Pacific Journal of Mathematics

# THE JACOBSON DESCENT THEOREM

DAVID J. WINTER

Vol. 104, No. 2 June 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

# THE JACOBSON DESCENT THEOREM

DAVID J. WINTER

**A direct proof of the Jacobson Descent Theorem is given and used to prove the Jacobson-Bourbaki Correspondence Theorem.**

The purpose of this paper is to give a proof of the *Jacobson Descent Theorem*, Theorem 1, which is direct in that it does not assume that $A = \mathrm{Hom}_{K^A} K$. This is then used to prove the *Jacobson-Bourbaki Correspondence Theorem*, Theorem 2. The approach simplifies earlier proofs.

A variation of a theme of Hochschild appearing in Jacobson [2] and Winter [3] recurs here in the concentrated form of the dual bases $x_i$, $R_j$ which thread their way through both proofs. Thus, this paper underlines the importance of this natural duality.

Throughout the paper, $K$ denotes a field, End $K$ denotes the ring of endomorphisms of $K$ as additive group, $A$ denotes a subring of End $K$ containing the $K$-span $KI$ of the identity $I$ of End $K$ and $V$ denotes a vector space over $K$ of finite or infinite dimension $V: K$. Regard $A$ as left $K$-vector space in the obvious way.

DEFINITION 1. An *A-product* on $V$ is a mapping $A \times V \to V$, denoted $(T, v) \to T(v)$, such that $V$ is an $A$-module and

$$(xT)(v) = x(T(v)) \quad (x \in K, T \in A, v \in V). \qquad \square$$

Clearly $T(v)$ $(T \in A, v \in K)$ is an $A$-product for $K$.

Suppose henceforth that $T(v)$ $(T \in A, v \in V)$ is an $A$-product for $V$, and $V^A = \{v \in V \mid T(xv) = T(x)v \text{ for } T \in A, x \in K\}$. In particular, we have then defined $K^A$.

DEFINITION 2. For $k$ a subfield of $K$, a *k-form* of $V$ is a $k$-subspace $V'$ of $V$ whose $k$-bases are $K$-bases of $V$. $\square$

THEOREM 1 (*Jacobson* [1]). *Let $A: K < \infty$, then $V^A$ is a $K^A$-form of $V$.*

*Proof.* $\hat{K} = \{\hat{x} \mid x \in K\}$ separates $A$ and therefore contains a basis $\hat{x}_1, \ldots, \hat{x}_n$ for the $K$-dual space $\mathrm{Hom}_K(A, K)$ of $A$ where $\hat{x} \in \mathrm{Hom}_K(A, K)$ is defined for $x \in K$ by $\hat{x}(T) = T(x)$ $(T \in A)$. Letting $R_1, \ldots, R_n$ be a dual basis for $A$, so that $R_i(x_j) = \delta_{ij}$ $(1 \leq i, j \leq n)$, we have $T(xR_i)(x_j) = T(x\delta_{ij}) = T(x)\delta_{ij} = (T(x)R_i)(x_j)$ $(1 \leq i, j \leq n)$ so that $T(xR_i) = T(x)R_i$ $(1 \leq i \leq n)$ for all $T$, since the $x_j$ separate $A$.

Letting $v \in V$, we therefore have $T(xR_i(v)) = (T(xR_i))(v) = T(x)R_i(v)$ for all $T \in A$, $x \in V$, so that $R_i(V) \subset V^A$ and, in particular, $R_i(K) \subset K^A$ $(1 \leq i \leq n)$.

It follows that the $K$-span $KV^A$ of $V^A$ is $V$. For we have $I = \sum_1^n y_i R_i$ for suitable $y_i \in K$, so that $v = \sum_1^n y_i R_i v \in KV^A$ for all $v \in V$.

Finally, let $v_i$ $(i \in I)$ be a $K^A$-basis for $V^A$. Suppose that $\sum_{i \in I} y_i v_i = 0$ with the $y_i$ in $K$. Then $0 = \sum_{i \in I} R_j(y_i)v_i$ with the $R_j(y_i) \in K^A$, so $R_j(y_i) = 0$ $(1 \leq i, j \leq n)$ and $y_i = 0$ $(1 \leq i \leq n)$. □

THEOREM 2 (*Jacobson* [2]). *Let* $A: K < \infty$. *Then* $A = \text{Hom}_{K^A} K$.

*Proof.* $A$ as left $A$-module satisfies $(xS)T = x(ST)$ $(x \in K, S, T \in A)$, so that $A^A$ is a $K^A$ form of $A$ and $A^A$ contains a basis $R_1, \ldots, R_n$ for $A$ over $K$. Choosing $x_i \in K$ so that $I = x_1R_1 + \cdots + x_nR_n$, we have $x = \sum_{i=1}^n x_i R_i(x)$, $R_i(x) \in K^A$, for $x \in K$, so that $K: K^A \leq A : K \leq \text{Hom}_{K^A} K: K \leq K : K^A$ and $A = \text{Hom}_{K^A} K$. □

It is clear, in retrospect, that the above $x_1, \ldots, x_n$ form a basis for $K$ over $K^A \stackrel{\text{def}}{=} k$ and that $A^A$ is the dual space $\text{Hom}_k(K, k) = K^*$ of $K$ over $k$. The equations $x_j = \sum_{i=1}^n x_i R_i(x_j)$ show that $R_i(x_j) = \delta_{ij}$, that is, $R_1, \ldots, R_n$ is a dual basis for $K^*$. Finally, $I = x_1R_1 + \cdots + x_nR_n$ shows that $T = \sum_1^n x_j R_j T$ and $T(x_i) = \sum_1^n x_j R_j T(x_i) = \sum_1^n (R_j \hat{x}_i(T))x_j$. Thus, the $X_{ij} = R_j \hat{x}_i$ (composite) $(1 \leq i, j \leq n)$ are the coordinate functions on the $T \in A$ relative to the basis $x_i$. They form a basis for the $k$-dual space $A^*$ of $A = \text{Hom}_k K$. Since we may identify $\hat{K} = \{\hat{x} \mid x \in K\}$ with $K$, it follows that the $k$-dual space $A^*$ of $A = \text{Hom}_k K = KK^*$ can be identified with $A$ whereby $X_{ij}$ corresponds to $X_i R_j$—that is, we have a nondegenerate bilinear $k$-pairing $\langle\, , \rangle$ on $A \times A$ such that $\langle x_i R_j, T \rangle = R_j T(x_i)$. This pairing is also characterized by the condition $\langle xR, yS \rangle = R(y)S(x)$ $(x, y \in K, R, S \in K^*)$. Since $(x_i R_j)(x_r R_s) = x_i(R_j x_r R_s) = x_i R_j(x_r)R_s = x_i \delta_{jr} R_s$, the $E_{ij} = x_i R_j$ form a system of matrix units for $A$. We have $\langle E_{ij}, E_{rs} \rangle = \langle x_i R_j, x_r R_s \rangle = R_j(x_r)R_s(x_i) = \delta_{jr}\delta_{is} = \text{Trace}(E_{ij}E_{rs})$. It follows that $\langle S, T \rangle = \text{Trace}\, ST (S, T \in A)$.

## REFERENCES

[1] Nathan Jacobson, *Forms of algebras*, Yeshiva Sci. Confs., 7 (1966), 41–71.

[2] ______, *Lectures in Abstract Algebra*, Vol. III Van Nostrand, Princeton (1964).

[3] David J. Winter, *Structures of Fields*, Graduate Texts in Mathematics No. 16, Springer-Verlag (1974).

Received May 8, 1981. This research was supported in part by the National Science Foundation.

UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $132.00 a year (6 Vol., 12 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

**The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.**

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

# Pacific Journal of Mathematics

Vol. 104, No. 2 June, 1983