INDICATOR FUNCTIONS WITH LARGE FOURIER TRANSFORMS

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We consider the question of when the function

\[ t \mapsto \hat{1}_F(t) \]

is bounded, where \( 1_F \) is the indicator function of a compact set \( F \) in \( \mathbb{R} \) and "\( \hat{\cdot} \)" denotes the Fourier transform.

We are concerned in this note with a question of P. R. Masani about the rate of decrease of certain Fourier transforms on the real line \( \mathbb{R} \). Throughout, all unexplained notation is as in [1]. For \( f \in \mathcal{S}(\mathbb{R}) \), we write

\[ \hat{f}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \exp(itx) \, dx \quad (t \in \mathbb{R}). \]

(It is convenient to use \( \exp(itx) \) in the integral in (1) in place of the equally common \( \exp(-itx) \).)

Masani has asked whether or not there exist compact subsets \( F \) of \( \mathbb{R} \) with Lebesgue measure \( \lambda(F) > 0 \) such that the function

\[ t \mapsto t\hat{1}_F(t) \]

is unbounded. By the Cantor-Bendixson theorem, we may suppose that \( F \) is perfect. For a bounded closed interval \( [a, b] \subset \mathbb{R} \), the function \( t\hat{1}_{[a,b]}(t) \) is

\[ -i(2\pi)^{-1/2}(\exp(itb) - \exp(it)) \]

which is trivially bounded. For \( a = \inf F \) and \( b = \sup F \), write \( U = [a, b] \setminus F \) and get

\[ t\hat{F}(t) + t\hat{U}(t) = -i(2\pi)^{-1/2}(\exp(itb) - \exp(it)) \]

so that the function (2) is bounded if and only if the function

\[ t \mapsto t\hat{U}(t) = h_U(t) \]

is bounded. Thus Masani's problem is equivalent to the problem of finding bounded open subsets \( U \) of \( \mathbb{R} \) whose complements contain no isolated points and for which the function \( h_U \) is unbounded.

We note a simple case in which \( h_U \) is bounded. Suppose that

\[ \lambda([\inf U, \sup U] \setminus U) = 0, \]
as happens for example if $U$ is the union of the complementary intervals in $[0, 1]$ of Cantor's ternary set. Then (4) and (6) give

$$(7) \quad t \cdot 0 + \hat{1}_U(t) = \exp(i(\sup U)t) - \exp(i(\inf U)t)).$$

The same holds if $U$ is the union of a finite family of open sets for each of which (6) holds.

We have no complete classification of the open subsets $U$ of $\mathbb{R}$ for which the function $h_U$ is bounded. However, there is one special case where the answer is clear, as a consequence of a theorem of L. H. Loomis [3].

Given a closed subset $F$ of $\mathbb{R}$, let $P(F)$ be the set of all condensation points $x$ of $F$ (every neighborhood of $x$ contains an uncountable subset of $F$). As is well known, $P(F)$ is perfect or void and $F \setminus P(F)$ is countable.

**Theorem A.** Suppose that the bounded open subset $U$ of $\mathbb{R}$ is the union of a countably infinite family of non-abutting open intervals $[a_j, b_j]_{j=1}^{\infty}$ and that the boundary $\partial U = U^- \setminus U$ has an accumulation point outside of the perfect set $P(\partial U)$.

Then the function $h_U$ is unbounded.

**Proof.** For convenience we will use $\mathcal{S}$, the usual space of rapidly decreasing complex-valued $C^\infty$ functions on $\mathbb{R}$. The Fourier transformation (1) maps $\mathcal{S}$ onto the corresponding space of functions on the dual line. The identity

$$(8) \quad (g')\hat{}(t) = -it\hat{g}(t) \quad (g \in \mathcal{S})$$

is standard.

We now assume that $h_U$ is bounded. We will ultimately obtain a contradiction. For all real-valued $g \in \mathcal{S}$, (8) and Parseval's identity give

$$(9) \quad ih_U * \hat{g}(0) = i(2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{1}_U(t) \hat{g}(-t) \ dt$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{1}_U(t) \hat{g}(-t) \ dt$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{1}_U(t) \hat{g}(-t) \ dt$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} 1_U(x) g'(x) \ dx$$

$$= (2\pi)^{-1/2} \sum_{j=1}^{\infty} \int_{a_j}^{b_j} g'(x) \ dx = (2\pi)^{-1/2} \sum_{j=1}^{\infty} (g(b_j) - g(a_j)).$$

Note that $P(\partial U)$ is void if and only if $\partial U$ is countable. In this case any accumulation point of $\partial U$ will serve our purpose.
Let \( f \) be any real-valued function in \( \mathcal{S} \) and let \( s \) be a fixed real number. Replace \( g \) in (9) by the function
\[
  t \to \hat{f} \ast \hat{g}(s + t).
\]

The identity (9) becomes
\[
  \text{(10)} \quad i h_U \ast \hat{f} \ast \hat{g}(s) = (2\pi)^{-1/2} \sum_{j=1}^{\infty} \left[ f(b_j)g(b_j)\exp(ib_j s) - f(a_j)g(a_j)\exp(ia_j s) \right].
\]

Now consider a point of accumulation \( x_0 \) of \( \partial U \) that does not lie in \( P(\partial U) \). There is a real valued function \( f \) in \( \mathcal{S} \) such that \( f(x_0) = 1 \) and \( f \) vanishes in an open neighborhood \( V \) of the set \( P(\partial U) \). We choose and fix such a function \( f \). Let \( g \) be a real-valued function in \( \mathcal{S} \) that vanishes on \( \partial U \setminus V \). For such a function \( g \), the function \( fg \) vanishes at all of the points \( a_j \) and \( b_j \), as a moment’s thought shows. Thus the identity (10) shows that
\[
  h_U \ast \hat{f} \ast \hat{g} = 0.
\]

For each \( x \) not in \( \partial U \setminus V \), we can define the real-valued function \( g \) in \( \mathcal{S} \) so that \( g(x) = 1 \) and so that \( g \) vanishes on \( \partial U \setminus V \). Therefore the spectrum of the function \( h_U \ast \hat{f} \) is contained in the countable closed set \( \partial U \setminus V \), which is contained in \( \partial U \setminus P(\partial U) \). Loomis ([3], Theorem 4) has shown that a bounded measurable function on a locally compact Abelian group \( G \) whose spectrum is compact and contains no nonvoid perfect subset is almost periodic. (For the present case, \( G = \mathbb{R} \), these are exactly the functions in \( \mathcal{S}^\infty(\mathbb{R}) \) with bounded countable spectrum.) Therefore the function \( h_U \ast \hat{f} \) is continuous and almost periodic for all functions \( f \) of the form described above.

Now let
\[
  t \to \sum_{k=1}^{n} \mu_k \exp(ic_k t) = p_f(t)
\]
be a trigonometric polynomial on \( \mathbb{R} \) such that
\[
  \text{(11)} \quad \| h_U \ast \hat{f} - p_f \|_\infty < \frac{1}{4}.
\]

Computing a convolution at 0, we use (11) and (10) to infer that
\[
  \text{(12)} \quad \frac{1}{4} \| g \|_1 \geq \left| \left( h_U \ast \hat{f} - p_f \right) \ast \hat{g}(0) \right|
\]
\[
  = \left| \sum_{j=1}^{\infty} \left[ f(b_j)g(b_j) - f(a_j)g(a_j) \right] - \sum_{k=1}^{n} \mu_k g(c_k) \right|.
\]
Since \( f(x_0) = 1 \), there is an open neighborhood \( W \) of \( x_0 \) with compact closure such that \( |f(x)| \geq \frac{1}{4} \) for all \( x \in W \). Plainly \( W^c \) and \( P(\partial U) \) are disjoint. Since \( W \cap (\partial U) \) is (countably) infinite and disjoint from \( P(\partial U) \), it contains a point \( x_1 \) of \( \partial U \) that is isolated in \( \partial U \) and is different from all of the points \( c_1, c_2, \ldots, c_n \). Note that \( x_1 \) cannot be \( x_0 \) and that the only possible isolated points of \( \partial U \) are endpoints \( a_j \) and \( b_j \) of the component intervals of \( U \).

Suppose that we have a real-valued function \( g \) in \( \mathcal{C} \) such that \( g(x_1) = 1 \), \( g \) vanishes in a neighborhood of the compact set \( (\partial U \setminus \{x_1\}) \cup \{c_1, c_2, \ldots, c_n\} \), and \( \|\hat{g}\|_1 = 1 \). Put this \( g \) into formula (12). Since \( f \) vanishes on \( P(\partial U) \) and \( g \) vanishes on \( \partial U \) except at \( x_1 \), the only surviving term in the second line of (12) is \( \pm f(x_1)g(x_1) \). Since \( |f(x_1)| \geq \frac{1}{4} \) by construction, (12) yields

\[
\frac{1}{4} \geq |f(x_1)g(x_1)| \geq \frac{3}{4} |g(x_1)| = \frac{3}{4},
\]

a contradiction. Therefore the function \( h_U \) is unbounded.

To finish the proof, we need only to find a function with the properties ascribed to \( g \) in the preceding paragraph. This is standard save for the requirement that \( g \) be in \( \mathcal{C} \). Imitating the standard construction, we suppose first that \( x_1 = 0 \). Let \( \delta \) be any positive real number, and take \( \psi \) to be an even nonnegative \( C^\infty \) function with support \([-\frac{1}{2}\delta, \frac{1}{2}\delta]\) for which

\[
(2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi^2(x) \, dx = 1.
\]

Define \( g \) as the convolution \( \psi \ast \psi \). Plainly \( g \) is in \( \mathcal{C} \) and has support \([-\delta, \delta]\). Since \( \psi \) is real-valued, we have

\[
g(0) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi(x)\psi(-x) \, dx = 1
\]

and

\[
\|\hat{g}\|_1 = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\psi}(t) \, dt = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\psi}^2(t) \, dt = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi^2(x) \, dx = 1.
\]

For \( x_1 \neq 0 \), use the translated function \( x \to g(-x_1 + x) \), whose support is \([x_1 - \delta, x_1 + \delta]\) and whose Fourier transform at \( t \) is \( \exp(ix_1 t)\hat{g}(t) \).

**Remarks.** Let \( (\gamma_j)_{j=1}^{\infty} \) be any bounded sequence of complex numbers such that \( |\gamma_j| \) is bounded away from zero. Consider the function

\[
\varphi = \sum_{j=1}^{\infty} \gamma_j 1_{[a_j, b_j]},
\]
where the open set \( U = \bigcup_{j=1}^{\infty} [a_j, b_j] \) satisfies the hypotheses of Theorem A. The proof of Theorem A can be repeated with an obvious modification in (11) to prove that the function
\[
t \mapsto t \hat{\phi}(t)
\]
is unbounded. If \( f \) is a continuous function on \( \mathbb{R} \) such that \( f' \) exists except possibly at a countable set of points and if both \( f \) and \( f' \) are in \( \mathcal{L}_1(\mathbb{R}) \), then \( f \) is absolutely continuous and
\[
(f')\hat{(t)} = -i \hat{f}(t)
\]
for all \( t \in \mathbb{R} \). Thus the function
\[
t \mapsto tf\hat{(t)}
\]
is not only bounded but is \( o(1) \). Adding to such \( f \) any function \( \phi \) of the form (14), we get more functions \( g \) in \( \mathcal{L}_1(\mathbb{R}) \) for which the function
\[
t \mapsto tg\hat{(t)}
\]
is unbounded.

**Example A.** Let \( \{[a_j, b_j]\}_{j=1}^{\infty} \) be a countably infinite family of non-void, non-abutting open intervals in \( \mathbb{R} \) and as above write \( U \) for the set \( \bigcup_{j=1}^{\infty} [a_j, b_j] \). Suppose that \( U \) is bounded. It is easy to see that \( \partial U \) is the closure of the countable set \( H = \{a_1, a_2, \ldots, a_n, \ldots\} \cup \{b_1, b_2, \ldots, b_n, \ldots\} \). If \( H^- \) is countable, then the open set \( U \) satisfies the hypotheses of Theorem A, since the perfect set \( P(\partial U) = P(H^-) \) is void. A continuum of such open sets exist and can be constructed \( \textit{ad libitum.} \) Thus open sets \( U \) for which \( h_U \) is unbounded exist in profusion.

**Example B.** We now present a construction that is roughly the antithesis of Example A, in that the set \( H \) consists solely of isolated points, while the set \( P(H^-) \) is equal to \( H^- \setminus H \) and is homeomorphic to Cantor's ternary set. At the same time the function \( h_U \) is unbounded for this set \( U \). Thus we will show that the hypotheses of Theorem A are not necessary in order for the function \( h_U \) to be unbounded.

For every positive integer \( n \), let \( E_n \) be the set of all sequences \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) where each entry \( \varepsilon_j \) is either 1 or \(-1\). Let \( C_n \) be the subset of \( E_n \) consisting of all \( \varepsilon \) with \( \varepsilon_1 = 1 \). For each \( \varepsilon \) in \( E_n \), let \( I(n, \varepsilon) \) be the open interval
\[
\sum_{j=1}^{n} \varepsilon_j 4^{-j} - \frac{1}{2} 4^{-n-1}, \quad \sum_{j=1}^{n} \varepsilon_j 4^{-j} + \frac{1}{2} 4^{-n-1}
\]
Let \( U \) be the union of all of the intervals \( I(n, \varepsilon) \) as \( \varepsilon \) runs through all of the \( 2^n \) elements of \( E_n \) and \( n \) runs through the set of all positive integers.
We find that
\begin{equation}
I(n, \varepsilon) \cap I(n', \varepsilon') = \emptyset
\end{equation}
unless \( n = n' \) and \( \varepsilon = \varepsilon' \). As in Example A, write \( H \) for the set of all endpoints of all of the intervals \( I(n, \varepsilon) \). Let \( D \) be the set of all numbers of the form
\begin{equation}
\sum_{j=1}^{\infty} \beta_j 4^{-j},
\end{equation}
where each \( \beta_j \) is either 1 or \(-1\). We find that
\begin{equation}
D = H^{-1} \setminus H = \partial U.
\end{equation}
The details of proving (16) and (17) are simple enough but are also somewhat tedious, and we omit them. Note that
\begin{equation}
\sup U = \frac{1}{3}, \quad \inf U = -\frac{1}{3}, \quad \text{and} \quad \lambda(U) = \frac{1}{4}.
\end{equation}
We now compute the function \( h_U \).

Given an interval \( [c - \gamma, c + \gamma] \) (\( c \in \mathbb{R}, \gamma > 0 \)), we have
\begin{equation}
\exp(i(c + \gamma)t) - \exp(i(c - \gamma)t) = 2i \sin(\gamma t) \exp(ict).
\end{equation}
For every positive integer \( n \), (3) and (19) show that
\begin{equation}
\sum_{\varepsilon \in E_n} i I_{(n, \varepsilon)}(t) = \sum_{\varepsilon \in E_n} 2 \sin\left(\frac{1}{2} 4^{-n-1} t\right) \exp\left(i \left( \sum_{j=1}^{n} \varepsilon_j 4^{-j} \right) t\right)
= \sum_{\varepsilon \in C_n} 2 \sin\left(\frac{1}{2} 4^{-n-1} t\right) \left[ \exp\left(i \left( \sum_{j=1}^{n} \varepsilon_j 4^{-j} \right) t\right) + \exp\left(i \left( -\sum_{j=1}^{n} \varepsilon_j 4^{-j} \right) t\right) \right]
= 2 \sin\left(\frac{1}{2} 4^{-n-1} t\right) \prod_{r=1}^{n} \left[ \exp(i 4^{-r} t) + \exp(-i 4^{-r} t) \right]
= 2^{n+1} \sin\left(\frac{1}{2} 4^{-n-1} t\right) \prod_{r=1}^{n} \cos(4^{-r} t).
\end{equation}
Add (20) over all positive integers \( n \) to obtain
\begin{equation}
h_U(t) = \sum_{n=1}^{\infty} 2^{n+1} \sin\left(\frac{1}{2} 4^{-n-1} t\right) \prod_{r=1}^{n} \cos(4^{-r} t).
\end{equation}
For a given positive integer \( p \), let us compute (21) for \( t = 2\pi 4^p \). For \( n = 1, 2, \ldots, p - 1 \), we have
\begin{equation}
\sin\left(\frac{1}{2} 4^{-n-1} 2\pi 4^p\right) = \sin(\pi 4^{p-n-1}) = 0.
\end{equation}
For \( n = p \), we have
\[
\sin\left(\frac{1}{2}4^{-p-1}2\pi4^p\right) = \sin\left(\frac{1}{2}\pi\right) = 2^{-1/2}.
\]
Also for \( n = p \), we have
\[
\prod_{r=1}^{p} \cos(4^{-r}2\pi4^p) = \prod_{r=1}^{p} \cos(2\pi4^p-r) = 1.
\]
For \( n \geq p + 1 \), we have
\[
\prod_{r=1}^{n} \cos(2\pi4^p-r) = 0,
\]
since
\[
\cos(2\pi4^p-p-1) = \cos\left(\frac{1}{2}\pi\right) = 0.
\]
Combining (21)–(25), we see that
\[
h_U(2\pi4^p) = 2^{p+1/2},
\]
so that \( h_U(t) \) is unbounded.

It is of some interest to examine the rate of growth of the function \( h_U(t) \) for \( U \)'s as in Theorem A.

**EXAMPLE C.** Let \( \varphi \) be any continuous nondecreasing function on \([1, \infty[\) such that \( \lim_{t \to \infty} \varphi(t) = \infty \). We can find a bounded open set \( U \) such that \( h_U(t) \) is unbounded and
\[
h_U(t) = O(\varphi(|t|)).
\]
To find such a set \( U \), let \( \psi = \psi(u) \) be the function defined on \([\varphi(1), \infty[\) such that: if \( \varphi \) assumes the value \( u \) at exactly one point \( t \), then \( \psi(u) = t \); if \( \varphi \) assumes the value \( u \) exactly in an interval \([a, b]\) with \( a < b \), then \( \psi(u) = b \). That is, \( \psi \) is as close to the inverse function of \( \varphi \) as one can get. It is plain that \( \lim_{u \to \infty} \psi(u) = \infty \) and that \( \psi \) is strictly increasing.

It is easy to construct an infinite series \( \sum_{n=1}^{\infty} r_n \) of positive terms such that
\[
\sum_{n=N+1}^{\infty} r_n = \frac{1}{\psi(N+1)}
\]
for all positive integers \( N \). Let \( \{a_n, b_n\}_{n=1}^{\infty} \) be a set of open intervals with the following properties for all \( n \):
\[
a_n < b_n; \quad b_n - a_n = r_n; \quad b_{n+1} < a_n;
\]
and
\[
\lim_{n \to \infty} a_n = 0.
\]
It is plain that \( P(\partial U) = \emptyset \), and so by Theorem A the function \( h_U(t) \) is unbounded. For every positive integer \( N \), we have

\[
|h_U(t)| \leq \left| \sum_{n=1}^{N} (\exp(ib_n t) - \exp(ia_n t)) \right| + \left| \sum_{n=N+1}^{\infty} (\exp(ib_n t) - \exp(ia_n t)) \right| \\
\leq 2N + |t| \sum_{n=N+1}^{\infty} (b_n - a_n) = 2N + |t| \frac{1}{\psi(N + 1)}.
\]

Given a real number \( t \) of absolute value at least 1, let \( N \) be the integer such that

\[ N \leq \varphi(|t|) < N + 1. \]

This gives us

\[ \psi(N) \leq \psi(\varphi(|t|)) < \psi(N + 1). \]

By our definition of \( \psi \), we have

\[ |t| \leq \psi(\varphi(|t|)), \]

\[ |h_U(t)| \leq 2\varphi(|t|) + \psi(N + 1) \frac{1}{\psi(N + 1)} = O(\varphi(|t|)). \]

Thus the function \( h_U(t) \) can go to infinity arbitrarily slowly.

Finally we compute the exact rate of growth of the function \( h_U(t) \) for the open set \( U \) of Example B. The equality (26) shows that

\[
|h_U(t)| \geq Ct^{1/2}
\]

for arbitrarily large positive values of \( t \). On the other hand, consider all of the intervals \( I(n, \varepsilon) \) for \( n \leq N, N \) being an arbitrary positive integer. There are exactly \( 2^{N+1} - 1 \) such intervals. The sum of the measures of all of the intervals \( I(n, \varepsilon) \) for \( n \geq N + 1 \) is \( 2^{-N-2} \). Accordingly, (28) shows that

\[
|h_U(t)| \leq 2(2^{N+1} - 1) + |t|2^{-N-2}.
\]

For a given \( t \) of absolute value at least 4, define \( N \) by

\[ 2^{2N+2} \leq |t| < 2^{2N+3}. \]

From (30) we get

\[ |h_U(t)| \leq 2|t|^{1/2} + 2^{-1/2}|t|^{1/2}, \]
so that
\begin{equation}
|h_U(t)| = O(|t|^{1/2}).
\end{equation}

The estimates (29) and (31) show that $|h_\nu(t)| = O(|t|^{\alpha})$ for $\alpha = \frac{1}{2}$ but for no smaller exponent $\alpha$.

We are indebted to Professor Masani for the following remarks on the origin of his problem.

**Question.** Let $x$ be a complex Banach space, and let $\{ (U(t): t \in \mathbb{R}) \}$ be a strongly continuous group of linear isometries of $x$ onto $x$ with infinitesimal generator $A$. For what bounded Borel subsets $S$ of $\mathbb{R}$ is it the case that
\begin{equation}
\text{Range } \int_S U(t) \, dt \subset \text{Dom } A?
\end{equation}

This question arises naturally in the theory of $x$-valued stationary measures over $\mathbb{R}$. See [4], page 303, Theorem 3.6. The inclusion (32) holds provided that $S$ is a closed interval. This is proved in [2], §10.3, page 307. Thus (32) holds if $S$ is a union of finitely many closed intervals.

Now suppose that $x$ is a Hilbert space. The problem of the inclusion (32) reduces to the problem of Masani stated in the second paragraph of this note. To see this, write
\begin{equation}
U(t) = \int_\mathbb{R} \exp(itx) \, d(E(x)), \quad \text{so that } \quad A = \int_\mathbb{R} ix \, d(E(x)).
\end{equation}

It is then easy to see that
\begin{equation}
\int_S U(t) \, dt = \int_\mathbb{R} i\hat{s}(x) \, d(E(x))
\end{equation}

and that
\begin{equation}
A \int_S U(t) \, dt \subset \int_\mathbb{R} ix\hat{s}(x) \, d(E(x)).
\end{equation}

Now (32) holds if and only if the operator on the left side of (33) is continuous on $x$, that is, if and only if the function $x \mapsto \hat{x}_S(x)$ is $E$-essentially bounded on $\mathbb{R}$. It is also easy to see that a bounded Borel set $S$ satisfies (32) for all $U(\cdot)$ if and only if the function $x \mapsto \hat{x}_S(x)$ is bounded on $\mathbb{R}$. Thus finding the bounded Borel sets satisfying (32) yields the problem stated in the second paragraph of this note.

Finally we remark that Masani [4], page 304, Proposition 3.8, has proved a special case of Example A.
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