Pacific Journal of Mathematics

FINITE-TO-ONE OPEN MAPPINGS ON CIRCULARLY CHAINABLE CONTINUA

PHILIP RICHARD BARTICK, II AND EDWIN DUDA

Vol. 106, No. 1 November 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol 106, No 1, 1983

FINITE-TO-ONE OPEN MAPPINGS ON CIRCULARLY CHAINABLE CONTINUA

PHILIP BARTICK AND EDWIN DUDA

The authors analyze the behavior of a finite-to-one open mapping on a hereditarily decomposable circularly chainable continuum. It is shown that such a mapping behaves similarly to an open mapping on a simple closed curve.

0. Introduction. In a previous paper (2), the authors completely described the behavior of a finite-to-one open mapping on a hereditarily decomposable chainable continuum. In this note we will completely describe the behavior of a finite-to-one open mapping on a hereditarily decomposable circularly chainable continuum. This result is a generalization of the following theorem of G. T. Whyburn (**8**): If X is a simple closed curve and $f(X) = Y$ is a non-constant open mapping onto a Hausdorff space, then Y is either a simple closed curve or an arc. If Y is a simple closed curve, then there is an integer n such that f is topologically equivalent to the mapping $w = z^n$ on the unit circle in the complex plane. If Y is a simple arc, then there exists an even integer k such that f is topologically equivalent to the mapping $f(1, x) = \sin(kx/2)$ for $0 \leq x \leq 2\pi$ from the unit circle $r = 1$ in the plane to the interval $[-1, 1]$. We use the fundamental results of E. S. Thomas (**6**) implicitly.

1. Notation and definitions. The word mapping is used to denote a continuous function and a metric continuum X is decomposable if it can be expressed as the union of two proper subcontinua. A continuum X is hereditarily decomposable if every non-degenerate subcontinuum is decomposable. The interior of a subset D is denoted by $\mathrm{int}(D)$ and the closure of D is represented by $\mathrm{cl}(D)$. Two mappings $f(X) = Y$ and $g(Z) = W$ are topologically equivalent if there exist homeomorphisms $h(X) = Z$ and $k(W) = Y$ such that $kgh(x) = f(x)$ for all x in X. A mapping $f(X) = Y$ is irreducible if no proper subcontinuum of X maps onto Y under f. Suppose X is a continuum which admits a monotone mapping t onto the unit interval such that no point inverse has interior points. A point inverse $t^{-1}(r)$, $r \neq 0$, is called an element of subcontinuity from the left in the upper-semicontinuous decomposition of X induced by t if $t^{-1}(r) \subset \mathrm{cl}(t^{-1}[0, r))$. If $r \neq 1$ and $t^{-1}(r) \subset \mathrm{cl}(t^{-1}(r, 1])$, then $t^{-1}(r)$ is an element of subcontinuity from the right. The collection of elements of subcontinuity from the left are known to be a dense set in X (**6**) and the same is true of the elements of subcontinuity from the right.

2. Preliminary results.

LEMMA 1. *Let X be a hereditarily decomposable circularly chainable continuum. There is a monotone upper-semicontinuous decomposition $G(X)$ defined on X whose quotient space is a simple closed curve and each element of which has empty interior.*

Outline of proof. Let $X = A \cup B$ be a decomposition of X into proper subcontinua. We can assume that $A \cap B$ has no interior points. The continua A and B are chainable and hereditarily decomposable, so that by R. H. Bing's characterization (**1**) they admit monotone upper-semicontinuous decompositions $G(A)$ and $G(B)$ respectively whose quotient spaces are simple arcs, say $[a, b]$ and $[c, d]$ respectively. Furthermore, no element of $G(A)$ or $G(B)$ has interior points relative to X. We can assume that the elements corresponding to a and c meet, as do the elements corresponding to b and d. The monotone decomposition $G(X)$ is obtained by taking the union of the elements corresponding to a and c as one element, the union of the elements corresponding to b and d as another element, and the rest of the elements of $G(A)$ and $G(B)$ for the remainder of $G(X)$. This decomposition is upper-semicontinuous, its quotient space is a simple closed curve, and no element of it has interior points.

The following theorem established in (**2**) is stated in complete detail even though only the first portion is used in this paper.

THEOREM A. *Let $f(X) = Y$ be a finite-to-one open mapping, where X is a non-degenerate hereditarily decomposable chainable continuum and Y is a Hausdorff space. Then $X = \bigcup_{j=1}^{n} K_j$, where each K_j is a continuum, $f(K_j) = Y$ for each j, f is a homeomorphism on any continuum which is interior to K_j relative to X, and if $K_i \cap K_j$ is non-empty for $i \neq j$, then the intersection is contained in a single element K of $G(X)$. If K is an element of subcontinuity from one side, then it is an element of subcontinuity from both sides. If not, then K is the union of two homeomorphic subcontinua which meet in a single element of $G(K)$, f is one-to-one on each of them, and they have the same image. If f is irreducible (i.e., there is only one K_j), then f is a homeomorphism.*

LEMMA 2. *If $f(X) = Y$ is a finite-to-one open mapping, where X is hereditarily decomposable and circularly chainable, and Y is Hausdorff, then Y is hereditarily decomposable and either circularly or linearly chainable.*

Proof. By well-known theorems, Y is a metric continuum. The space Y is hereditarily decomposable, for if Y contained a non-degenerate indecomposable continuum L, then by an application of the Brouwer Reduction Theorem there would exist an indecomposable continuum in X which maps onto L. The space Y is circularly or linearly chainable by a theorem of E. Duda and J. Kell (**3**) or by applying a result of W. T. Ingram (**5**).

3. Main result.

THEOREM 1. *Let $f(X) = Y$ be a finite-to-one open mapping, where X is hereditarily decomposable and circularly chainable and Y is a Hausdorff space. Then Y is hereditarily decomposable and either linearly or circularly chainable. If Y is circularly chainable, then f is exactly n-to-one for some n and a local homeomorphism, and $X = \bigcup_{j=1}^{n} K_j$, where each K_j is a continuum, $f(K_j) = Y$ for each j, and if $K \subset \operatorname{int}(K_j)$ is a continuum, then f is one-to-one on K. If Y is linearly chainable, then there is an even integer k such that $X = \bigcup_{j=1}^{k} K_j$, where each K_j is a continuum, $f(K_j) = Y$ for each j, and if $K \subset \operatorname{int}(K_j)$ is a continuum, then f is one-to-one on K.*

Proof. By Lemma 2, Y is a hereditarily decomposable linearly or circularly chainable continuum. By Lemma 1, there is an upper-semicontinuous decomposition $G(X)$ generating a monotone mapping g_1 onto a unit circle S_1, such that each point-inverse of g_1 is an element of $G(X)$. If Y is circularly chainable, there is a similar upper-semicontinuous decomposition $G(Y)$ generating a monotone map g_2 onto a unit circle S_2. If Y is linearly chainable, then by Bing's theorem (**1**), there is a monotone decomposition $G(Y)$ generating a monotone map g_3 of Y onto the interval $I = [-1, 1]$. Each point-inverse of g_1, g_2, or g_3, or equivalently, each element of $G(X)$ or $G(Y)$, has empty interior.

It was proved in (**2**) that if $T \in G(X)$, then $f(T) \in G(Y)$, and if $L \in G(Y)$, then $f^{-1}(L) = \bigcup_{i=1}^{m} L_i$, where $L_i \cap L_j = \varnothing$ if $i \neq j$ and each $L_i \in G(X)$.

If Y is circularly chainable, define a mapping f_1 of S_1 onto S_2 by $f_1(z) = g_2 f g_1^{-1}(z)$. The mapping f_1 is well-defined, since f preserves decomposition elements, and f_1 is continuous, finite-to-one and open, since f has these properties. By the quoted theorem of Whyburn, f_1 is a local homeomorphism which is topologically equivalent to z^n for some integer n. If I_1 is an interval in S_1 of length less than $2\pi/n$, then $f_1 \mid I_1$ is a homeomorphism of I_1 onto $f_1(I_1)$. The set $g_1^{-1}(I_1)$ is a continuum, and it is a component of $f^{-1}f(g_1^{-1}(I_1))$, so $f \mid g_1^{-1}(I_1)$ is a finite-to-one open mapping. Furthermore, $f \mid g_1^{-1}(I_1)$ is irreducible, and hence by Theorem A, it is a homeomorphism. We now know that f is a local homeomorphism and exactly n-to-one.

Let S_1 be divided into n non-overlapping arcs $I_j, j = 1,\ldots,n$, each of length $2\pi/n$, and let $K_j = g_1^{-1}(I_j)$ for each j. Then X is the union of the $K_j, f(K_j) = Y$ for each j, and if $K \subset \operatorname{int}(K_j)$ then $f|K$ is one-to-one.

Suppose Y is linearly chainable. Define the map f_2 from S_1 to I by $f_2(z) = g_3 f g_1^{-1}(z)$. As before, f_2 is a continuous finite-to-one open mapping of S_1 onto I, and by Whyburn's theorem, f_2 is topologically equivalent to the mapping $f(1, x) = \sin(kx/2)$ for x between 0 and 2π and k an even integer. In this case, $S_1 = \bigcup_{j=1}^{k} I_j$ and $f_2 | I_j$ is a homeomorphism of I_j onto I. Let $K_j = g_1^{-1}(I_j)$ for each j. Then $f(K_j) = Y$ for each j, and if L is any continuum in the interior of K_j, then $f|L$ is a finite-to-one open mapping of L onto $f(L)$. Furthermore, it is irreducible, so that $f|L$ is a homeomorphism by Theorem A.

The following example illustrates that when Y is linearly chainable, $f|K_j$ is not necessarily a homeomorphism. Let X be the closure of the graph of $y = \sin(1/x)$, $x \neq 0$, $x \in [-1, 1]$. Let the endpoints of X be a and b. In the unique minimal monotone decomposition $G(X)$ given by Bing's theorem, the only non-degenerate element is the interval from $(0, -1)$ to $(0, 1)$ on the y-axis. Let q be the quotient map obtained by identifying each point (x, y) in X with $(-x, -y)$. The mapping q is open and exactly 2-to-one except at the origin. Let Y be a disjoint copy of X with endpoints c and d corresponding to a and b, respectively. Let Z be the circularly chainable continuum obtained by identifying c with a and b with d. Let p be the mapping of Z onto X obtained by folding Z. The mapping $f = q \circ p$ is a finite-to-one open mapping of Z onto $q(X)$. The continuum $q(X)$ is chainable. There are four continua K_1, K_2, K_3, K_4 given by Theorem 1 such that $Z = \bigcup_{j=1}^{4} K_j$ and $f(K_j) = q(X)$ but $f|K_j$ is not one-to-one, even though $f|\operatorname{int}(K_j)$ is one-to-one.

We thank the referee for pointing out that Lemma 2 of this paper appears in a more general form in (**4**) and that Lemma 1 is also true in a more general form in (**7**).

References

1. R. H. Bing, *Snakelike continua*, Duke Math. J., **18** (1951), 653–663.
2. E. Duda and P. Bartick, *Finite-to-one Open Mappings on Chainable Continua*, Proceedings of Warsaw Topology Conference 1978.
3. E. Duda and J. Kell, *Monotone and Open Mappings on Circularly Chainable Continua*, Proceedings of Warsaw Topology Conference 1978.
4. E. E. Grace and E. J. Vought, *Semi-confluent and weakly confluent images of tree-like and atriodic continua*, Fund. Math., **101** (1978), 151–158.
5. W. T. Ingram, *Decomposable circle-like continua*, Fund. Math., **63** (1968), 193–198.
6. E. S. Thomas, *Monotone decompositions of irreducible continua*, Rozprawy Matematycne, **50** (1966), 1–73.

7. E. J. Vought, *Monotone decompositions of continua not separated by any subcontinua*, TAMS, **192** (1974), 67–78.
8. G. T. Whyburn, *Interior transformations on compact sets*, Duke Math. J., **3** (1937), 370–381.

Received September 29, 1980 and in revised form December 29, 1981.

UNIVERSITY OF MIAMI
CORAL GABLES, FL 33124

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of California
Los Angeles, CA 90024

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR OGUS
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113

R. FINN and H. SAMELSON
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS　E. F. BECKENBACH (1906–1982)　B. H. NEUMANN　F. WOLF　K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

Pacific Journal of Mathematics

Vol. 106, No. 1 November, 1983