FINITE-TO-ONE OPEN MAPPINGS ON CIRCULARLY CHAINABLE CONTINUA

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The authors analyze the behavior of a finite-to-one open mapping
on a hereditarily decomposable circularly chainable continuum. It is
shown that such a mapping behaves similarly to an open mapping on a
simple closed curve.

0. Introduction. In a previous paper (2), the authors completely
described the behavior of a finite-to-one open mapping on a hereditarily
decomposable chainable continuum. In this note we will completely
describe the behavior of a finite-to-one open mapping on a hereditarily
decomposable circularly chainable continuum. This result is a generaliza-
tion of the following theorem of G. T. Whyburn (8): If $X$ is a simple
closed curve and $f(X) = Y$ is a non-constant open mapping onto a
Hausdorff space, then $Y$ is either a simple closed curve or an arc. If $Y$ is
a simple closed curve, then there is an integer $n$ such that $f$ is topologically
equivalent to the mapping $w = z^n$ on the unit circle in the complex plane.
If $Y$ is a simple arc, then there exists an even integer $k$ such that $f$ is
topologically equivalent to the mapping $f(1, x) = \sin(kx/2)$ for $0 \leq x \leq
2\pi$ from the unit circle $r = 1$ in the plane to the interval $[-1, 1]$. We use
the fundamental results of E. S. Thomas (6) implicitly.

1. Notation and definitions. The word mapping is used to denote a
continuous function and a metric continuum $X$ is decomposable if it can
be expressed as the union of two proper subcontinua. A continuum $X$ is
hereditarily decomposable if every non-degenerate subcontinuum is de-
composable. The interior of a subset $D$ is denoted by $\text{int}(D)$ and the
closure of $D$ is represented by $\text{cl}(D)$. Two mappings $f(X) = Y$ and
g($Z) = W$ are topologically equivalent if there exist homeomorphisms
$h(X) = Z$ and $k(W) = Y$ such that $kgh(x) = f(x)$ for all $x$ in $X$. A
mapping $f(X) = Y$ is irreducible if no proper subcontinuum of $X$ maps
onto $Y$ under $f$. Suppose $X$ is a continuum which admits a monotone
mapping $t$ onto the unit interval such that no point inverse has interior
points. A point inverse $t^{-1}(r), r \neq 0$, is called an element of subcontinuity
from the left in the upper-semicontinuous decomposition of $X$ induced by
t if $t^{-1}(r) \subseteq \text{cl}(t^{-1}[0, r))$. If $r \neq 1$ and $t^{-1}(r) \subseteq \text{cl}(t^{-1}(r, 1])$, then $t^{-1}(r)$
is an element of subcontinuity from the right. The collection of elements
of subcontinuity from the left are known to be a dense set in $X$ (6) and the
same is true of the elements of subcontinuity from the right.
2. Preliminary results.

**Lemma 1.** Let $X$ be a hereditarily decomposable circularly chainable continuum. There is a monotone upper-semicontinuous decomposition $G(X)$ defined on $X$ whose quotient space is a simple closed curve and each element of which has empty interior.

*Outline of proof.* Let $X = A \cup B$ be a decomposition of $X$ into proper subcontinua. We can assume that $A \cap B$ has no interior points. The continua $A$ and $B$ are chainable and hereditarily decomposable, so that by R. H. Bing's characterization (1) they admit monotone upper-semicontinuous decompositions $G(A)$ and $G(B)$ respectively whose quotient spaces are simple arcs, say $[a, b]$ and $[c, d]$ respectively. Furthermore, no element of $G(A)$ or $G(B)$ has interior points relative to $X$. We can assume that the elements corresponding to $a$ and $c$ meet, as do the elements corresponding to $b$ and $d$. The monotone decomposition $G(X)$ is obtained by taking the union of the elements corresponding to $a$ and $c$ as one element, the union of the elements corresponding to $b$ and $d$ as another element, and the rest of the elements of $G(A)$ and $G(B)$ for the remainder of $G(X)$. This decomposition is upper-semicontinuous, its quotient space is a simple closed curve, and no element of it has interior points.

The following theorem established in (2) is stated in complete detail even though only the first portion is used in this paper.

**Theorem A.** Let $f(X) = Y$ be a finite-to-one open mapping, where $X$ is a non-degenerate hereditarily decomposable chainable continuum and $Y$ is a Hausdorff space. Then $X = \bigcup_{j=1}^{n} K_j$, where each $K_j$ is a continuum, $f(K_j) = Y$ for each $j$, $f$ is a homeomorphism on any continuum which is interior to $K_j$ relative to $X$, and if $K_i \cap K_j$ is non-empty for $i \neq j$, then the intersection is contained in a single element $K$ of $G(X)$. If $K$ is an element of subcontinuity from one side, then it is an element of subcontinuity from both sides. If not, then $K$ is the union of two homeomorphic subcontinua which meet in a single element of $G(K)$, $f$ is one-to-one on each of them, and they have the same image. If $f$ is irreducible (i.e., there is only one $K_j$), then $f$ is a homeomorphism.

**Lemma 2.** If $f(X) = Y$ is a finite-to-one open mapping, where $X$ is hereditarily decomposable and circularly chainable, and $Y$ is Hausdorff, then $Y$ is hereditarily decomposable and either circularly or linearly chainable.
Proof. By well-known theorems, Y is a metric continuum. The space Y is hereditarily decomposable, for if Y contained a non-degenerate indecomposable continuum L, then by an application of the Brouwer Reduction Theorem there would exist an indecomposable continuum in X which maps onto L. The space Y is circularly or linearly chainable by a theorem of E. Duda and J. Kell (3) or by applying a result of W. T. Ingram (5).

3. Main result.

Theorem 1. Let \( f(X) = Y \) be a finite-to-one open mapping, where X is hereditarily decomposable and circularly chainable and Y is a Hausdorff space. Then Y is hereditarily decomposable and either linearly or circularly chainable. If Y is circularly chainable, then \( f \) is exactly n-to-one for some \( n \) and a local homeomorphism, and \( X = \bigcup_{i=1}^n K_i \), where each \( K_i \) is a continuum, \( f(K_i) = Y \) for each \( i \), and if \( K \subseteq \text{int}(K_i) \) is a continuum, then \( f \) is one-to-one on \( K \). If Y is linearly chainable, then there is an even integer \( k \) such that \( X = \bigcup_{j=1}^k K_j \), where each \( K_j \) is a continuum, \( f(K_j) = Y \) for each \( j \), and if \( K \subseteq \text{int}(K_j) \) is a continuum, then \( f \) is one-to-one on \( K \).

Proof. By Lemma 2, Y is a hereditarily decomposable linearly or circularly chainable continuum. By Lemma 1, there is an upper-semicontinuous decomposition \( G(X) \) generating a monotone mapping \( g_1 \) onto a unit circle \( S_1 \), such that each point-inverse of \( g_1 \) is an element of \( G(X) \). If Y is circularly chainable, there is a similar upper-semicontinuous decomposition \( G(Y) \) generating a monotone map \( g_2 \) onto a unit circle \( S_2 \). If Y is linearly chainable, then by Bing’s theorem (1), there is a monotone decomposition \( G(Y) \) generating a monotone map \( g_3 \) of Y onto the interval \( I = [-1, 1] \). Each point-inverse of \( g_1 \), \( g_2 \), or \( g_3 \), or equivalently, each element of \( G(X) \) or \( G(Y) \), has empty interior.

It was proved in (2) that if \( T \in G(X) \), then \( f(T) \in G(Y) \), and if \( L \in G(Y) \), then \( f^{-1}(L) = \bigcup_{i=1}^m L_i \), where \( L_i \cap L_j = \emptyset \) if \( i \neq j \) and each \( L_i \in G(X) \).

If Y is circularly chainable, define a mapping \( f_1 \) of \( S_1 \) onto \( S_2 \) by \( f_1(z) = g_2 f g_1^{-1}(z) \). The mapping \( f_1 \) is well-defined, since \( f \) preserves decomposition elements, and \( f_1 \) is continuous, finite-to-one and open, since \( f \) has these properties. By the quoted theorem of Whyburn, \( f_1 \) is a local homeomorphism which is topologically equivalent to \( z^n \) for some integer \( n \). If \( I_i \) is an interval in \( S_1 \) of length less than \( 2\pi/n \), then \( f_1|I_i \) is a homeomorphism of \( I_i \) onto \( f_1(I_i) \). The set \( g_1^{-1}(I_i) \) is a continuum, and it is a component of \( f^{-1}(g_1^{-1}(I_i)) \), so \( f|g_1^{-1}(I_i) \) is a finite-to-one open mapping. Furthermore, \( f|g_1^{-1}(I_i) \) is irreducible, and hence by Theorem A, it is a homeomorphism. We now know that \( f \) is a local homeomorphism and exactly \( n \)-to-one.
Let $S_1$ be divided into $n$ non-overlapping arcs $I_j, j = 1, \ldots, n$, each of length $2\pi/n$, and let $K_j = g_{i}^{-1}(I_j)$ for each $j$. Then $X$ is the union of the $K_j, f(K_j) = Y$ for each $j$, and if $K \subset \text{int}(K_j)$ then $f|K$ is one-to-one.

Suppose $Y$ is linearly chainable. Define the map $f_2$ from $S_1$ to $I$ by $f_2(z) = g_3f^{-1}_{g_1}(z)$. As before, $f_2$ is a continuous finite-to-one open mapping of $S_1$ onto $I$, and by Whyburn's theorem, $f_2$ is topologically equivalent to the mapping $f(1, x) = \sin(kx/2)$ for $x$ between 0 and $2\pi$ and $k$ an even integer. In this case, $S_1 = \bigcup_{j=1}^{k} I_j$ and $f_2|I_j$ is a homeomorphism of $I_j$ onto $I$. Let $K_j = g_{i}^{-1}(I_j)$ for each $j$. Then $f(K_j) = Y$ for each $j$, and if $L$ is any continuum in the interior of $K_j$, then $f|L$ is a finite-to-one open mapping of $L$ onto $f(L)$. Furthermore, it is irreducible, so that $f|L$ is a homeomorphism by Theorem A.

The following example illustrates that when $Y$ is linearly chainable, $f|K_j$ is not necessarily a homeomorphism. Let $X$ be the closure of the graph of $y = \sin(1/x)$, $x \neq 0$, $x \in [-1,1]$. Let the endpoints of $X$ be $a$ and $b$. In the unique minimal monotone decomposition $G(X)$ given by Bing's theorem, the only non-degenerate element is the interval from $(0, -1)$ to $(0, 1)$ on the $y$-axis. Let $q$ be the quotient map obtained by identifying each point $(x, y)$ in $X$ with $(-x, -y)$. The mapping $q$ is open and exactly 2-to-one except at the origin. Let $Y$ be a disjoint copy of $X$ with endpoints $c$ and $d$ corresponding to $a$ and $b$, respectively. Let $Z$ be the circularly chainable continuum obtained by identifying $c$ with $a$ and $b$ with $d$. Let $p$ be the mapping of $Z$ onto $X$ obtained by folding $Z$. The mapping $f = q \circ p$ is a finite-to-one open mapping of $Z$ onto $q(X)$. The continuum $q(X)$ is chainable. There are four continua $K_1, K_2, K_3, K_4$ given by Theorem 1 such that $Z = \bigcup_{j=1}^{4} K_j$ and $f(K_j) = q(X)$ but $f|K_j$ is not one-to-one, even though $f|\text{int}(K_j)$ is one-to-one.

We thank the referee for pointing out that Lemma 2 of this paper appears in a more general form in (4) and that Lemma 1 is also true in a more general form in (7).

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