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All spaces considered are completely regular and X^* denotes $\beta X - X$. The point $x \in X^*$ is called *a remote point of* X if $x \notin \mathrm{Cl}_{\beta X} A$ for each nowhere dense subset A of X. If $y \in Y$, then the space Y is said to be *extremally disconnected at* y if $y \notin \overline{U} \cap \overline{V}$ whenever U and V are disjoint open sets. In this paper we construct two noncompact σ-compact spaces X, one locally compact and one nowhere locally compact, such that X has no remote points, and in fact such that βX is not extremally disconnected at any point.

Our examples were motivated by the following results from [**6**]:

(1) *X has remote points if X has countable π-weight, in particular if X is separable and first countable, and is not pseudocompact*, [**6**, 1.5]; see also [**7**] for an earlier consistency result, and [**1**] for a more general result.

(2) *βX is extremally disconnected at each remote point of X*, [**6**, 5.2]. Via the observation that

(3) *if Y is dense in Z, and $y \in Y$, then Y is extremally disconnected at y iff Z is extremally disconnected at y*,

these results and the following imply a nonhomogeneity result, which applies for example to the rationals and the Sorgenfrey line

(4) *if X is a nowhere locally compact nonpseudocompact space which has a remote point and if $\{x \in X$: X is not extremally disconnected at $x\}$ is dense in X, e.g. if X is first countable, then X^* is not homogeneous because X^* is extremally disconnected at some but not at all points.*

(This is a special case of Frolík's theorem that X^* is not homogeneous if X is not pseudocompact, [**8**]. The proof of Frolík's theorem does not yield a simple "because" as in (4). X is called nowhere locally compact if no point of X has a compact neighborhood, or, equivalently, if X^* is dense in βX.)

In this paper we produce two closely related examples which show that the condition on the π-weight cannot be omitted altogether in (1), thus answering a question of [**6**].

Our two examples are rather big: they have cellularity at least ω_3. This suggests the question of whether every nonpseudocompact separable space has a remote point. (This would generalize (1).) It follows from a construction in [**7**] that the answer is affirmative under CH.

EXAMPLES. *There are two noncompact σ-compact spaces X, one locally compact and one nowhere locally compact, such that X has no remote points, and in fact such that βX is not extremally disconnected at any point.*

Because of (3) the nowhere locally compact example shows that the condition on the π-weight cannot be omitted altogether in the nonhomogeneity result (4). We will show that an older nonhomogeneity proof, involving far points, still applies.

No remote points.

A subset P of a space X is called a *P-set* if for each F_σ-subset F of X, if $F \cap P = \varnothing$ then $\overline{F} \cap P = \varnothing$. A subset T of a space X is called a 2-*set* if there are disjoint open U and V in X with $T \subseteq \overline{U} \cap \overline{V}$.

LEMMA 1. *There is a compact space U such that for each $q \in U$ there is a decreasing ω_1-sequence $\langle P_\xi: \xi \in \omega_1\rangle$ of clopen sets such that $\bigcap_{\xi\in\omega_1} P_\xi$ is a nowhere dense set of U which contains q.*

□ Give ω_2 the discrete topology. Identify ω_2^* with the space of free ultrafilters on ω_2. Then

$$U = \left\{q \in \omega_2^*: |Q| = \omega_2 \text{ for all } Q \in q\right\},$$

the space of uniform ultrafilters on ω_2, is a closed, hence compact, subspace of ω_2^* of course. We need the following result due to Čudnovskiĭ and Čudnovskiĭ, [3] and, independently, to Kuen and Prikry, [11], and earlier, but with GCH to Chang [2]:

$$(*)\quad \text{for each } q \in U \text{ there is a decreasing } \omega_1\text{-sequence } \langle Q_\xi: \xi \in \omega_1\rangle \text{ in } q \text{ such that } \bigcap_{\xi\in\omega_1} Q_\xi = \varnothing.$$

As usual, let $\hat{A}$ denote $U \cap \overline{A}$ (closure in $\beta\omega_2$), for $A \subseteq \omega_2$. For a given $q \in U$ let $\langle Q_\xi: \xi \in \omega_1\rangle$ be as in $(*)$, and define $\langle P_\xi: \xi \in \omega_1\rangle$ by $P_\xi = \hat{Q}_\xi$ for $\xi \in \omega_1$. Clearly $\langle P_\xi: \xi \in \omega_1\rangle$ is a decreasing ω_1-sequence of clopen subsets of U such that $P = \bigcap_{\xi\in\omega_1} P_\xi$ contains q. Now recall that $\{\hat{B}: B \subseteq \omega_2 \text{ and } |B| = \omega_2\}$, being the collection of all nonempty clopen subsets of U, is a base for U. Consider any $B \subseteq \omega_2$ with $|B| = \omega_2$. There is an $\eta \in \omega_1$ with $|B - Q_\eta| = \omega_2$. Then $\varnothing \neq (B - Q_\eta)\hat{} = \hat{B} - \hat{Q}_\eta \subseteq \hat{B} - P$. It follows that P is nowhere dense. □

REMARK. Instead of ω_1 we can take any regular cardinal κ, and then U will be the space of uniform ultrafilters on κ^+.

Clearly Lemma 1 implies that there is a compact space which is covered by the collection of its nowhere dense closed P-sets. Since evidently each 2-set is nowhere dense the following is a stronger assertion.

LEMMA 2. *There is a compact space H such that for each $q \in H$ there is a closed P in H with $q \in P$ such that P is both a P-set and a 2-set.*

□ Let U be as in Lemma 1, and let $H = U \times U$. Consider any $q_0, q_1 \in U$. For $i \in 2$ choose a decreasing ω_1-sequence $\langle P_{i,\xi}\colon \xi \in \omega_1\rangle$ of clopen sets in U such that $P_1 = \bigcap_{\xi \in \omega_1} P_{i,\xi}$ is a nowhere dense subset of U which contains q_i. Then $P_0 \times P_1$ is a nowhere P-set in H which contains $\langle q_0, q_1\rangle$. We show that $P_0 \times P_1$ is also a 2-set

For $i \in 2$ define an open $V_{i,\xi}$ with recursion on $\xi \in \omega_1$ by

$$V_{i,\xi} = (U - P_{i,\xi}) - \left(\bigcup_{\eta \in \xi} V_{i,\eta}\right)^- \qquad \left(\bigcup_{\nu \in 0} V_{i,\eta} = \varnothing \text{ of course}\right).$$

Then evidently $(\bigcup_{\eta \leq \xi} V_{i,\eta})^- = U - P_{i,\xi}$ for $i \in 2$ and $\xi \in \omega_1$. Since P_0 and P_1 are nowhere dense it follows that

$$(\dagger) \qquad \left(\bigcup_{\xi \in \omega_1} V_{i,\xi}\right)^- = (U - P_i)^- = U, \qquad \text{for } i \in 2.$$

Define open subsets W_0 and W_1 of H by

$$W_0 = \bigcup_{\xi \in \omega_1} P_{0,\xi} \times V_{1,\xi} \quad \text{and} \quad W_1 = \bigcup_{\xi \in \omega_1} V_{0,\xi} \times P_{1,\xi}.$$

Then $W_0 \cap W_1 = \varnothing$ since if $\xi \leq \eta < \omega_1$ then $V_{i,\xi} \subseteq U - P_{i,\xi} \subseteq U - P_{i,\eta}$, for $i \in 2$ (so that $(P_{0,\xi} \times V_{1,\xi}) \cap (V_{0,\eta} \times P_{1,\xi}) = \varnothing$ for all $\xi, \eta \in \omega_1$). To prove that $P_0 \times P_1 \subseteq \overline{W_0} \cap \overline{W_1}$ we have only to prove that $P_0 \times P_1 \subseteq \overline{W_0}$, because of symmetry. We have

$$W_0 \supseteq \bigcup_{\xi \in \omega_1}\left(\left(\bigcap_{\eta \in \omega_1} P_{0,\eta}\right) \times V_{1,\xi}\right) = P_0 \times \bigcup_{\xi \in \omega_1} V_{1,\xi},$$

hence $\overline{W_0} \supseteq P_0 \times U \supseteq P_0 \times P_1$ as required. □

REMARK 2. We do not know if the space U of Lemma 1 can be used for the space H of Lemma 2. We are indebted to the referee for pointing out that the set $P = \bigcap_{\xi \in \omega_1} P_\xi$ obtained in Lemma 1 is not a 2-set: P has character ω_1, but in U the closure of every open F_{ω_1}-set ($\equiv$ union of ω_1 many closed sets) is easily seen to be open, [**CoN**, Thm. 14.9], which implies that no closed set in U of character ω_1 is a 2-set. To see this let F be a closed set in U of character ω_1 and let V and W be disjoint open sets in U such that $F \subseteq \overline{V}$. Since F has character ω_1 there is an open F_{ω_1}-set $T \subseteq V$ such that $\overline{T} \cap \overline{F} \neq \varnothing$. Now $\overline{T} \cap W = \varnothing$ since $T \cap W = \varnothing$, and $\overline{T}$ is clopen. It follows that $F \not\subseteq \overline{W}$.

SUBREMARK. It is at least consistent that $U = U(\omega_2)$ has a closed P-set that is a 2-set. There is a closed nowhere dense $P \subseteq U$ which is a P_{ω_2}-set ($\equiv$ for every F_{ω_2}-set F in U, if $F \cap P = \varnothing$ then $\overline{F} \cap P = \varnothing$), namely $\bigcap \{C: C \subseteq \omega_2$ is a cub$\}$, and if $2^{\omega_2} = \omega_3$ then every nowhere dense P_{ω_3}-set in U (or in any space of weight ω_3) is a 2-set. However, if $2^{\omega_2} = \omega_3$ then U is not covered by the collection of its nowhere dense closed P_{ω_3}-sets, by **[10**, 1.1].

REMARK 3. After this paper had been written another proof of Lemma 1 was discovered by Kunen, van Mill and Mills: the space of nondecreasing functions $\omega_2 \to \omega_1 + 1$, **[10**, 3.1]. It is easy to see that the P-sets obtained there are 2-sets. The example of Lemma 2 has the additional feature that each P-set has character ω_1.

REMARK 4. The above remarks suggest the question of whether there is a compact space which is covered by the collection of its closed nowhere dense P-sets but which has no nonempty closed P-set which is also a 2-set. This question can be answered quite easily. Let E be the projective cover of the example of Lemma 1, i.e. E is the unique extremally disconnected compact space that admits an irreducible map, say π, onto U. As is well known, $\pi^{\leftarrow}(D)$ is nowhere dense in E iff D is nowhere dense in E. Since it is easily seen that $\pi^{\leftarrow}(P)$ is a P-set of E iff P is a P-set of U, we conclude that E can be covered by nowhere dense closed P-sets. Since E is extremally disconnected, there are no nonempty 2-sets in E. The following question however remains open:

Question. Is there (in ZFC) a compact space which is covered by the collection of its closed nowhere dense P-sets but which has no nonempty nowhere dense P_{ω_2}-set?

LEMMA 3. *Let K be a compact space, and let P be a P-set in K. Furthermore, let Y be a countable space, let π: $K \times Y \to K$ be the projection, and let $\beta\pi$: $\beta(K \times Y) \to K$ be the Stone extension of π. Then for each $x \in \beta(K \times Y)$, if $\beta\pi(x) \in P$ then $x \in (P \times Y)^-$.*

$\square$ Consider any $x \in \beta(K \times Y) - (P \times Y)^-$. Let V be a closed neighborhood of x which misses $P \times Y$. Then $x \in ((K \times Y) \cap V)^-$, hence

$$\beta\pi(x) \in (\beta\pi^{\rightarrow}((K \times Y) \cap V))^- = (\pi^{\rightarrow}((K \times Y) \cap V))^-.$$

Also, $\pi^{\leftarrow}((K \times Y) \cap V)$ is an F_σ (since $(K \times Y) \cap V$ is σ-compact) in K which misses the P-set P, hence $(\pi^{\leftarrow}((K \times Y) \cap V))^{-} \cap P = \varnothing$. Consequently $\beta\pi(x) \notin P$. □

CORollary 1. *If K is a compact space which is covered by nowhere dense P-sets, then $K \times Y$ has no remote points, for each countable space Y.* □

COROLLARY 2. *If K is a compact space which is covered by P-sets which are 2-sets, then $\beta(K \times Y)$ is not extremally disconnected at any point, for each countable space K.*

□ The key observation is that if D is dense in a space X, then the closure in X of each 2-set in D is a 2-set in X. □

If H is as in Lemma 2, if ω is the integers and if Q is the rationals, then our examples are $H \times \omega$ and $H \times Q$.

Far points.

A point p of X^* is called a *far* (or *ω-far*) *point* of X if $p \notin \mathrm{Cl}_{\beta X} D$ for each (countable) closed discrete subset D of X. Clearly, if X has no isolated points then each remote point of X is a far point; the converse of this is generally false, [**6**, 4.8]. There is a nonhomogeneity result involving far points, or ω-far points, similar to (4) of the introduction, but less attractive since it involves $X^{**} = (X^*)^*$: If X is nowhere locally compact, and is not countably compact, and has a far (ω-far) point, then X^* is not homogeneous because for some but not for all $x \in X^*$ there is a (countable) closed discrete D in the space X^{**} such that $x \in \mathrm{Cl}_{\beta X^*} D$, [**5**, 2,4.3].

One might hope that our examples can be used to answer the question of [**5**] of whether every noncompact Lindelöf space has an ω-far point (which would be a far point). (It is easy to see that every normal nonLindelöf space has an ω-far point, [**5**, 4.3].) This is not the case: both our examples have far points. This follows from the following result.

THEOREM. *If X has a countably infinite discrete collection K of compact subspaces without isolated points, and if X is normal, or, more generally, if K can be separated by a discrete open family, then X has a far point.*

Before we proceed to the proof we point out an attractive corollary:

COROLLARY. *Every locally compact (or, more generally, Čech-complete) nonpseudocompact space has a far-point.*

□ If X is nonpseudocompact it has a countably infinite family $\mathcal{U}$ consisting of nonempty open sets. By a well-known tree argument one finds for each $U \in \mathcal{U}$ a compact $K_U \subseteq U$ that admits a continuous map f_U onto the Cantor discontinuum ${}^{\omega}2$. For $U \in \mathcal{U}$ choose a compact $L_U \subseteq K_U$ such that $f_U \restriction L_U$ is an irreducible map onto ${}^{\omega}2$, then L_U has no isolated points. □

Proof of Theorem. First recall that $\mathbf{R}$ has a far point, by an elegant argument due to Eberlein [7, Thm. 1.3]. It follows that $Y = U\mathcal{K}$ has a far point. As in the proof of the Corollary, each member of $\mathcal{K}$ admits a (necessarily closed) map onto the Cantor discontinuum, hence on the closed unit interval. Since $\mathcal{K}$ is countably infinite it follows that Y admits a closed map onto $\mathbf{R}$. The Stone extension βf of f maps ϕY onto $\beta \mathbf{R}$, hence there is $y \in Y^*$ such that $\beta f(y)$ is a far point of R. Since $f^{\rightarrow} D$ is closed discrete in $\mathbf{R}$ for each closed discrete D in Y this y is a far point of Y, cf. [**5**, §2, Fact 3].

We now point out that

(*) For any two disjoint closed F and G in X, if $F \subseteq Y$ then $\mathrm{Cl}_{\beta X} F \cap \mathrm{Cl}_{\beta X} G = \varnothing$.

The proof is similar to the known case, [**9**, 3L], that $\mathcal{K}$ consists of singletons. From (*) we see that $\mathrm{Cl}_{\beta X} Y = \beta Y$. Since Y is closed in X it follows that X contains a far point of Y. This point is a far point of X since, by (*), for each closed discrete subset D of Y we have $\mathrm{Cl}_{\beta X}(D - Y) \cap \mathrm{Cl}_{\beta X} Y = \varnothing$. □

REMARK 5. Dow [**4**] has shown that every separable nonpseudocompact space has a remote point under MA.

REMARK 6. After this paper was written there has been much progress on the question of whether every Lindelöf space has a far point: It is known that the answer is affirmative under MA, [**12**, 9.1].

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