EXTREME POINTS IN THE HAHN-BANACH-KANTOROVIČ SETTING

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This paper presents an existence and characterization theorem for the extreme points of the convex set of all extensions of a linear operator from a real vector space into an order complete real vector lattice which are dominated by a sublinear operator. This result is applied to positive extensions, contractions, and dominated invariant extensions.

The paper falls into four sections.

Section 1 is reserved for preliminaries.

In §2 we consider the convex set of all extensions of a linear operator defined on a vector subspace of a real vector space $X$ with values in an order complete real vector lattice $Y$ which are dominated by a sublinear operator $P$ from $X$ into $Y$. We present a characterization of the extreme points of this set being useful for applications. This part is related to papers of Kutateladze [7],[8] and Portenier [15].

In §3 we give two applications of the preceding result. The first one yields another proof of an existence and characterization theorem due to Lipecki [10], [11] concerning extreme positive extensions of a linear operator which is defined on a subspace of an ordered vector space. The second one yields a new characterization theorem for extreme contractions from a separable Banach space into the space of real valued continuous functions on a compact extremally disconnected space.

In §4 the results of §2 are extended to $P$-dominated extensions which are positive on a given cone in $X$, and we apply them to $P$-dominated extensions which are invariant with respect to a set of mappings from $X$ into $X$. Furthermore, we obtain a refinement of a dominated extension theorem for positive linear operators due to Luxemburg and Zaanen.

1. Preliminaries. We adhere to the notation of Schaefer's monograph [16]. Throughout $X$ stands for a real vector space, $M$ for its vector subspace and $Y$ for an order complete real vector lattice. $P: X \to Y$ denotes a sublinear mapping, i.e. $P$ is positively homogeneous and subadditive. The space of all linear operators from $M$ into $Y$ is denoted by $L(M, Y)$. Given a vector subspace $N$ of $X$ with $M \subset N$ and $T \in L(M, Y)$, we put

$$E_N(P, T) = \{S \in L(N, Y): S \leq P \mid N \text{ and } S \mid M = T\}.$$
The notation $E_X(P, T)$ is abbreviated to $E(P, T)$ and $E(P)$ stands for $E_X(P, T)$, provided $M = \{0\}$, i.e.

$$E(P) = \{S \in L(X, Y) : S \leq P\}.$$ 

Finally, $E_N(P, T)$ denotes the set of all extreme points of the convex set $E_N(P, T)$.

**Proposition 1.1.** (Hahn-Banach-Kantorovič Theorem, [6, 2.5.7, 2.5.8]).

If $T \in L(M, Y)$ and $T \leq P | M$, then $E(P, T) \neq \emptyset$. In particular, given $x \in X$ and $y \in [-P(-x), P(x)]$, then there exists an operator $S \in E(P)$ such that $Sx = y$.

Bonnice, Silverman [3] and To [21] (cf. also Ioffe [5]) have proved that a preordered vector space is order complete, if it has the Hahn-Banach extension property according to Proposition 1.1. Thus the order completeness of $Y$ is indispensable in our investigation of extreme extensions.

2. Existence and characterization of extreme extensions. With $P: X \to Y$ sublinear and $S \in L(X, Y)$ (and the linear subspace $M$) we associate the map $P^S: X \to Y \cup \{-\infty\}$ defined by

$$P^S(x) = \inf\{(P - S)(u + z + x) + (P - S)(u - z - x) : z \in M, u \in X\}.$$ 

**Lemma 2.1.** The following conditions are equivalent.

(i) $S \leq P$,

(ii) $0 \leq P^S \leq 2(P - S)$,

(iii) $P^S | M = 0$,

(iv) $P^S(0) > -\infty$,

(v) $P^S: X \to Y$ is sublinear.

The simple proof is left to the reader. (The fact that matters subsequently is that (i) implies the other statements.) Moreover, if $S \leq P$ and $S | M = P | M$, then the definition of $P^S$ reduces to

$$P^S(x) = \inf\{(P - S)(u + x) + (P - S)(u - x) : u \in X\}$$

for all $x \in X$.

If $T \in L(M, Y)$ and $T \leq P | M$, then the existence of extreme points of $E(P, T)$ is known, see the sophisticated result of Vincent-Smith [22, Addendum to Theorem 1] and recent results of Kutateladze [7], [8] and Lipecki [11]. We shall give a proof using the characterization of extreme extensions stated in part (b) of the following theorem. This characterization turns out to be useful for applications. For $Y = \mathbb{R}$ and
\[ M = \{0\} \] it follows from Proposition 2.2 in Portenier [15] and for \( M = \{0\} \) some other characterizations may be found in [7], [8].

**Theorem 2.2.** Let \( T \in L(M, Y) \).

(a) If (and only if) \( T \leq P \upharpoonright M \), then \( \text{ex } E(P, T) \neq \emptyset \).

(b) Suppose \( S \in E(P, T) \). Then \( S \in \text{ex } E(P, T) \) if and only if \( P^S(x) = 0 \) for each \( x \in X \).

**Proof.** Both parts (a) and (b) are proved simultaneously. In Step 1 and Step 2 we show the existence of \( S \in E(P, T) \) such that \( P^S = 0 \) by means of the Kuratowski-Zorn lemma; Step 3 proves the "if" part and Step 4 proves the "only if" part of (b) which completes the proof.

**Step 1.** By \( \mathcal{M} \) we denote the class of all pairs \((N, R)\), where \( N \) is a vector subspace of \( X \) with \( M \subseteq N \) and \( R \in L(N, Y) \) such that \( R \upharpoonright M = T \), \( R \leq P \upharpoonright N \) and \( P^R = 0 \) (\( P^R(x) := \inf \{(P - R)(u + z + x) + (P - R)(u - z - x) : z \in M, u \in N\}, x \in N\)). Let \( \ll \) be an ordering in \( \mathcal{M} \) defined by \( (N_1, R_1) \ll (N_2, R_2) \) if and only if \( N_1 \subseteq N_2 \) and \( R_2 \upharpoonright N_1 = R_1 \). Given \( (N, R) \in \mathcal{M} \) and \( x_0 \in X \), we shall show that \( (N_0, R_0) \in \mathcal{M} \) and \( (N, R) \ll (N_0, R_0) \), where \( N_0 = \text{lin}(N \cup \{x_0\}) \) and \( R_0 : N_0 \to Y \) is defined by

\[ R_0(v + tx_0) = Rv + ty_0 \quad (v \in N, t \in \mathbb{R}) \]

with

\[ y_0 = \inf \{ P(v + x_0) - Rv : v \in N \}. \]

As easily seen we have \( R_0 \in L(N_0, Y) \) and \( R_0 \upharpoonright N = R \) and \( R_0 \leq P \upharpoonright N_0 \). It remains to show that \( P^{R_0} = 0 \). In view of \( 0 \leq P^{R_0} \upharpoonright N \leq P^R = 0 \) we get

\[ 0 \leq P^{R_0}(v + tx_0) \leq P^{R_0}(v) + P^{R_0}(tx_0) = |t| P^{R_0}(x_0) \]

\[ = |t| P^{R}(w + x_0) \leq 2 |t| (P - R_0)(w + x_0) \]

for all \( v, w \in N \) and \( t \in \mathbb{R} \). This yields \( P^{R_0} = 0 \), since

\[ \inf \{(P - R_0)(v + x_0) : v \in N\} = 0 \]

by the definition of \( y_0 \).

**Step 2.** Let \( \mathcal{M}_0 \subseteq \mathcal{M} \) be a chain and put \( N_0 = \bigcup \{ N : (N, R) \in \mathcal{M}_0 \} \) and \( R_0 \upharpoonright N = R \) for all \( (N, R) \in \mathcal{M}_0 \). Then \( (N_0, R_0) \) in \( \mathcal{M} \) is an upper bound for \( \mathcal{M}_0 \), since \( 0 \leq P^{R_0}(x) \leq P^R(x) \) for all \( x \in N \) and \( (N, R) \in \mathcal{M}_0 \). Thus, by the Kuratowski-Zorn lemma, \( \mathcal{M} \) has a maximal element \((N, R)\) and we obtain \( N = X \) by Step 1.
Step 3. Let $S \in E(P, T)$ with $P_S = 0$ and $S_0 \in L(X, Y)$ such that $S \pm S_0 \in E(P, T)$. Then $S_0 | M = 0$ and $\pm S_0 \leq P - S$. Hence, for each $u, x \in X$ and $z \in M$ we have

$$2S_0x = S_0(u + z + x) - S_0(u - z - x)$$

$$\leq (P - S)(u + z + x) + (P - S)(u - z - x)$$

which implies $2S_0 \leq P^S = 0$. Therefore, $S_0 = 0$ whence $S \in \text{ex} E(P, T)$.

Step 4. Given $S \in \text{ex} E(P, T)$ and $x_0 \in X$, there exists $S_0 \in L(X, Y)$ such that $S_0(x_0) = P_S(x_0)$ and $S_0 \leq P^S$ by Proposition 1.1. Thus,

$$\pm S_0x = S_0(\pm x) \leq P^S(\pm x) = P^S(x) \leq 2(P - S)(x)$$

for all $x \in X$ which implies $S \pm \frac{1}{2}S_0 \leq P$. Moreover, $\pm S_0z \leq P^S(z) = 0$ for all $z \in M$ whence $S \pm \frac{1}{2}S_0 \in E(P, T)$. Therefore, $P^S(x_0) = 0$.

The proof of Theorem 2.2 suggests to associate with $P: X \to Y$ sublinear and $T \in L(M, Y)$ the map $P_T: X \to Y \cup \{-\infty\}$ defined by

$$P_T(x) = \inf \{P(x + z) - Tz: z \in M\}.$$

Lemma 2.3. Suppose $T \in L(M, Y)$.

(a) The following conditions are equivalent.

(i) $T \leq P | M$,

(ii) $T = P_T | M$,

(iii) $P_T(0) > -\infty$,

(iv) $P_T: X \to Y$ is sublinear.

(b) Suppose $S \in L(X, Y)$. Then $S \in E(P, T)$ if and only if $S \leq P_T$.

The simple proof is left to the reader. It is worth mentioning that $P_S$ and $(P_T)^S$ coincide for each $S \in E(P, T)$. Especially, this implies

$$P^S(x) = \inf \{(P_T - S)(u + x) + (P_T - S)(u - x): u \in X\}$$

for all $x \in X$. Moreover, if $T \in L(M, Y)$ and $T \leq P | M$, then $E(P, T) = E(P_T)$ holds by Lemma 2.3 and according to Proposition 1.1 we obtain

$$\{Sx: S \in E(P, T)\} = [-P_T(-x), P_T(x)]$$

for all $x \in X$. Following up these ideas, we obtain

Corollary 2.4. Suppose $T \in L(M, Y)$ with $T \leq P | M$ and $x \in X$. Then

$$\text{ex}[-P_T(-x), P_T(x)] \subseteq \{Sx: S \in \text{ex} E(P, T)\} \subseteq [-P_T(-x), P_T(x)].$$
Proof. We only have to show the first inclusion. Given \( y \in \text{ex}\left[-P_T(-x), P_T(x)\right] \), we define
\[
H = \{ S \in E(P, T) : Sx = y \}
\]
and
\[
Q(u) = \sup\{ Su : S \in H \}, \quad u \in X.
\]
Then \( H \neq \emptyset \), \( Q : X \to Y \) is sublinear, and \( E(Q) = H \). By Theorem 2.2, \( \text{ex} H \neq \emptyset \). Moreover, \( H \) is an extreme subset of \( E(P, T) \) by virtue of \( y \in \text{ex}\left[-P_T(-x), P_T(x)\right] \). Thus \( \text{ex} H \subset \text{ex} E(P, T) \) which implies the assertion.

Remark 2.5. Both inclusions are proper in general. They provide precise bounds for \( \{ Sx : S \in \text{ex} E(P, T) \} \) as will be shown by the following examples. For this let \((\Omega, \emptyset, \mu)\) be a probability space. We put \( X = L_1(\mu), Y = \mathbb{R}, P(u) = \int |u| \, d\mu \) and \( x = 1_\Omega \). Then
\[
E(P) = \{ f \in L_\infty(\mu) : |f| \leq 1_\Omega \}
\]
and
\[
\text{ex} E(P) = \{ f \in L_\infty(\mu) : |f| = 1_\Omega \},
\]
and we have \( P(x) = 1 \) and \( -P(-x) = -1 \).

If \( \mu \) is the one-point measure \( \delta_\omega \) in \( \omega \in \Omega \), then \( \{ \int f \, d\mu : f \in \text{ex} E(P) \} = \{-1, 1\} \subset [-1, 1] \) which shows that the first inclusion turns into equality and the second inclusion is proper.

If \( \mu \) is non-atomic, then \( \{ \int f \, d\mu : f \in \text{ex} E(P) \} = [-1, 1] \). Indeed, given \( |y| \leq 1 \), let \( y_0 = (1/2)(|y| + 1) \in [0, 1] \) and \( A \in \emptyset \) with \( \mu(A) = y_0 \). Then \( f = 1_A - 1_{A^c} \) is in \( \text{ex} E(P) \) with \( \int f \, d\mu = |y| \). Thus the first inclusion is proper and the second inclusion turns into equality.

Remark 2.6. Obviously
\[
\{-P_T(-x), P_T(x)\} \subset \text{ex}\left[-P_T(-x), P_T(x)\right]
\]
and this inclusion is proper in general. Indeed, let \((\Omega, \emptyset, \mu)\) be a probability space, where \( \mu \) is not \( \{0, 1\}\)-valued, let \( X = \mathbb{R}, Y = L_\infty(\mu), P(u) = |u| \cdot 1_\Omega \) for all \( u \in X \) and \( x = 1 \). Then \( \{-P(-x), P(x)\} = \{-1_\Omega, 1_\Omega \} \subset \text{ex}\{-1_\Omega, 1_\Omega\} = \{ f \in L_\infty(\mu) : |f| = 1_\Omega \} \).

Suppose now that \( M, N \) are linear subspaces of \( X \) with \( M \subset N \subset X \) and \( R \in E_N(P, T) \). Theorem 2.7 and the succeeding counterexamples clear up the connections between the sets \( \text{ex} E_N(P, T), \text{ex} E(P, T) \) and...
ex $E(P, R)$. In particular, it follows that $S \in \text{ex } E(P, R)$ implies $S \in \text{ex } E(P, S \cap N)$, whereas $S \mid N \notin \text{ex } E_N(P, T)$ in general. Conversely, $S \in \text{ex } E(P, S \cap N)$ and $S \mid N \in \text{ex } E_N(P, T)$ imply $S \in \text{ex } E(P, T)$.

**Theorem 2.7.** Let $M, N$ be subspaces of $X$ with $M \subset N \subset X$. Suppose $T \in L(M, Y)$ and $R \in E_N(P, T)$. Then
(a) $\text{ex } E(P, R) \supset E(P, R) \cap \text{ex } E(P, T)$.
(b) Suppose $R \in \text{ex } E_N(P, T)$. Then

$$\text{ex } E(P, R) = E(P, R) \cap \text{ex } E(P, T).$$

**Proof.** (a) is obvious and as regards (b), $R \in \text{ex } E_N(P, T)$ implies that $E(P, R)$ is an extreme subset of $E(P, T)$ which proves the assertion.

**Remark 2.8.** The inclusion in (a) is proper in general. Indeed, let $N = X = Y = \mathbb{R}$, $M = \{0\}$, $P(x) = |x|$ and $Rx = 0$ for all $x \in \mathbb{R}$. Then $\{R\} = E(P, R) = \text{ex } E(P, R)$ and $\text{ex } E(P) = \{\text{id}_\mathbb{R}, -\text{id}_\mathbb{R}\}$.

The converse in (b) does not hold as the following example shows (cf. also Singer [19, p. 106]). Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $M = \{(0, 0)\}$, $N = \mathbb{R} \times \{0\}$, $Rx = 0$ for all $x \in N$ and let $P((x_1, x_2)) = \max\{|x_1|, |x_2|\}$ and $\text{pr}_i((x_1, x_2)) = x_i$ for all $(x_1, x_2) \in \mathbb{R}^2$. Then $\text{ex } E(P, R) = \{\text{pr}_2, -\text{pr}_2\} \subset \text{ex } E(P) = \{\text{pr}_1, -\text{pr}_1, \text{pr}_2, -\text{pr}_2\}$, whereas $R \notin \text{ex } E_N(P) = \{\text{pr}_1 \mid N, -\text{pr}_1 \mid N\}$.

In particular, this example yields an operator $S \in \text{ex } E(P)$ such that $S \mid N \notin \text{ex } E_N(P)$; put $S = \text{pr}_2$.

**3. Applications.** By Theorem 2.2 we obtain immediately a result due to Lipecki concerning extreme positive extensions of an operator defined on a subspace of an ordered vector space with values in an order complete vector lattice.

A1. (Lipecki [10, Theorem 1], [11, Theorem 2 and Remark 2]). Let $X$ be an ordered vector space with positive cone $C$, $M$ a majorizing vector subspace of $X$ and $T: M \to Y$ a positive linear operator. Then
(a) $\text{ex } \{S \in L(X, Y): S \mid M = T \text{ and } S \mid C \geq 0\} \neq \emptyset$.
(b) Suppose $C$ is generating. Then $S \in \{S' \in L(X, Y): S' \mid M = T, S' \mid C \geq 0\}$ is an extreme point of this set if and only if

$$\inf\{Su: \pm (x + z) \leq u \in X, z \in M\} = 0$$

for each $x \in X$.

**Proof.** Obviously, $P: X \to Y$ defined by

$$P(x) = \inf\{Tz: x \leq z \in M\}$$
is sublinear and we have

\[ E(P, T) = \{ S \in L(X, Y) : S|M = T, S|C \geq 0 \}. \]

Moreover, it is readily verified that

\[ P^S(x) = 2 \cdot \inf \{ Su : \pm (x + z) \leq u \in X, z \in M \} \]

holds for all \( x \in X \) and \( S \in E(P, T) \). Thus the assertion follows from Theorem 2.2.

We now apply Theorem 2.2 to prove a new result concerning extreme contractions into spaces of continuous functions. This subject was started by Blumenthal et al. [2].

Let \( X \) be a normed vector space and \( Y = C(H) \), where \( H \) is an extremally disconnected compact space. Recall that a compact space is extremally disconnected (i.e. open subsets have an open closure) if and only if the space of continuous real valued functions is order complete (cf. [16, II. 7.7]). \( X' \) denotes the continuous dual of \( X \), \( U(X') \) is the unit ball of \( X' \) and \( U(X, C(H)) \) is the unit ball of \( L(X, C(H)) \). An operator \( S \in U(X, C(H)) \) is called almost nice, if \( S' : H \to U(X') \) defined by \( S'(h) = \delta_h \circ S \) maps a dense subset of \( H \) into \( \text{ex} \ U(X') \).

It is immediate from Theorem 2.2 (with \( M = \{ 0 \} \) and \( P(x) = \| x \| \)) that \( x' \in U(X') \) is an extreme point of this set if and only if

\[ \inf \{ \| u + x \| + \| u - x \| + 2x'(u) : u \in X \} = 0 \]

for each \( x \in X \). (Incidentally, this characterization may be used to prove the well known result that \( \text{ex} U(C'(K)) = \{ \alpha \delta_k : |\alpha| = 1, k \in K \} \), where \( K \) is a compact space [4, V.8.6]). Furthermore, let us point out that an almost nice operator \( S \in U(X, C(H)) \) is an extreme point of this set, since \( S' \) is weak* continuous [4, VI.7.1]. The converse does not hold in general (cf. Remark 3.1). In addition, we note (cf. Oates [14, Theorem 1.2]) that \( U(X, C(H)) \) is the closed convex hull of its extreme points with respect to the strong (or equivalently weak) operator topology on the space of all continuous linear operators from \( X \) into \( C(H) \). More general Krein-Milman type theorems may be found in Morris and Phelps [13] and (without proofs) in Levashov [9].

A2. Let \( X \) be a separable normed space and let \( H \) be an extremally disconnected compact space. Then \( S \in U(X, C(H)) \) is an extreme point of this set if and only if \( S \) is almost nice.
Proof. We only have to show the "only if" part. Suppose $S \in \text{ex } U(X, C(H))$. By Theorem 2.2 (with $M = \{0\}$ and $P(x) = \|x\|_H$), we have

$$\inf\{\|u + x\|_H + \|u - x\|_H - 2Su : u \in X\} = 0$$

for each $x \in X$. Let $\varphi: X \times H \to \mathbb{R}$ be defined by

$$\varphi(x, h) = \inf\{\|u + x\| + \|u - x\| - 2\delta_h(Su) : u \in X\}.$$ 

The set $\{\varphi(x, \cdot) > 0\} \subset H$ is meager for each $x \in X$, since the lattice infimum and the point infimum in $C(H)$ differ on a meager subset (cf. Stone [20]). If $A$ is a countable dense subset of $X$, then

$$K = \bigcap_{x \in A} \{\varphi(x, \cdot) = 0\}$$

is a dense subset of $H$ by Baire's category theorem. Since $\varphi(\cdot, h)$ is continuous for all $h \in H$, we obtain $\varphi(\cdot, h) = 0$ for all $h \in K$. Thus by Theorem 2.2 $T$ is almost nice.

Without proof we note the following slight generalization of A2.

A3. Let $X$ be a separable normed space, $M$ a vector subspace of $X$ and $T \in U(M, C(H))$, where $H$ is an extremally disconnected compact space. Then

$$S \in \{R \in U(X, C(H)) : R | M = T\}$$

is an extreme point of this set if and only if

$$S'(h) \in \text{ex } U(X') : x' | M = T'(h)$$

for all $h$ in some dense subset of $H$.

Remark 3.1. A2 and A3 fail for non-separable normed spaces $X$. Indeed, let $H$ be a compact extremally disconnected space such that the set $H_0$ of isolated points of $H$ is not dense (e.g. if $(\Omega, \mathcal{F}, \mu)$ is a positive $\sigma$-finite non-atomic measure space and $\mathcal{K}$ the ideal of $\mu$-null sets, then the Stone representation space of $\mathcal{F}/\mathcal{K}$ is extremally disconnected and has no isolated points [18, p. 28 and p. 86]). By virtue of a result due to Blumenthal et al. [2, Theorem 2] there is a (non-separable) Banach space $X$ and an operator $S \in \text{ex } U(X, C(H))$ such that $\{h \in H : S'(h) \in \text{ex } U(X')\} = H_0$. Hence $S$ is not almost nice.

Nevertheless, the separability of $X$ can be removed if $X = C(K)$ for some compact space $K$. More generally, the separability assumption in A2 can be replaced by the assumption that $\text{ex } U(X')$ is weak* closed. This result is due to Sharir [17] and it may also be proved by an application of Theorem 2.2 and Theorem 2.7.
4. Generalizations. In this section the preceding results are generalized to $P$-dominated extensions, which are positive on a (pointed convex) cone $C \subset X$. With $P : X \to Y$ sublinear and a cone $C \subset X$ we associate the map $P_C : X \to Y \cup \{-\infty\}$ defined by

$$P_C(x) = \inf\{P(x + u) : u \in C\}$$

and, given $T \in L(M, Y)$, we put

$$E(P, T, C) = \{S \in E(P, T) : S|C \geq 0\}.$$

**Lemma 4.1.** Let $C$ be a cone in $X$.
(a) The following conditions are equivalent.
(i) $P|C \geq 0$,
(ii) $P_C(0) > -\infty$,
(iii) $P_C : X \to Y$ is sublinear.
(b) Suppose $S \in L(X, Y)$. Then $S \leq P$ and $S|C \geq 0$ if and only if $S \leq P_C$.

The simple proof will be omitted (for $Y = \mathbb{R}$ compare Anger and Lembcke [1, Lemma 1.9, Lemma 3.2]). In view of the preceding lemma the following corollary is merely a restatement of Theorem 2.2.

**Corollary 4.2.** Let $T \in L(M, Y)$ and $C$ be a cone in $X$.
(a) If (and only if) $T \leq P_C|M$, then $\text{ex } E(P, T, C) \neq \emptyset$.
(b) An operator $S \in E(P, T, C)$ is an extreme point of this set if and only if $(P_C)^S(x) = 0$ for each $x \in X$.

**Remark 4.3.** Lemma 2.3 yields equivalent assertions for the statement $T \leq P_C|M$. Furthermore, $T \leq P_C|M$ yields

$$E(P, T, C) = E(P_C, T) = E(P_T, C) = E(P_T, C),$$

where $P_T$ stands for both (coinciding) operators $(P_C)_T$ and $(P_T)_C$. Finally, we note that $T \leq P_C|M$ readily implies $T \leq P|M$, $T|M \cap C \geq 0$, $P|C \geq 0$, but the converse does not hold in general. Indeed, let $X = \mathbb{R}^2$, $M = \mathbb{R} \times \{0\}$, $C = \{0\} \times \mathbb{R}$, $Y = \mathbb{R}$, $P((x_1, x_2)) = |x_1 + x_2|$ for all $(x_1, x_2) \in X$ and $T(x_1, 0) = x_1$ for all $(x_1, 0) \in M$ and note that $P_C|M = 0$.

An application of Corollary 4.2 yields a refinement of a result due to Luxemburg and Zaanen.

**A4.** (Luxemburg and Zaanen; cf. [6, 2.6.3]). Suppose that $X$ is a vector lattice with positive cone $C$, $M$ is a vector sublattice, and $P$ is lattice-increasing, i.e. $|x_1| \leq |x_2|$ implies $P(x_1) \leq P(x_2)$, $x_i \in X$. Let $T \in L(M, Y)$.
(a) If $T \leq P|M$ and $T|M \cap C \geq 0$, then $\text{ex } E(P, T, C) \neq \emptyset$. 


(b) An operator $S \in E(P, T, C)$ is extreme in this set if and only if
\[
\inf\{p\left((u + z + x)^+ + P\left((u - z - x)^+\right) - 2Su: z \in M, u \in X\}\} = 0
\]
for each $x \in X$.

Proof. First note that $P_C(x) = P(x^+)$ holds for all $x \in X$. Indeed, $x \leq x^+$ implies $P_C(x) \leq P(x^+)$; conversely, $x \leq u \in X$ implies $x^+ \leq u^+$, and since $P$ is lattice-increasing we obtain $P(x^+) \leq P(u^+) \leq P(u)$ whence $P(x^+) \leq P_C(x)$. Moreover, we have $Tz \leq Tz^+ \leq P(z^+)$ for all $z \in M$. Hence the assertions follow from Corollary 4.2.

In the remaining part of this section we deal with invariant $P$-dominated extensions. Let $\emptyset$ be a set of mappings from $X$ into $X$. A linear operator $S: X \to Y$ is called invariant if $SV = S$ for all $V \in \emptyset$. The vector space of all invariant linear operators from $X$ into $Y$ is denoted by $L(X, Y)_\emptyset$ and, given $T \in L(M, Y)$, we put
\[
E(P, T)_\emptyset = E(P, T) \cap L(X, Y)_\emptyset.
\]
Furthermore, let $G$ denote the linear hull of the set $\{Vx - x: V \in \emptyset, x \in X\}$. Obviously, $S \in L(X, Y)$ is invariant if and only if $S | G = 0$, i.e. $G \subset \ker S$.

A5. Let $T \in L(M, Y)$ and $\emptyset$ be a set of mappings from $X$ into $X$.
(a) If (and only if) $T \leq P_G | M$, then $\ex E(P, T)_\emptyset \neq \emptyset$.
(b) $S \in E(P, T)_\emptyset$ is an extreme point of this set if and only if
\[
\inf\{(P - S)(u + z + x) + (P - S)(u - z - x): z \in M + G, u \in X\} = 0
\]
for each $x \in X$.

Proof. Obviously, $E(P, T)_\emptyset = E(P, T, G)$, and it is easily seen that $(P_G)^S = (P^S)_G$ holds for all $S \in E(P, T, G)$. Hence the assertions follow from Corollary 4.2.

A6. Let $T$ and $\emptyset$ be as in A5. Assume that there exists an operator $R \in L(X, X)$ such that $RM \subset M, PR \leq P$ and $P_{\ker R} \leq P_G$.
(a) If $TR = T$ on $M$ and $T \leq P | M$, then $\ex E(P, T)_\emptyset \neq \emptyset$.
(b) If additionally $R$ is a projection with $P_{\ker R} = P_G$, then an operator $S$ in $E(P, T)_\emptyset$ is an extreme point of this set if and only if $P^S(x) = 0$ for each $x$ in the fixed space of $R$.

Proof. (a) For $z \in M$ and $u \in \ker R$ we get $Tz = TRz \leq P(Rz) = P(R(z + u)) \leq P(z + u)$ whence $T \leq P_{\ker R} | M$. The assertion follows from A5.
(b). Suppose \( S \in E(P, T)_g \). Then \( S \leq P_G = P_{\ker R} \) implies \( SR = S \) whence \((PR)^S = P^S R\). Furthermore, it is readily verified that \( PR = P_{\ker R} \). Therefore \((P_G)^S = P^S R\). By virtue of Corollary 4.2 \( S \) is an extreme point of \( E(P, T)_g \) if and only if \( P^S(Rx) = 0 \) for each \( x \in X \), i.e. \( P^S(x) = 0 \) for each \( x \) in the fixed space of \( R \).

We conclude our considerations by applying this result to a topological setting. A similar result for positive invariant extensions was stated by the first-named author in [12].

A7. Suppose that \( X \) is a locally convex space, that \( \mathcal{G} \) is a mean ergodic semigroup of continuous linear operators on \( X \) [16, III.7.1] and that the order complete vector lattice \( Y \) is a topological vector space with a closed normal positive cone (e.g. we may assume that \( Y \) is an order complete topological vector lattice). Let \( M \) be a closed subspace and let \( P \) be continuous such that \( VM \subseteq M \) and \( PV \leq P \) for all \( V \in \mathcal{G} \). Suppose \( T \in L(M, Y) \).

(a) If \( T \) is invariant and \( T \leq P | M \), then \( \text{ex} \ E(P, T)_g \neq \emptyset \).

(b) An operator \( S \) in \( E(P, T)_g \) is an extreme point of this set if and only if \( P^S(x) = 0 \) for each \( x \) in the fixed space of \( \mathcal{G} \).

Proof. Let \( R \) be the zero element of the closed convex hull of \( \mathcal{G} \) in the space of all continuous linear operators on \( X \) equipped with the topology of pointwise convergence. \( R \) is a continuous linear projection onto the fixed space of \( \mathcal{G} \) with kernel the closure of \( G \) [16, III.7.2]. In view of the properties of \( P \) and \( \mathcal{G} \) we obtain \( P_{\ker R} = P_G \), \( PR \leq P \) and \( RM \subseteq M \). Furthermore, each \( S \in E(P) \) is continuous since \( P \) is continuous and the positive cone of \( Y \) is normal. Thus \( T \) is continuous by virtue of \( E(P, T) \neq \emptyset \) and employing the invariance of \( T \) we obtain \( TR = T \) on \( M \). The assertions follow now from A6.

Remark 4.4. If \( \mathcal{G} \) is a set of mappings from \( X \) into \( X \), then invariant versions of A2 and A3 are valid.

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Spiros Argyros, On compact spaces without strictly positive measure .......... 257
Steven Robert Bell, Regularity of the Bergman projection in certain nonpseudoconvex domains ......................................................... 273
Carlos R. Borges and Gary Fred Gruenhage, Sup-characterization of stratifiable spaces .......................................................... 279
Giuseppe Ceresa and Alberto Collino, Some remarks on algebraic equivalence of cycles ............................................................... 285
Charles Kam-Tai Chui and Maurice Hasson, Degree of uniform approximation on disjoint intervals ........................................ 291
Gary Gundersen, Meromorphic functions that share two finite values with their derivative .......................................................... 299
Richard I. Hartley, Lifting group homomorphisms .............................................. 311
Gerald William Johnson and David Lee Skoug, Notes on the Feynman integral. III. The Schroedinger equation ..................................... 321
John Cronan Kieffer, Some topologies on the set of discrete stationary channels ................................................................. 359
Harald Luschgy and Wolfgang Thomsen, Extreme points in the Hahn-Banach-Kantorovič setting .................................................. 387
Zbigniew Piotrowski, A. Rosłanowski and Brian M. Scott, The pinched-cube topology ................................................................. 399
Elias Saab and Paulette Saab, A dual geometric characterization of Banach spaces not containing \( l_1 \) ................................................................. 415
Walter Schachermayer, Norm attaining operators on some classical Banach spaces ........................................................................ 427
Martin Scharlemann, Essential tori in 4-manifold boundaries .......................... 439
Jacques C. H. Simon, Nonlinear representations of Poincaré group and global solutions of relativistic wave equations .......................... 449
Adrian R. Wadsworth, \( p \)-Henselian field: \( K \)-theory, Galois cohomology, and graded Witt rings .......................................................... 473