(m-j)} \otimes e_{j+(m-j)+1}(w_m e_{m+1})$$

$$= S a_{m-j-1} e_{m+n-1} - a_{m-j} w_m e_{m+n}$$

$$= (w_{m+n-1} a_{m-j-1} - a_{m-j} w_m) e_{m+n}$$

$$= \begin{cases} 
0 & m - j - 1 \neq -1 \\
0 - a_0 w_j e_{j+n} & m - j = 0 \\
-e_i & m = j.
\end{cases}$$

Thus $ST - TS = -e_i \otimes e_j$ and $\delta_S(-T) = e_i \otimes e_j$. \hfill \Box

**Lemma 1.** If $S e_n = w_n e_{n+1}$ $n \in \mathcal{N}(\mathcal{Z})$ is a unilateral (bilateral) weighted shift and $f \in \mathcal{B}(\mathcal{H})^*$ is in the annihilator of $\mathcal{R}(\delta_S)$, then

$$f(e_{i+k} \otimes e_{j+k}) = \frac{w_j \cdots w_{j+k-1}}{w_i \cdots w_{i+k-1}} f(e_i \otimes e_j)$$

for $i, j \in \mathcal{N}(\mathcal{Z})$ and $k \in \mathcal{N}$.

**Proof.** Since $f$ annihilates $\mathcal{R}(\delta_S)$,

$$0 = f(S(e_i \otimes e_{j+1}) - (e_{i} \otimes e_{j+1})S) = w_j f(e_{i+1} \otimes e_{j+1}) - w_j f(e_i \otimes e_j).$$

Thus $f(e_{i+1} \otimes e_{j+1}) = (w_j / w_i) f(e_i \otimes e_j)$ for all $i, j$ and the lemma follows by induction. \hfill \Box

**Corollary 1.** If $S e_n = w_n e_{n+1}$, $n \in \mathcal{N}(\mathcal{Z})$ is a unilateral (bilateral) weighted shift and $e_n \otimes e_m \in \mathcal{R}(\delta_S)^-$, then $e_i \otimes e_j \in \mathcal{R}(\delta_S)^-$ for all $i, j \in \mathcal{N}(\mathcal{Z})$ satisfying the condition $m - n = j - i$. \hfill \Box
THEOREM 2. Let $S$ be the unilateral (bilateral) weighted shift $S_n = w_n e_{n+1}, n \in \mathbb{N}$ or $\mathbb{Z}$. For $i \in \mathbb{N}$ or $\mathbb{Z}$ and $n \in \mathbb{N}$, we have $e_i \otimes e_{i+n} \in \mathfrak{R}(\delta_S)^-$ if and only if $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} = \infty$ where the sum is taken over $\mathbb{N}$ or $\mathbb{Z}$ as $S$ is unilateral or bilateral.

Proof. By Corollary 1, it suffices to consider $e_1 \otimes e_{n+1}$.

Suppose that $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$. If $J$ is a trace class operator that commutes with $S$, the equation

$$\text{trace}((SA - AS)J) = \text{trace}(SAJ - AJS) = \text{trace}(SAJ) - \text{trace}(SAJ) = 0$$

shows that trace$(\cdot J)$ annihilates $\mathfrak{R}(\delta_S)^-$. Since $S^n$ commutes with $S$ and $\text{trace}(S^n(e_1 \otimes e_{n+1})) = \text{trace}(w_1 \cdot w_2 \cdots \cdot w_n e_{n+1} \otimes e_{n+1}) = w_1 \cdot w_2 \cdots \cdot w_n \neq 0$, it follows that $S^n$ cannot be of trace class. Hence

$$\infty = \Sigma_k \{ S^n | e_k, e_k \} = \Sigma_k w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1}.$$

Conversely, suppose that $\Sigma_k w_k \cdot w_{k+1} \cdots w_{k+n-1} = \infty$ and that $f \in \mathfrak{B}(\mathcal{H})^*$ annihilates $\mathfrak{R}(\delta_S)^-$. Then $\Sigma_{k=0}^\infty w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1} = \infty$ or (in the bilateral case) $\Sigma_{k=-\infty}^{\infty} w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1} = \infty$. In the first case define $T_N = \Sigma_{k=n}^{N+n} e_k \otimes e_{n+k}$. Then $\|T_N\| = 1$ and using Lemma 1,

$$\|f\| \geq |f(T_N)| = \left| \left[ \frac{w_{n+1} \cdot w_{n+2} \cdots \cdot w_{n+k-1}}{w_1 \cdot w_2 \cdots \cdot w_k} f(e_1 \otimes e_{n+1}) \right] \right| = \left| \sum_{k=n}^{N+n} \frac{w_k \cdots w_{n+k-1}}{w_1 \cdots w_n} f(e_1 \otimes e_{n+1}) \right|$$

$$= \left| \frac{f(e_1 \otimes e_{n+1})}{w_1 \cdots w_n} \sum_{k=n}^{N+n} w_k w_{k+1} \cdots \cdot w_{k+n-1} \right|.$$ 

Since $\Sigma_{k=n}^{N+n} w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1} \to \infty$ as $N \to \infty$, we see that $f(e_1 \otimes e_{n+1}) = 0$ and $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$. Now suppose that $\Sigma_{k=-\infty}^{\infty} w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1} = \infty$. If $l < 0$, we can apply Lemma 1 to $k = -l + 1$ to show that

$$f(e_1 \otimes e_{n+1}) = \frac{w_{n+l} \cdots w_n}{w_l \cdots w_0} f(e_l \otimes e_{n+l})$$

or

$$f(e_1 \otimes e_{n+1}) = \frac{w_l \cdots w_0}{w_{n+l} \cdots w_n} f(e_1 \otimes e_{n+1}).$$
Defining \( R_N = \sum_{i=-n}^{-N} e_i \otimes e_{n+i} \), we see that

\[
\| f \| \geq |f(R_N)| = \left| \sum_{l=-n}^{-N-n} \frac{w_l \cdots \cdot w_{n+l-1}}{w_1 \cdots \cdot w_n} f(e_1 \otimes e_{n+1}) \right|
\]

\[
= \left| \sum_{l=-n}^{-N-n} \frac{w_l \cdots \cdot w_{n+l-1}}{w_1 \cdots \cdot w_n} f(e_1 \otimes e_{n+1}) \right|
\]

\[
= \left| \frac{f(e_1 \otimes e_{n+1})}{w_1 \cdots \cdot w_n} \sum_{l=-n}^{-N-n} w_l \cdots \cdot w_{n+l-1} \right|
\]

As before, the fact that \( \sum_{l=-n}^{-N-n} w_l \cdots \cdot w_{n+l-1} \to \infty \) implies that \( f(e_1 \otimes e_{n+1}) = 0 \) and \( e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^- \).

**Remark.** Note that if we take \( n = 0 \) in the proof of Theorem 1 then the \( a_n \) become \( 1/w_n \). Thus \( e_i \otimes e_i \in \mathcal{R}(\delta_S) \) if the \( w_n \) are bounded away from zero. If the weights are not bounded away from zero, then taking \( n = 0 \) in the proof of Theorem 2 we find that \( \| f \| \geq \sum_{k=0}^N |f(e_1 \otimes e_1)| \) and thus \( e_1 \otimes e_i \in \mathcal{R}(\delta_S)^- \).

**Corollary 2.** Let \( S \) be the unilateral (bilateral) weighted shift \( S e_n = w_n e_{n+1} \), \( n \in \mathbb{N}(\mathbb{Z}) \). Then the following are equivalent.

(a) \( \mathcal{R}(\delta_S)^- \) contains the n-lower triangular compact operators.

(b) \( e_1 \otimes e_{i+n} \in \mathcal{R}(\delta_S)^- \)

(c) \( e_i \otimes e_{i+n} \in \mathcal{R}(\delta_S)^- \) for some \( i \in \mathbb{N}(\mathbb{Z}) \).

(d) \( \sum_k w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1} = \infty \).

**Proof.** The equivalence of (b), (c) and (d) has already been established and (b) follows from (a) since \( e_1 \otimes e_{i+n} \) is compact and n-lower triangular. It remains to be shown that (b) implies (a). From the proof of Theorem 2, we see that if \( e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^- \), then \( S^n \) is not trace class. Hence \( S^n \) is not trace class for \( 0 \leq m < n \). Thus \( \sum_k w_k \cdot w_{k+1} \cdots \cdot w_{k+m-1} = \infty \) and all operators of the form \( e_i \otimes e_{i+m} \) are elements of \( \mathcal{R}(\delta_S)^- \). Since by Theorem 1, and the above remark, \( e_i \otimes e_{i+m} \in \mathcal{R}(\delta_S)^- \) for \( m \leq 0 \), it follows that \( \mathcal{R}(\delta_S)^- \) contains the closed linear span of \( \{ e_i \otimes e_{i+m} : m \leq n \} \) (i.e., the n-lower triangular compact operators). \( \Box \)

**Remark.** It is not true that if \( \mathcal{R}(\delta_S)^- \) contains an n-lower triangular compact operator which is not \( (n-1) \)-lower triangular then \( \mathcal{R}(\delta_S)^- \) contains all n-lower triangular compact operators. In fact \( \mathcal{R}(\delta_S)^- \) will always contain such an operator; namely \( \delta_S(e_1 \otimes e_{n+2}) = w_1 e_2 \otimes e_{n+2} - w_{n+1} e_1 \otimes e_{n+1} \).
**DEFINITION.** A weighted shift satisfies the total products condition if \( \sum w_k \cdot w_{k+1} \cdots \cdot w_{k+n} = \infty \) for all \( n \in \mathbb{N} \).

**Corollary 3.** Let \( S_n = w_n w_{n+1}, n \in \mathbb{N} (\mathbb{Z}) \) be a unilateral (bilateral) weighted shift. Then \( \mathcal{H} \subseteq \mathcal{R}((\delta_S)^-) \) if and only if \( S \) satisfies the total products condition.

We now make application to the question: which weighted shifts are d-symmetric? Recall that an operator \( T \) is d-symmetric if \( \mathcal{R}(\delta_T)^- = \mathcal{R}(\delta_T)^-* \). In [2] it is proved that an operator \( T \) is d-symmetric if and only if \( TT^* - T^*T \in \mathcal{C}(T) = \{ C \in \mathcal{B}(\mathcal{H}) : C \mathcal{B}(\mathcal{H}) + \mathcal{B}(\mathcal{H})C \subseteq \mathcal{R}(\delta_T)^- \} \).

**Theorem 3.** The weights of a d-symmetric weighted shift \( S \) satisfy the total products condition.

**Proof.** By Theorem 1, \( e_i \otimes e_j \in \mathcal{R}((\delta_S)^-) \) for \( i \geq j \). By the d-symmetry of \( S \), we see that \( e_j \otimes e_i = (e_i \otimes e_j)^* \in \mathcal{R}((\delta_S)^-) \) for \( j \leq i \). Thus \( \mathcal{H} \), the linear span of all \( e_i \otimes e_j \), is contained in \( \mathcal{R}((\delta_S)^-) \) and so by Corollary 3, the weights of \( S \) satisfy the total products condition.

The total products condition is not sufficient for d-symmetry else any weighted shift with weights bounded away from zero would be d-symmetric. However the weighted shift with weights alternating between 1 and 2 has an irreducible representation as the operator \( (\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}) \) on \( \mathbb{C}^2 \), while in [2] it is shown that any irreducible representation of a d-symmetric operator must be over a Hilbert space of dimension 1 or \( S_0 \). There are, however, natural conditions under which the total products condition is sufficient.

**Theorem 4.** An essentially normal weighted shift \( S \) is d-symmetric if and only if it satisfies the total products condition.

**Proof.** The necessity of the total products condition follows from Theorem 3 and sufficiency follows from the facts that \( SS^* - S^*S \) is compact and that \( \mathcal{H} \subseteq \mathcal{R}((\delta_S)^-) \) implies \( \mathcal{H} \subseteq \mathcal{C}(S)^- \).

**Corollary 4.** A hyponormal (in particular subnormal) weighted shift \( S_n = w_n e_{n+1} \) is d-symmetric.

**Proof.** If \( S \) is hyponormal, then its weights are increasing and bounded. Thus

\[
SS^* - S^*S = \text{diag}(w_{i-1}^2 - w_i^2)
\]

is compact and \( \sum_{k=1}^{\infty} w_k \cdot w_{k+1} \cdots \cdot w_{k+n-1} \geq \sum_{k=1}^{\infty} w_1^n = \infty \) for all \( n \in \mathbb{N} \).
REFERENCES


Received March 9, 1981 and in revised form September 9, 1981. Some of these results are contained in the author's Ph.D. thesis written under the direction of J. P. Williams at Indiana University. This research was supported in part by a Westmont College Alumni Faculty Development Grant.

WESTMONT COLLEGE
SANTA BARBARA, CA 93108
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $132.00 a year (6 Vol., 12 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Copyright © 1983 by Pacific Journal of Mathematics
Leo James Alex, Simple groups and a Diophantine equation .......................... 257
Herbert James Alexander and John Wermer, On the approximation of singularity sets by analytic varieties .......................... 263
Waleed A. Al-Salam and Mourad Ismail, Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction .......... 269
J. L. Brenner and Roger Conant Lyndon, Permutations and cubic graphs .................................................. 285
Ian George Craw and Susan Ross, Separable algebras over a commutative Banach algebra .................................................. 317
Jesus M. Dominguez, Non-Archimedean Gel’fand theory ......................... 337
David Downing and Barry Turett, Some properties of the characteristic of convexity relating to fixed point theory .......................... 343
James Arthur Gerhard and Mario Petrich, Word problems for free objects in certain varieties of completely regular semigroups ............ 351
Moses Glasner and Mitsuru Nakai, Surjective extension of the reduction operator .................................................. 361
Takesi Isiwata, Ultrafilters and mappings .................................. 371
Lowell Duane Loveland, Double tangent ball embeddings of curves in $E^3$ .................................................. 391
Douglas C. McMahon and Ta-Sun Wu, Homomorphisms of minimal flows and generalizations of weak mixing .......................... 401
P. H. Maserick, Applications of differentiation of $L_p$-functions to semilattices .................................................. 417
Wayne Bruce Powell and Constantine Tsinakis, Free products in the class of abelian $I$-groups .................................................. 429
Bruce Reznick, Some inequalities for products of power sums ................. 443
C. Ray Rosentrater, Compact operators and derivations induced by weighted shifts .................................................. 465
Edward Silverman, Basic calculus of variations ................................ 471
Charles Andrew Swanson, Criteria for oscillatory sublinear Schrödinger equations .................................................. 483
David J. Winter, The Jacobson descent theorem ................................ 495