BASIC CALCULUS OF VARIATIONS

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For the classical one-dimensional problem in the calculus of variations, a necessary condition that the integral be lower semicontinuous is that the integrand be convex as a function of the derivative. We shall see that, if the problem is properly posed, then this condition is also necessary for the k-dimensional problem. For the one-dimensional problem this condition is also sufficient. For the k-dimensional problem this condition is shown to be sufficient subject to an additional hypothesis. For the one-dimensional problem there is an existence theorem if the integrand grows sufficiently rapidly with respect to the derivative, and this result also holds for the k-dimensional problem, subject to an additional hypothesis. Some of these additional hypotheses are automatically satisfied for the one-dimensional problem.

Let \( G \) be a bounded domain in \( \mathbb{R}^k \), \( A = G \times \mathbb{R}^N \), \( Z \) be the space of \((N \times k)\)-matrices and \( F \in C(A \times Z) \). If \( y : G \to \mathbb{R}^N \) is smooth, let \( I_F(y) = \int_G F(x, y(x), y'(x)) \, dx \) where \( y'(x) \) is the matrix of partial derivatives of \( y \).

If \( k = N = 2 \) and if \( F(a, b, p) = | \text{det } p | \) then \( I_F \) is the area integral which is lower semicontinuous though \( F \) is not convex in \( p \) for fixed \((a, b)\). Thus the one-dimensional results do not, apparently, generalize.

There are \( r = (\frac{N+k}{k}) - 1 \) Jacobians of orders 1, \ldots, \min\{k, N\}. Let \( Y = \mathbb{R}^r \). There exists \( \tau : Z \to Y \) such that \( \tau \circ y'(x) = J(y, x) \), where \( J(y, x) = [J(y)](x) \), and \( J(y) \) is the collection of all Jacobians of \( y \), whenever \( y \) is a smooth map. If \( f : A \times Y \to \mathbb{R} \) and if \( f(\theta, \tau(p)) = F(\theta, p) \) for all \((\theta, p)\), then, evidently, \( I(y) = I_F(y) \) where \( I(y) = \int_G f(y_*(x), J(y, x)) \, dx \) and \( y_*(x) = (x, y(x)) \).

If \( u : V \times W \to X \) and if \( v \in V \) let \( u_\circ w = u(v, w) \) for each \( w \in W \).

We define a class \( AC \) of transformations \( y \) for which each component of \( y \) and each component of \( J(y) \), defined in a distribution sense, is in \( L = L(G) \). We consider \( I(y) \) to be the basic integral, not \( I_F(y) \).

Let \( T = \text{range } \tau \). If \( k = 1 \) then \( T = Y \) and \( T \) can be identified with \( Z \) so that \( f = F \). In general, however, setting \( f_\theta \circ \tau = F_\theta \) defines \( f_\theta \) on \( T \subset Y \) where \( T \neq Y \). Let us say that \( f \) is \( T \)-convex if \( f_\theta \) can be extended to a function which is convex over all of \( Y \) for each \( \theta \in A \). Please notice that we do not require that \( f_\theta \) be convex. What we do require is that there exist a convex function over all of \( Y \) which extends \( f_\theta \). Then a necessary condition that \( I \) be lower semicontinuous is that \( f \) be \( T \)-convex. If the extended function is also continuous over \( A \times Y \), then the condition is also sufficient.

In some applications \( f \), rather than \( F \), may be given initially [1].
If $k > 1$ then the parametric problem is not covered by the existence theorem. Even worse, the dichotomy into parametric and non-parametric problems no longer seems feasible. If $k = N = 2$ and if $F(\theta, p) = |\det p|^2$ then $I$ is not parametric. Since it is invariant under smooth area-preserving changes of variables, it has something of the distinguishing feature of parametric integrals. Here $r = 5$ and $f_\theta(t)$ depends upon a single component of $t$. Thus $f_\theta$ does not grow with $\|t\|$.

The starting point of this paper is [5]. Morrey’s sufficiency condition for quasiconvexity gave the idea of using $f$ rather than $F$. That idea, together with the notion of the Cesari-Weierstrass integral [2] and the ideas used in [7] and [8] led to the sufficient condition. The compactness results are familiar [6]. The consistent use of quasilinear functions to approximate continuous functions, rather than Lipschitzian or smoother functions, is standard in area theory, especially in Cesari’s papers.

2. If $y$ is smooth then each component of $J(y)$ is the determinant of a submatrix of order $k$ of $y'_\alpha$, except possibly for sign. One of these submatrices is the identity. Its determinant does not correspond to any component of $J(y)$. Thus $J(y)$ has $r$ components. Let $Y = \mathbb{R}^r$.

If $M \geq m$ let $\Lambda(M, m)$ be the collection of all strictly increasing $m$-termed sequences taken from $\{1, \ldots, M\}$. Let $s = \min\{k, N\}$. If $j \leq s$, if $i \in \Lambda(N, j)$ and $\alpha \in \Lambda(k, j)$, let $p^i_{\alpha} = \det[(p_{\alpha m}^i)_1^{m,n\leq s}]$ and define $\tau: Z \to Y$ by $\tau(p) = \{p^i_{\alpha} \mid (i, \alpha) \in \bigcup_{j=1}^s(\Lambda(N, j) \times \Lambda(k, j))\}$. We may write $[^i_\alpha]$ for $\tau(p)$. Similarly, if $\phi$ is a $(k \times k)$-matrix then the determinants of the $(k \times k)$-submatrices of $[^i_\alpha]$ are in 1-1 correspondence with those of $[^i_\alpha]$. (We delete the determinant of the top matrix, of course.)

Evidently there exists a unique linear map $\phi: Y \to Z$ such that $\Psi \circ \tau(p) = p$ for each $p \in Z$.

If $(i, \alpha) \in \bigcup_{j=1}^s(\Lambda(M, j) \times \Lambda(k, j))$ then there exists $\lambda$, $1 \leq \lambda \leq r$, such that

$$\frac{\partial(y^{i_1}, \ldots, y^{i_s})}{\partial(x^{\alpha_1}, \ldots, x^{\alpha_r})} = \frac{dy^i}{dx^\alpha} = \pm \tau(y')^\lambda.$$

We can suppose that, if $N \geq k$ and $j = s = k$, then $r_0 = (N+k) - (\binom{N}{k}) \leq \lambda \leq r$.

The components of $J(y)$ are, except possibly for sign, the components of $\tau(y')$. Thus there is no loss in generality in ordering the rows of the submatrices in such a way that we can identify $J(y)$ with $\tau(y')$.

3. To obtain the necessary condition for lower semicontinuity we require some information about $\tau$. 


LEMMA 3.1. Let \( \mu_n \in \mathbb{R}, \ n = 1, \ldots, m, \) with \( \Sigma \mu_n = 1. \) If \( p_n, p \) and \( q \in \mathbb{Z} \) with \( \Sigma \mu_n \tau(p_n) = \tau(p) \) then \( \Sigma \mu_n (p_n + q)^j = (p + q)^j \) for \( j = 1, \ldots, k. \)

Proof. We expand and get \( (p + q)^j \) where \( \Sigma' \) is the sum over \( \alpha \in \Lambda(j, i) \) and \( \gamma \in (1, \ldots, j) \) \( \sim \{ \alpha \}. \) Also, \( e_{\alpha, i} = \pm 1. \) Then

\[
\Sigma \mu_n (p_n + q)^j \leq p^j + \sum_{n=1}^{m} \sum_{i=1}^{j-1} e_{\alpha, i} p_{n_i}^j \gamma_1 \cdots \gamma_{j-i} + q^j.
\]

COROLLARY 3.2. \( \tau(p + q) = \Sigma \mu_n \tau(p_n + q). \)

LEMMA 3.3. Let \( y: \mathbb{R}^k \to \mathbb{R}^N \) be quasilinear with compact support \( K \) and simplexes of linearity \( \delta_1, \ldots, \delta_m. \) Let \( p_n = y'(x) \) for \( x \in \text{Int} \delta_n \) and let \( \mu_n = |\delta_n|/|K| \). Then \( \mu_n > 0, \Sigma \mu_n = 1 \) and \( \Sigma \mu_n \tau(p_n) = 0. \)

Except for notation, this is Lemma 4.4 [6].

It is not hard to verify that \( Y \) is the convex hull of \( T. \)

Let us say that \( I \) is lsc if \( I(y) \leq \liminf I(y_n) \) whenever \( y_n \) converges uniformly to \( y, y_n \) and \( y \) satisfy a uniform Lipschitz condition (which may depend upon the sequence) and \( y_n - y \) is quasilinear with support contained in a cube contained in \( G. \) (See Def. 4.4.2, [6].)

If \( N \geq k \) and if \( f(\theta, q) = f(\theta, (0, \ldots, 0, q^{r_0}, \ldots, q^r)) \) for each \( \theta \in A \) then we say that \( f \) depends only upon Jacobians of maximum rank.

LEMMA 3.4. Let \( f \) depend only upon Jacobians of maximum rank and suppose that \( f_{\theta} \in C' \) for each \( \theta \in A. \) If \( I \) is lsc then then \( f \) is T-convex.

Proof. If \( f_{\theta}(\tau(p)) \leq \Sigma \lambda_{\beta} f_{\theta}(\tau(p_{\beta})) \) whenever \( \theta \in A, p, p_{\beta} \in \mathbb{Z}, \lambda_{\beta} > 0, \Sigma \lambda_{\beta} = 1 \) and \( \Sigma \lambda_{\beta} \tau(p_{\beta}) = \tau(p) \), then \( t \mapsto \inf(\Sigma \lambda_{\beta} \tau(p_{\beta}) | \Sigma \lambda_{\beta} \tau(p_{\beta}) = t) \) is an extension of the required type. If

\[
f_{\theta}(\tau(q)) \geq f_{\theta}(\tau(p)) + f_{\theta}(\tau(p)) \tau(q - p)
\]

for all \( \theta \in A, p \) and \( q \in \mathbb{Z}, \) then by Corollary 3.2, \( \Sigma \lambda_{\beta} \tau(p_{\beta} - p) = \tau(0) \) so \( \Sigma \lambda_{\beta} f_{\theta}(\tau(p_{\beta})) \geq \Sigma \lambda_{\beta} f_{\theta}(\tau(p)) + f_{\theta}(\tau(p)) \Sigma \lambda_{\beta} \tau(p_{\beta} - p) = \Sigma \lambda_{\beta} f_{\theta}(\tau(p)) = f_{\theta}(\tau(p)). \)
Let \( Q = \mathbb{R}^k \cap \{ x \mid -\frac{1}{2} \leq x^1, \ldots, x^k \leq \frac{1}{2} \} \) and let \( h > 0 \). Let \( p \in \mathbb{Z} \).

Then \( Q \) is partitioned into \( 3^k \) cells by the hyperplanes \( x^\alpha = \pm h/2, \alpha = 1, \ldots, k \). Each of these cells, except \( hQ \), is then subdivided into \( k! \) simplexes whose vertices are contained in the set of vertices of the containing cell. Let \( S \) be the set of all these simplexes. Now we define \( \alpha \) continuous (quasilinear) function \( \xi \) on \( Q \) into \( \mathbb{R}^k \) by putting \( \zeta(x) = px \) if \( x \in hQ \), \( \zeta(x) = 0 \) if \( x \in \partial Q \) and \( \xi \mid \sigma \) is linear (affine) if \( \sigma \in S \). If \( x \in \text{Int} \sigma \) let \( \zeta'(x) = p_\sigma \). Thus, by Lemma 3.3, \( \tau(p)h^k + \sum_{\sigma \in S} \tau(p_\sigma) \mid \sigma \mid = 0 \). Also, for each \( \sigma \in S \) there exists \( j \in \{1, \ldots, k\} \) such that \( j \) columns of \( p_\sigma \) are \( O(h) \) and \( \mid \sigma \mid = O(h^{k-j}) \).

By Theorem 4.4.2 [6],

\[
\begin{align*}
f_\theta(\tau(0)) &\leq \int_Q f_\theta(\tau(\xi'(x))) \, dx = f_\theta(\tau(p))h^k + \sum_{\sigma \in S} f_\theta(\tau(p_\sigma)) \mid \sigma \mid \\
&= f_\theta(\tau(p))h^k + \sum_{\sigma \in S} \left[ f_\theta(\tau(0)) + f_\theta(\tau(0))\tau(p_\sigma) + o(\tau(p_\sigma)) \right] \mid \sigma \mid \\
&= f_\theta(\tau(p))h^k + f_\theta(\tau(0))(1 - h^k) - f_\theta(\tau(0))\tau(p)h^k \\
&\quad + \sum_{\sigma \in S} O(\tau(p_\sigma)) \mid \sigma \mid
\end{align*}
\]

so that \( f_\theta(\tau(0))h^k + f_\theta(\tau(0))\tau(p)h^k \leq f_\theta(\tau(p))h^k + \sum_{\sigma \in S} O(\tau(p_\sigma)) \mid \sigma \mid \). If \( f \) depends only upon Jacobians of rank \( k \), then the last term on the right is \( o(O(h^k)) = o(h^k) \) so that \( f_\theta(\tau(p)) \geq f_\theta(\tau(0)) + f_\theta(\tau(0))\tau(p)h^k \).

**Corollary 3.5.** The lemma remains valid if the differentiability condition is dropped.

**Proof.** Let \( F_\theta = f_\theta \circ \tau \) and suppose that \( F_\theta \in C' \). Then \( f_\theta = F_\theta \circ \Psi \), \( f_\theta' = (F_\theta' \circ \Psi')\Psi' \) and \( f_\theta \in C' \). If \( F_\theta \not\in C' \) we mollify. Let \( B \) be the unit sphere in \( \mathbb{Z} \), let \( \mu \in C^\infty(\mathbb{Z}) \) be nonnegative with support contained in \( B \) and \( \int B \mu(\xi) \, d\xi = 1 \). If \( \rho > 0 \) let \( \mu_\rho(\xi) = 1/\rho^N \mu(\xi/\rho) \).

If \( y_n \to y \) then \( y_n - \xi \to y - \xi \) where, because of the definition of lsc, we can suppose that \( y_n - \xi \) and \( y - \xi \) differ only on a compact subset of \( G \). A routine argument shows that \( y \mapsto \int_G F(y_*(x), y'(x) - \xi) \, dx \) is lsc. Thus

\[
y \mapsto \int_R F_\rho(y_*(x), y'(x)) \, dx
\]

is lsc where \( F_\rho(\theta, p) = \int_{B^\rho} F((\theta, p - \xi)\mu_\rho \mid \xi \mid) \, d\xi \). Let \( f_\rho(\theta, q) = F_\rho(\theta, \Psi q) \). Then \( (f_\rho)_\theta \in C' \) since \( (F_\rho)_\theta \in C' \). Thus, by the lemma, \( f_\rho \) is \( T \)-convex and the corollary follows by letting \( \rho \to 0 \).

**Theorem 3.6.** Let \( I \) be lsc. Then \( f \) is \( T \)-convex.
Proof. If \( \theta \in A \) let \( g(\theta, [\phi]) = g_\theta([\phi]) = f_\theta([l]) \). (See §2.) Now let

\[
h\left( \theta, \left[ \begin{array}{c} I \\ \phi \\ p \end{array} \right] \right) = g\left( \theta, \left[ \begin{array}{c} \phi \\ p \end{array} \right] \right).
\]

Let \( Z_0, Y_0 \) and \( \Psi_0 \) correspond to \( Z, Y \) and \( \Psi \) with \( \mathbb{R}^{N+k} \) replacing \( \mathbb{R}^N \). Let \( h_\theta \) be defined over all of \( Y_0 \) by \( h_\theta(q) = h_\theta(r) \) if \( \Psi_0 q = \Psi_0 r \). By this construction \( h \in C(A \times Y_0) \), \( h \) is nonnegative and \( h \) depends only upon Jacobians of maximum rank.

If \((\xi, y) : G \to \mathbb{R}^k \times \mathbb{R}^N\) then let

\[
I_h(\xi, y) = \int_G h\left( y_*(x), \left[ \begin{array}{c} I \\ \xi'(x) \\ y'(x) \end{array} \right] \right) \, dx
\]

and \( I_h \) is lsc. Thus \( h \) is \( T \)-convex. In a natural way \( Y = \text{dom } f_\theta \subseteq \text{dom } h_\theta \). Furthermore, \( h_\theta \) extends \( f_\theta | T \). Thus \( g_\theta = h_\theta | Y \) is an extension of \( f_\theta | T \) which is convex over all of \( Y \).

4. In this section we define a class of transformations, which we call \( AC \), on which \( I \) is defined. This class is probably not a vector space.

Let \( \mathcal{O} = C^\infty(G) \), \( L = L_1(G) \) and \( L_p = L_p(G) \) for \( p > 1 \). If \( B \) is one of these spaces let \( F_0 B = B \), \( F_j B = 0 \) if \( j > k \) and, if \( 1 \leq j \leq k \), let

\[
F_j B = \left\{ \omega \mid \omega = \sum_{\lambda \in \Lambda(k,j)} \omega_\lambda \, dx^\lambda \text{ where each } \omega_\lambda \in B \right\}.
\]

As usual, \( dx^\lambda = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_j} \).

If \( \omega \in F_j L \) and if there exists \( \zeta \in F_{j+1} L \) such that

\[
\int \omega \wedge d\phi = (-1)^{j+1} \int \zeta \wedge \phi
\]

for each \( \phi \in F_{k-j-1} \mathcal{O} \), then we say that \( \omega \in \mathcal{F}_j H \) and write \( d\omega \) for \( \zeta \). If \( d\omega \) exists, then \( d\omega \) is unique.

By putting an appropriate norm on \( \mathcal{F}_o H \) we can identify this space with \( H = H^1_1(G) \). Also, \( H_o = H^1_0(G) \) is the closure, in \( H \), of \( \mathcal{F} \), of \( \mathcal{F}_o \).

If \( \omega_n = \sum \omega_n^\lambda \, dx^\lambda \) and \( \omega = \sum \omega_\lambda \, dx^\lambda \) are in \( F_j L \) then \( \omega_n \rightharpoonup \omega \) in \( F_j L \) if \( \omega_n^\lambda \rightharpoonup \omega_\lambda \) in \( L \) for each \( \lambda \), where \( \rightharpoonup \) denotes weak convergence on compact subsets of \( G \).

**Lemma 4.1.** If \( \omega_n \rightharpoonup \omega \) in \( F_j L \), if \( \omega_n \in \mathcal{F}_j H \) and if \( d\omega_n \rightharpoonup \zeta \) in \( F_{j+1} L \) then \( \omega \in \mathcal{F}_j H \) and \( d\omega = \zeta \).
Proof. Let $\phi \in F_{k-j-1}$). Then
\[
\int \omega \wedge d\phi = \lim \int \omega_n \wedge d\phi = (-1)^{j+1} \lim \int d\omega_n \wedge \phi = (-1)^{j+1} \int \xi \wedge \phi.
\]

**Lemma 4.2.** If $\omega \in \mathcal{F}_j H$ then $x^\alpha \omega \in \mathcal{F}_j H$ and
\[
d(x^\alpha \omega) = dx^\alpha \wedge \omega + x^\alpha d\omega.
\]

**Proof.** Let $\phi \in F_{k-j-1}$ and $\psi = x^\alpha \phi$ so that $d\psi = dx^\alpha \wedge \phi + x^\alpha d\phi$ and
\[
\int x^\alpha \wedge d\phi = \int \omega \wedge [d\psi - dx^\alpha \wedge \phi] = \int \omega \wedge d\psi + (-1)^{j+1} \int dx^\alpha \wedge \omega \wedge \phi = (-1)^{j+1} \int (x^\alpha d\omega + dx^\alpha \wedge \omega) \wedge \phi.
\]

**Lemma 4.3.** If $\omega \in \mathcal{F}_j H$ then $d^2 \omega = 0$.

**Proof.** Let $\xi = d\omega$ and $\phi \in F_{k-j-2}$. Then $\int \xi \wedge d\phi = (-1)^j \int \omega \wedge d^2 \phi = 0 = (-1)^j 0$ so that $d^2 \omega = d\xi = 0$.

If $z \in H$ then $dz = \sum_{\alpha} z^\alpha dx^\alpha$ where $\{z^\alpha\}$ is the set of distribution derivatives of $z$. Let $M$ be a positive integer and $s = \min\{k, M\}$. Suppose that $dz^i$ has been defined for $i \in \Lambda(M, j), j \leq s - 1$. If $h \in \Lambda(M, j + 1), m = h_1$ and $i = h \sim \{m\} \in \Lambda(M, j)$ then we define $dz^h$, if $z^m dz^i \in \mathcal{F}_j H$, by $dz^h = d(z^m dz^i)$.

If $dz^j$ is defined for $i \in \Lambda(M, j)$ and $\alpha \in \Lambda(k, j)$ then we define $z^i_\alpha$ by
\[
dz^i = \sum_{\alpha \in \Lambda(k, j)} z^i_\alpha dx^\alpha
\]
so that, if $z$ is smooth, $z^i_\alpha = (\partial(z^i_1, \ldots, z^i_j)/\partial(x^\alpha_1, \ldots, x^\alpha_j))$.

Let $y \in L^N$ and suppose that $dy^i$ is defined for each $i \in \Lambda(M, s)$, where $s = \min\{N, k\}$, and thus for each $i \in \bigcup_{j=1}^s \Lambda(M, j)$. Then we can suppose that $J(y) = \{y^i_\alpha | (i, \alpha) \in \bigcup_{j=1}^s (\Lambda(N, j) \times \Lambda(k, j))\}$ is an element of $L^\prime$.

If $J(y)$ is defined and if $J(y) = \tau(y')$ almost everywhere then we say that $y \in AC$. By the definition of $\mathcal{F}_j H$, the components of $J(y)$ are functions.

The following lemmas are immediate.
Lemma 4.4. \( y^* \in AC \) if and only if \( y \in AC \) and \( J(y) = \{ y_{*\beta}^i | i \in \Lambda(k + N, j) \) and \( \beta = (1, \ldots, k) \}).

Lemma 4.5. Let \( j \leq s = \min\{N, k\} \) and \( y \in AC \). If \((i, \alpha) \in \Lambda(N, j) \times \Lambda(k, j)\) for \( 1 \leq j \leq s \) then there exists \( h \in \Lambda(k + N, k) \) such that, except possibly for sign, \( y_{*h}^i = y_{\alpha}^i \).

Let \( y_n \in AC \) and \( y \in L^N \) with \( y_{*m}^n \to y_{*m}^* \) in \( L \) for each \( m \in \Lambda(k + N, 1) \). Suppose that if \( j \leq k \) and \( i \in \Lambda(k + N, j) \) there exists \( \xi^i \in F_j L \) such that \( dy_{*n}^i \to \xi^i \) in \( F_j L \). If, in addition, \( y_{*m}^n d\gamma_n^i \to y_{*m}^* \gamma^i \) in \( F_j L \) whenever \( i \in \Lambda(k + N, j) \), \( j < k \), \( m \in \Lambda(k + N, 1) \), and \( m \notin i \) then we say that \( y_n \Rightarrow y \).

Theorem 4.6. If \( y_n \to y \) then \( y \in AC \) and \( J(y_n) \to J(y) \) in \( L \).

Proof. By Lemma 4.1, \( J(y) \) is defined. By Theorem 3.4.4 [6], \( y_{*m}^n \to y_{*m}^* d\gamma_n^i \to y_{*m}^* \gamma^i \) in \( L(K) \) for each compact set \( K \subset G \). Hence we can suppose that \( y_{*m}^n \to y_{*m}^* \gamma^i \) almost everywhere in \( G \). We can also suppose that \( i \neq (1, 2, \ldots, k) \). Hence there exists \( m \in \{1, \ldots, k\} \), \( m \notin i \), such that \( x^*m \gamma_n^i \to x^*m \gamma^i \) so that \( dy_{*n}^i \to dy^i \) almost everywhere.

Lemma 4.7. If \( p \) and \( q \) are Lebesgue conjugate, if \( f_n \to f \) in \( L_p \) and \( g_n \to g \) in \( L_q \) then \( f_n g_n \to fg \) in \( L \).

Proof. Let \( E \) be a measurable subset of a compact subset of \( G \). Then

\[
\int_E (f_n g_n - f_g) \, dx = A_n + B_n
\]

where \( A_n = \int_E f(g_n - g) \, dx \) and \( B_n = \int_E (f_n - f)g_n \, dx \). By the weak convergence, \( A_n \to 0 \) and \( \{ \int_E |g_n(x)|^q \, dx \}^{1/q} \) is bounded independently of \( n \). Thus \( B_n \to 0 \) by the Hölder inequality.

If \( y \in AC \) and \( y^i_{*\beta} \in L_p \) for each \( i \in \Lambda(k \times N, k) \), where \( \beta = (1, \ldots, k) \), then we set \( \| J(y) \|_p = \sum_{i \in \Lambda(k \times N, k)} \| y_{*\beta}^i \|_p \).

If \( y_o \in AC \) let \( \mathcal{M}(y_o) = AC \cap \{ y | y - y_o \in (H_o)^N \} \).

Theorem 4.8. Suppose that there exists \( M > 0 \) such that for each \( y \in \mathcal{M}(y_o) \) either

(i) \( \| y \|_\infty \leq M \) and \( \| J(y) \|_p \leq M \) for some \( p > 1 \), or

(ii) \( \| J(y) \|_q \leq M \) where \( q = 2k/(k + 1) \). Then \( \mathcal{M}(y_o) \) is \( \Rightarrow \) sequentially compact.

Proof. If (i) holds then \( \| y \|_1 \) is uniformly bounded so that there exists a sequence \( \{ y_n \} \) in \( \mathcal{M}(y_o) \) and \( \xi \in (H_o)^N \) such that \( y_n - y_o \to \xi \) in \( (H_o)^N \).
Thus $y_n - y_0 \to \xi$ in $L$. Let $y = y_0 + \xi$. By passing to a subsequence we can suppose that $y_n(x) \to y(x)$ a.e. By the bounded convergence theorem, $y_n^* \to y^*$ in $(L_s)^N$ where $s = r/(r - 1)$ is Lebesgue conjugate to $p$. If (ii) holds then there exists a sequence $\{y_n\}$ in $\mathcal{M}(y_0)$ and $\xi \in (H_{q,0})^N$ such that $y_n - y_0 \to \xi$ in $(H_{q,0})^N$. Thus, by Th. 3.5.3, [6], $y_n \to y$ in $L_t$ where $1/t = 1/q - 1/k = (k - 1)/2k$ so that $t$ is conjugate to $q$. The theorem follows by induction, Lemma 4.1 and Lemma 4.7.

5. We make use of a type of convexity studied by Tonelli to show that $T$-convexity is sufficient for lower semicontinuity.

According to Tonelli, a $T$-convex function $f$ is semi-regular positive semi-normal if for each $\theta \in A$, $p, q \in Y$ with $q \neq 0$, there exists $\lambda \in \mathbb{R}$ such that $2f(\theta, p) < f(\theta, p + \lambda q) + f(\theta, p - \lambda q)$.

For the following lemma see Turner [10].

**Lemma 5.1.** A necessary and sufficient condition that $f$ be semi-regular positive semi-normal is that for each $\varepsilon > 0$ and each $(\theta, p) \in A \times Y$, there exists $\delta > 0$, $\nu > 0$, $\xi \in Y^*$ and $\rho \in \mathbb{R}$ such that for all $\phi \in A$ with $\|\phi - \theta\| < \delta$,

(a) $f(\phi, q) \geq \xi q + \rho + \nu\|q - p\|$ for each $q \in Y$ and

(b) $f(\phi, q) \leq \xi q + \rho + \varepsilon$ if $\|q - p\| < \delta$.

Let $f$ be semi-regular positive. If $\xi \in Y^*$ let

$$
\rho_\xi(\theta) = \inf\{f(\theta, q) - \xi q \mid q \in Y\}
$$

for each $\theta \in A$. Thus $f(\theta, p) = \sup\{\xi p + \rho_\xi(\theta) \mid \xi \in Y^*\}$.

Let $\sigma_\xi(\phi) = \liminf_{\theta \to \phi} \rho_\xi(\theta)$ where $\theta$ and $\phi$ belong to $A$, of course.

Then $\rho_\xi$ is upper semicontinuous, $\sigma_\xi$ is lower semicontinuous and $\sigma_\xi \leq \rho_\xi$.

**Theorem 5.2.** If $f$ is semi-regular positive semi-normal, then $f(\theta, p) = \sup\{\xi p + \sigma_\xi(\theta) \mid \xi \in Y^*\}$.

**Proof.** Let $\varepsilon > 0$. By Lemma 5.1 there exist $\delta > 0$, $\nu > 0$, $\xi \in Y^*$ and $\rho \in \mathbb{R}$ such that if $\phi \in A$ and $\|\phi - \theta\| < \delta$, then

(a) $f(\phi, q) \geq \xi q + \rho + \nu\|q - p\|$ for each $q \in Y$, and

(b) $f(\phi, q) \leq \xi q + \rho + \varepsilon$ if $\|q - p\| < \delta$.

Hence $\rho_\xi(\phi) \geq \rho$ for each $\phi \in A$ with $\|\phi - \theta\| < \delta$ so that $\sigma_\xi(\theta) \geq \rho$ and $f(\theta, p) \leq \xi p + \sigma_\xi(\theta) + \varepsilon$.

We say that $f$ is $V$-convex if $f(\theta, p) = \sup\{\xi p + \sigma_\xi(\theta) \mid \xi \in Y^*\}$ for each $\theta \in A$. Thus $f$ is $V$-convex if $f$ is semi-regular positive semi-normal.

6. In this section we show that if $f \in C(A \times Y)$ is nonnegative and $T$-convex, then $I$ is lower semicontinuous.
Let \( \{e^\lambda\} \) be a dual basis for \( Y^* = e^\lambda e_\mu = \delta^\lambda_\mu \) for \( e_\mu \in Y \). If \( \xi \in Y^* \) there exist \( \xi_\lambda \in \mathbb{R} \) such that \( \xi = \sum \xi_\lambda e^\lambda \).

Let \( \mathcal{S} \) be the collection of all finite families \( \sigma \) of compact subsets contained in \( G \) such that if \( K \in \sigma \) and \( L \in \sigma \), \( |K \cap L| = 0 \) whenever \( K \neq L \).

If \( y \in AC \), \( \xi \in Y^* \) and \( K \) is a compact subset of \( G \), let \( A(\xi, y, K) = \xi(\int_K J(y, x) \, dx) = \int_K \xi(J(y, x)) \, dx \) and

\[
B(\xi, y, K) = \left( \inf \left\{ \sigma_\xi(y_*(x)) \right\} \mid x \in K \right) |K|.
\]

Now we define \( \mathcal{J} \) on \( AC \) by

\[
\mathcal{J}(y) = \sup_{\sigma \in \mathcal{S}} \sup_{K \in \sigma} \left[ A(\xi, y, K) + B(\xi, y, K) \right].
\]

**Lemma 6.1.** Let \( y_n \) and \( y_0 \) belong to \( AC \) with \( y_n - y_0 \in (H_0)^N \). If \( y_n - y_0 \to \xi \) in \( H^N \) and if we set \( y = y_0 + \xi \) then \( y - y_0 \in (H_0)^N \) and \( y_n \to y \) in \( (L_1(K))^N \) for each compact subset \( K \) of \( G \).

This lemma follows from Theorems 3.2.1 and 3.4.4 [6].

**Lemma 6.2.** Let \( X \) be a measurable subset of \( G \) and \( \{f_n\} \) be a sequence of measurable functions with \( f_n(x) \to f(x) \) a.e. in \( X \). Let \( \varepsilon > 0 \). Then there exists a compact set \( K \subset X \) with \( |X \sim K| < \varepsilon \), \( f_n \mid K \) continuous for each \( n \) and \( f_n \mid K \to f \mid K \) uniformly.

This lemma follows from Egoroff's Theorem and Lusin's Theorem.

**Theorem 6.3.** Let \( f \) be \( V \)-convex and suppose that \( y_n \) and \( y \) are in \( \mathcal{C}(y_0) \). If \( (y_n, J(y_n)) \to (y, J(y)) \) in \( L^N \times L' \) then \( \mathcal{J}(y) \leq \lim \inf \mathcal{J}(y_n) \).

**Proof.** Let \( K \) be a compact subset of \( G \). By Lemma 6.1 we can suppose that \( y_n \to y \) in \( L(K)^N \) so that (passing to a subsequence if necessary) \( y_n(x) \to y(x) \) for almost all \( x \in K \). Let \( M > 0 \), \( M(\theta) = \min\{\sigma_\xi(\theta), M\} \) and let \( f^M(\theta, p) = \sup\{\xi p + \sigma_\xi(\theta) \mid \xi \in Y^*\} \). It is sufficient to show that the theorem holds with \( f \) replaced by \( f^M \). Hence we can suppose that \( \sigma_\xi(\theta) \leq M \) for all \( (\theta, \xi) \in \mathbb{R} \times Y^* \). Let \( \varepsilon > 0 \). There exists \( \eta \in (0, \varepsilon/M) \) such that \( \int_E \xi(y_*(x)) \, dx < \varepsilon \) if \( E \) is a measurable subset of \( K \) with \( |E| < \eta \). By Lemma 6.2 there exists a compact set \( C \subset K \) such that \( |K \sim C| < \eta \), \( y_n \mid C \) is continuous and \( y_n \to y \) uniformly on \( C \). Hence

\[
B(\xi, y, C) = \left( \inf_{x \in C} \sigma_\xi(y_*(x)) \right) |C| \geq \left( \inf_{x \in K} \sigma_\xi(y_*(x)) \right) |C| \geq B(\xi, y, K) - \varepsilon.
\]
Also there exist $x_n \in C$ such that $\sigma_\xi(y_n(x_n)) = \inf_{x \in C} \sigma_\xi(y_n(x))$. We can suppose that $x_n \to x \in C$. Now $y_n(x_n) \to y(x)$ so that $\sigma_\xi(y_\star(x)) \leq \liminf \sigma_\xi(y_n(x_n))$. Thus $B(\xi, y, C) \leq \liminf B(\xi, y_n, C)$ while $A(\xi, y, C) = \lim A(\xi, y_n, C)$. The theorem follows.

**Theorem 6.4.** Let $f$ be $V$-convex. If $y \in AC$ then $\mathcal{I}(y) = I(y)$.

**Proof.** Let $K$ be a compact subset of $G$ and $\xi \in Y^*$. Then
\[
\int_K f(y_\star(x), J(y, x)) \, dx \\
\geq \int_K \left[ \xi(J(y, x)) + \sigma_\xi(y_\star(x)) \right] \, dx \geq A(\xi, y, K) + B(\xi, y, K)
\]
so that $I(y) \geq \mathcal{I}(y)$ and we can suppose that $\mathcal{I}(y) < \infty$. If $L$ is an interval contained in $G$ let $\mathcal{S}_L = \mathcal{S} \cap \{\sigma \mid \bigcup_{K \in \sigma} K \subset L\}$ and let
\[
\mathcal{\Phi}(L) = \sup_{\sigma \in \mathcal{S}_L} \sup_{K \in \sigma} \left[ A(\xi, y, K) + B(\xi, y, K) \right].
\]
Then $\Phi$ is nonnegative, superadditive and of bounded variation. Let $D\Phi$ be the Lebesgue derivative of $\Phi$ with respect to cubes. Then $D\Phi(x) \geq \xi(J(y, x)) + \sigma_\xi(y_\star(x))$ so that $D\Phi(x) \geq f(y_\star(x), J(y, x))$ almost everywhere in $G$. Evidently $\mathcal{I}(y) \geq \sup_{\sigma \in \mathcal{S}} \sum_{L \in \sigma} \Phi(E)$ where $\mathcal{S}' = \mathcal{S} \cap \{\sigma \mid \sigma$ is a family of finitely many non-overlapping intervals}. Thus $\mathcal{I}(y) \geq \sup_{\sigma \in \mathcal{S}} \sum_{L \in \sigma} \int_L f(y_\star(x), J(y, x)) \, dx = I(y)$.

**Corollary 6.5.** The theorem holds if $f \in C(A \times Y)$ and $f_\theta$ is convex for each $\theta \in A$. Thus $I$ is lsc if $f$ is continuous and $T$-convex.

**Proof.** Let $\epsilon > 0$ and $g(\theta, q) = f(\theta, q) + \epsilon \|q\|$ for each $(\theta, q) \in A \times Y$. Let $I_g(y) = \int_Y g(y_\star(x), J(y, x)) \, dx$. If $J(y_n) \to J(y)$ in $L'$ then there exists $m > 0$ such that $\|J(y_n)\| < m$ for each $n$. Hence $I(y) \leq I_g(y) \leq \liminf I_g(y_n) = \liminf I(y_n) + \epsilon \|I(y_n)\| \leq \liminf I(y_n) + m \epsilon$ since $g$ is semi-regular positive semi-normal and hence $V$-convex.

The construction in Theorem 3.5 can be used to show that not only is $T$-convexity a necessary condition that $I$ be lower semi-continuous with respect to the convergence of that theorem, but also with respect to the convergence of Corollary 6.5.

The gap between the necessary and sufficient conditions for lower semi-continuity can now be described by the fact that $f$ can be $T$-convex without being continuous (but see the paragraph preceding Corollary 7.3).

Since $\Rightarrow$ is stronger than $\to$, the following corollary is immediate.

**Corollary 6.6.** If $y_n \Rightarrow y$ in $\mathfrak{M}(y_\theta)$ then $I(y) \leq \liminf I(y_n)$.
7. We conclude with an existence theorem and some minor generalizations.

**Theorem 7.1.** Let \( f \in C(A \times Y) \) be nonnegative and \( f_\theta \) be convex for each \( \theta \in A \). If \( \mathcal{M}(y_o) \) is \( \Rightarrow \) compact and if \( \inf \{ I(y) \mid y \in \mathcal{M}(y_o) \} < \infty \) then \( I \) attains its minimum on \( \mathcal{M}(y_o) \).

This result follows from Corollary 6.6.

**Corollary 7.2.** Suppose that there exists \( m > 0 \) such that for each \((\theta, s) \in A \times Y\) either

(i) There exists \( M > 0 \) and \( p > 1 \) such that \( \| y \|_\infty < M \) and \( f(\theta, s) \geq m\| s \|^p \), or

(ii) \( f(\theta, s) \geq m\| s \|^q \) where \( q = 2k/(k + 1) \). If \( \inf \{ I(y) \mid y \in \mathcal{M}(y_o) \} < \infty \) then \( I \) attains its minimum on \( \mathcal{M}(y_o) \).

The corollary follows from Theorem 4.8.

Let \( Y_1 \) be a compact convex subset of \( Y \). If \( y_o \in AC \) and if \( J(y_o, x) \in Y_1 \) for almost all \( x \in G \), then let

\[
\mathcal{M}_1(y_o) = \mathcal{M}(y_o) \cap \{ y \mid J(y, x) \in Y_1 \text{ for almost all } x \in G \}.
\]

Let \( f \in C(A \times Y_1) \). If \( I \) is lower semicontinuous on \( \mathcal{M}_1(y_o) \) then, as before, \( f \) must be \( T \)-convex, i.e., there exists \( g_\theta : Y_1 \rightarrow \mathbb{R} \) where \( g_\theta \) is convex and extends \( f_\theta \) for each \( \theta \in A \). Since \( Y_1 \) is compact, it follows that \( g \) is continuous so, for this case, a necessary and sufficient condition that \( I \) be lower semicontinuous is that \( f \) be \( T \)-convex. Thus the next corollary follows from the preceding one.

**Corollary 7.3.** Let \( Y_1 \) be a compact convex subset of \( Y \) and \( f \in C(A \times Y_1) \) be \( T \)-convex. If, in addition, \( f \) satisfies (i) or (ii) and \( \inf \{ I(y) \mid y \in \mathcal{M}_1(y_o) \} < \infty \) then \( I \) attains its minimum on \( \mathcal{M}_1(y_o) \).

Let \( Y_2 \) be a compact subset of \( Y \) and \( f \in C(A \times Y_2) \). Let \( Y_1 \) be the convex hull of \( Y_2 \) and let \( g \) be defined on \( A \times Y_1 \) by

\[
g(\theta, q) = \inf \left\{ \sum_{i=1}^{n} \lambda_i f(\theta, p_i) \mid p_i \in Y_2, \lambda_i > 0, \sum \lambda_i = 1, \text{ and } \sum \lambda_i p_i = q \right\}.
\]

If \( g \in C(A \times Y_1) \) is \( T \)-convex and if

\[
\inf \{ I_g(y) \mid y \in \mathcal{M}_1(y_o) \} < \infty,
\]

then \( I \) attains its minimum on \( \mathcal{M}_1(y_o) \).
where $I_g(y) = \int_G g(y_*(x), J(y, x)) \, dx$, then, by Corollary 7.3, there exists $z \in \mathcal{N}_1(y_o)$ such that $g(z) = \min\{I_g(y) \mid y \in \mathcal{N}_1(y_o)\}$. Then $z$ is called a relaxed minimizer for $f$ on $Y_2$.

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