

Pacific Journal of Mathematics

THE JACOBSON DESCENT THEOREM

DAVID J. WINTER

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A direct proof of the Jacobson Descent Theorem is given and used to prove the Jacobson-Bourbaki Correspondence Theorem.

The purpose of this paper is to give a proof of the *Jacobson Descent Theorem*, Theorem 1, which is direct in that it does not assume that $A = \text{Hom}_{K^A} K$. This is then used to prove the *Jacobson-Bourbaki Correspondence Theorem*, Theorem 2. The approach simplifies earlier proofs.

A variation of a theme of Hochschild appearing in Jacobson [2] and Winter [3] recurs here in the concentrated form of the dual bases x_i, R_j which thread their way through both proofs. Thus, this paper underlines the importance of this natural duality.

Throughout the paper, K denotes a field, $\text{End } K$ denotes the ring of endomorphisms of K as additive group, A denotes a subring of $\text{End } K$ containing the K -span KI of the identity I of $\text{End } K$ and V denotes a vector space over K of finite or infinite dimension $V: K$. Regard A as left K -vector space in the obvious way.

DEFINITION 1. An A -product on V is a mapping $A \times V \rightarrow V$, denoted $(T, v) \rightarrow T(v)$, such that V is an A -module and

$$(xT)(v) = x(T(v)) \quad (x \in K, T \in A, v \in V). \quad \square$$

Clearly $T(v)$ ($T \in A, v \in K$) is an A -product for K .

Suppose henceforth that $T(v)$ ($T \in A, v \in V$) is an A -product for V , and $V^A = \{v \in V \mid T(xv) = T(x)v \text{ for } T \in A, x \in K\}$. In particular, we have then defined K^A .

DEFINITION 2. For k a subfield of K , a k -form of V is a k -subspace V^k of V whose k -bases are K -bases of V . □

THEOREM 1 (Jacobson [1]). Let $A: K < \infty$, then V^A is a K^A -form of V .

Proof. $\hat{K} = \{\hat{x} \mid x \in K\}$ separates A and therefore contains a basis $\hat{x}_1, \dots, \hat{x}_n$ for the K -dual space $\text{Hom}_K(A, K)$ of A where $\hat{x} \in \text{Hom}_K(A, K)$ is defined for $x \in K$ by $\hat{x}(T) = T(x)$ ($T \in A$). Letting R_1, \dots, R_n be a dual basis for A , so that $R_i(x_j) = \delta_{ij}$ ($1 \leq i, j \leq n$), we have $T(xR_i)(x_j) = T(x\delta_{ij}) = T(x)\delta_{ij} = (T(x)R_i)(x_j)$ ($1 \leq i, j \leq n$) so that $T(xR_i) = T(x)R_i$ ($1 \leq i \leq n$) for all T , since the x_j separate A .

Letting $v \in V$, we therefore have $T(xR_i(v)) = (T(xR_i))(v) = T(x)R_i(v)$ for all $T \in A$, $x \in V$, so that $R_i(V) \subset V^A$ and, in particular, $R_i(K) \subset K^A$ ($1 \leq i \leq n$).

It follows that the K -span KV^A of V^A is V . For we have $I = \sum_1^n y_i R_i$ for suitable $y_i \in K$, so that $v = \sum_1^n y_i R_i v \in KV^A$ for all $v \in V$.

Finally, let v_i ($i \in I$) be a K^A -basis for V^A . Suppose that $\sum_{i \in I} y_i v_i = 0$ with the y_i in K . Then $0 = \sum_{i \in I} R_j(y_i) v_i$ with the $R_j(y_i) \in K^A$, so $R_j(y_i) = 0$ ($1 \leq i, j \leq n$) and $y_i = 0$ ($1 \leq i \leq n$). \square

THEOREM 2 (*Jacobson [2]*). *Let $A: K < \infty$. Then $A = \text{Hom}_{K^A} K$.*

Proof. A as left A -module satisfies $(xS)T = x(ST)$ ($x \in K$, $S, T \in A$), so that A^A is a K^A form of A and A^A contains a basis R_1, \dots, R_n for A over K . Choosing $x_i \in K$ so that $I = x_1 R_1 + \dots + x_n R_n$, we have $x = \sum_{i=1}^n x_i R_i(x)$, $R_i(x) \in K^A$, for $x \in K$, so that $K: K^A \leq A: K \leq \text{Hom}_{K^A} K: K \leq K: K^A$ and $A = \text{Hom}_{K^A} K$. \square

It is clear, in retrospect, that the above x_1, \dots, x_n form a basis for K over $K^A = k$ and that A^A is the dual space $\text{Hom}_k(K, k) = K^*$ of K over k . The equations $x_j = \sum_{i=1}^n x_i R_i(x_j)$ show that $R_i(x_j) = \delta_{ij}$, that is, R_1, \dots, R_n is a dual basis for K^* . Finally, $I = x_1 R_1 + \dots + x_n R_n$ shows that $T = \sum_1^n x_j R_j T$ and $T(x_i) = \sum_1^n x_j R_j T(x_i) = \sum_1^n (R_j \hat{x}_i(T)) x_j$. Thus, the $X_{ij} = R_j \hat{x}_i$ (composite) ($1 \leq i, j \leq n$) are the coordinate functions on the $T \in A$ relative to the basis x_i . They form a basis for the k -dual space A^* of $A = \text{Hom}_k K$. Since we may identify $\hat{K} = \{\hat{x} \mid x \in K\}$ with K , it follows that the k -dual space A^* of $A = \text{Hom}_k K = KK^*$ can be identified with A whereby X_{ij} corresponds to $X_i R_j$ —that is, we have a nondegenerate bilinear k -pairing $\langle \cdot, \cdot \rangle$ on $A \times A$ such that $\langle x_i R_j, T \rangle = R_j T(x_i)$. This pairing is also characterized by the condition $\langle xR, yS \rangle = R(y)S(x)$ ($x, y \in K$, $R, S \in K^*$). Since $(x_i R_j)(x_r R_s) = x_i (R_j x_r R_s) = x_i R_j(x_r) R_s = x_i \delta_{jr} R_s$, the $E_{ij} = x_i R_j$ form a system of matrix units for A . We have $\langle E_{ij}, E_{rs} \rangle = \langle x_i R_j, x_r R_s \rangle = R_j(x_r) R_s(x_i) = \delta_{jr} \delta_{is} = \text{Trace}(E_{ij} E_{rs})$. It follows that $\langle S, T \rangle = \text{Trace } ST$ ($S, T \in A$).

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