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REGULARITY OF THE BERGMAN PROJECTION IN CERTAIN NONPSEUDOCONVEX DOMAINS

STEVEN ROBERT BELL

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Suppose D is a smooth bounded domain contained in C^n ($n \geq 2$) whose Bergman projection satisfies global regularity estimates, and suppose K is a compact subset of D such that $D - K$ is connected. The purpose of this note is to prove that, under these circumstances, the Bergman projection associated to the domain $D - K$ satisfies global regularity estimates.

This result is presently known only in very special cases when both D and K have a particularly simple form. For example, the fundamental paper of Kohn [5] reveals that if Ω_1 and Ω_2 are two smooth bounded strictly pseudoconvex domains in C^n ($n > 2$) such that $\Omega_2 \subset\subset \Omega_1$, then the $\bar{\partial}$ -Neumann problem for the domain $\Omega_1 - \bar{\Omega}_2$ is subelliptic. Kohn's formula, $P = I - \bar{\partial}^* N \bar{\partial}$, which relates the Bergman projection P to the $\bar{\partial}$ -Neumann operator N , shows that the Bergman projection associated to $\Omega_1 - \bar{\Omega}_2$ satisfies global regularity estimates. Recently, Derridj and Fornaess [3] have shown that if Ω_1 and Ω_2 are two pseudoconvex domains with real analytic boundaries in C^n with $n \geq 3$ and $\Omega_2 \subset\subset \Omega_1$, then the $\bar{\partial}$ -Neumann operator for $\Omega_1 - \bar{\Omega}_2$ satisfies subelliptic estimates. Hence, the Bergman projection associated to $\Omega_1 - \bar{\Omega}_2$ satisfies global estimates in this case, also.

In Bell and Boas [2], it is proved that the Bergman projection associated to a smooth bounded complete Reinhardt domain satisfies global regularity estimates. Thus, there are more subtle examples of non-pseudoconvex domains for which regularity of the Bergman projection holds than those addressed by the theorem of the present work. Recently, the techniques used in [2] have been refined by David E. Barrett [1] to prove that the Bergman projection associated to a smooth bounded domain with a Lie group of transverse symmetries satisfies global regularity estimates.

The question as to whether or not the Bergman projection associated to a domain satisfies global regularity estimates is very important in problems relating to boundary behavior of holomorphic mappings (see [2]).

The Bergman projection P associated to a bounded domain D contained in C^n is the orthogonal projection of $L^2(D)$ onto $H(D)$, the closed subspace of $L^2(D)$ consisting of L^2 holomorphic functions. The space $C^\infty(\bar{D})$ is defined to be the set of functions in $C^\infty(D)$, all of whose

derivatives are bounded functions on D . The family of derivative sup-norms exhibits the Frechet space topology on $C^\infty(\bar{D})$.

We shall say that a bounded domain D satisfies *condition R* whenever P is a continuous operator from $C^\infty(\bar{D})$ to $C^\infty(\bar{D})$. We can now state the main result of this paper.

THEOREM. *If $D \subset \mathbb{C}^n$ ($n \geq 2$) is a smooth bounded domain which satisfies condition R and K is a compact subset of D such that $D - K$ is connected, then $D - K$ satisfies condition R.*

Examples of domains for which condition R is known to hold include smooth bounded strictly pseudoconvex domains (Kohn [5]), smooth bounded pseudoconvex domains with real analytic boundaries (Kohn [6], Diederich and Fornæss [4]), and smooth bounded complete Reinhardt domains (Bell and Boas [2]).

Before we prove the theorem, we must define some Sobolev norms and spaces. If D is a smooth bounded domain and s is a positive integer, the space $W^s(D)$ is the usual Sobolev space of complex valued functions on D whose distributional derivatives up to order s are contained in $L^2(D)$. The Sobolev s -norm of a function u is defined via

$$\|u\|_s^2 = \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2(D)}^2$$

where the symbol ∂^α is the standard differential operator of order α . If $v \in L^2(D)$, we define the *negative* Sobolev s -norm of v via

$$\|v\|_{-s} = \text{Sup} \left\{ \left| \int_D v \phi \right| : \phi \in C_0^\infty(D); \|\phi\|_s = 1 \right\}.$$

If $g \in H(D)$ we define the *special* Sobolev s -norm of g to be

$$\| \| g \| \|_s = \text{Sup} \left\{ \left| \int_D g \bar{h} \right| : h \in H(D); \| h \|_{-s} = 1 \right\}.$$

REMARK. It is always true that if D is a smooth bounded domain, then there is a constant C such that

$$\| \| g \| \|_s \leq C \| g \|_s$$

for all $g \in H(D)$. This can be proved using techniques similar to those used in [2]. The reverse inequality $\| g \|_s \leq C \| \| g \| \|_s$ only holds if the Bergman projection associated to D satisfies an estimate of the form $\| P\phi \|_s \leq C \| \phi \|_s$. The norm $\| \| \|_s$ has found fruitful application in the theory of boundary behavior of holomorphic mappings.

Our main theorem is a relatively simple consequence of the following lemma.

LEMMA. *Suppose that D is a smooth bounded domain contained in \mathbb{C}^n which satisfies condition R and that s is a positive integer. There exists a positive integer $M = M(s)$ and a constant $C = C(s)$ such that*

$$\|g\|_s \leq C \| \|g\|_{s+M}$$

for all g in $H(D)$.

We shall now prove the theorem, assuming the lemma.

Proof of the Theorem. Let P denote the Bergman projection associated to $D - K$. Let u be a function in $C^\infty(\overline{D - K})$. The function Pu extends to be holomorphic on all of D by Hartog's theorem. We will prove the theorem by showing that for each positive integer s , there are constants $c = c(s)$ and $N = N(s)$ which are independent of u such that

$$\|Pu\|_{W^s(D)} \leq c \text{Sup} \{ |\partial^\alpha u(x)| : z \in D - K; |\alpha| \leq N \}.$$

Let s be a fixed positive integer, and let $M = M(s)$ be the constant of the lemma associated to D and s . According to the lemma, $\|Pu\|_s \leq C \| \|Pu\|_{s+M}$. Let g be a test function in $H(D)$. To complete the proof of the theorem, we must bound $|\int_D Pu\bar{g}|$ by a constant times

$$\|g\|_{-s-M} \text{Sup} \{ |\partial^\alpha u(z)| : z \in D - K; |\alpha| \leq N \}$$

for some integer N , where the constant is independent of g and u . Let Ω be a smooth bounded domain such that $K \subset \subset \Omega \subset \subset D$. Now

$$\int_D Pu\bar{g} = \int_{D-K} Pu\bar{g} + \int_K Pu\bar{g}.$$

The second integral in this sum can be ignored for our purposes because $\|g\|_{L^2(D-K)} \leq (\text{constant}) \|g\|_{-s-M}$ and $\|Pu\|_{L^2(K)} \leq (\text{constant}) \|u\|_{L^2(D-K)}$. The first integral can be further decomposed:

$$\int_{D-K} Pu\bar{g} = \int_{D-K} u\bar{g} = \int_{D-\Omega} u\bar{g} + \int_{\Omega-K} u\bar{g}.$$

Once again, the second integral in the sum can be ignored because $\|g\|_{L^2(\Omega)} \leq (\text{constant}) \|g\|_{-s-M}$. Thus, it remains only for us to estimate the integral $\int_{D-\Omega} u\bar{g}$.

Let $\partial/\partial n$ denote the normal derivative operator on $b(D - \bar{\Omega})$. If ψ is a function such that $\psi = 0 = \partial\psi/\partial n$ on $b(D - \bar{\Omega})$, then $\Delta\psi$ is orthogonal to holomorphic functions on $D - \bar{\Omega}$. This can be seen by performing an

integration by parts. We now solve the following elliptic boundary value problem on $D - \bar{\Omega}$:

$$\Delta^m \phi = 0 \quad \text{on } D - \bar{\Omega}$$

where $m = s + M + 2$, and ϕ satisfies the boundary conditions:

$$\begin{cases} \phi = \frac{\partial \phi}{\partial n} = 0, \\ \Delta \phi = u, \\ \left(\frac{\partial}{\partial n}\right)^t \Delta \phi = \left(\frac{\partial}{\partial n}\right)^t u \quad \text{for } t = 1, 2, \dots, m - 3, \end{cases}$$

on bD and $b\Omega$.

The solution ϕ to this problem is such that $u - \Delta \phi$ belongs to the $W^{s+M}(D - \bar{\Omega})$ closure of $C_0^\infty(D - \bar{\Omega})$. To complete the proof of the theorem, observe that

$$\int_{D-\Omega} u \bar{g} = \int_{D-\Omega} (u - \Delta \phi) \bar{g}.$$

The absolute value of this last integral is less than or equal to

$$\|u - \Delta \phi\|_{W^{s+M}(D-\Omega)} \|g\|_{-s-M}.$$

Finally, we must estimate $\|u - \Delta \phi\|_{W^{s+M}(D-\Omega)}$. Now, for each positive integer t , there is a constant C_t which does not depend on u such that $\|\phi\|_t \leq C_t \|u\|_{t+Q}$ where Q can be taken to be equal to $(m - 3)(m + 2)/2$ (see [7]). Hence,

$$\begin{aligned} \|u - \Delta \phi\|_{s+M} &\leq C(\|u\|_{s+M} + \|\phi\|_{s+M+2}) \\ &\leq C \text{Sup}\{|\partial^\alpha u(z)| : z \in D - K; |\alpha| \leq N\} \end{aligned}$$

where $N = s + M + 2 + Q$. This completes the proof of the theorem. \square

The proof of the theorem will be legitimate, once we establish the truth of the lemma.

Proof of the Lemma. Since P maps $C^\infty(\bar{D})$ to $C^\infty(\bar{D})$ continuously, there is a positive integer $M = M(s)$ such that $\|P\phi\|_s \leq (\text{constant}) \|\phi\|_{s+M}$ for all ϕ in $W^{s+M}(D)$.

Let Ω be a relatively compact subset of D , and let g be a function in $H(D)$. The linear functional L on $H(D)$ defined by

$$Lh = \sum_{|\alpha| \leq s} \int_{\Omega} \partial^\alpha h \bar{\partial}^\alpha g$$

is continuous. Hence, There is a function G in $H(D)$ such that $Lh = \langle h, G \rangle_{L^2(D)}$ for all h in $H(D)$. Now

$$\|g\|_{W^s(\Omega)}^2 = Lg = \langle g, G \rangle_{L^2(D)} \leq \|g\|_{s+M} \|G\|_{-s-M}.$$

The proof of the lemma will be finished when we prove that $\|G\|_{-s-M} \leq (\text{constant}) \|g\|_{W^s(\Omega)}$ where the constant is independent of g and Ω . Indeed, if $\phi \in C_0^\infty(D)$, then

$$\begin{aligned} \left| \int_D G \bar{\phi} \right| &= \left| \int_D G \bar{P}\phi \right| = \left| \sum_{|\alpha| \leq s} \int_\Omega \partial^\alpha g \bar{\partial}^\alpha P\phi \right| \leq \|g\|_{W^s(\Omega)} \|P\phi\|_s \\ &\leq C \|g\|_{W^s(\Omega)} \|\phi\|_{s+M}. \end{aligned}$$

Hence, $\|g\|_{W^s(\Omega)} \leq C \|g\|_{s+M}$. Since the constant C is independent of g and Ω , we obtain that $\|g\|_s \leq C \|g\|_{s+M}$.

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