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Sterling K. Berberian

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S. K. BERBERIAN

Every semifinite von Neumann algebra A possesses an expectation $\natural: A \to W$, where W is a commutative von Neumann subalgebra of A containing the center of A, and where $\natural$ extends the trace of a "large" finite subalgebra of A. An AW^*-algebraic proof yields applications to the embedding of semifinite AW^*-algebras in algebras of type I.

1. Uniform algebras. An algebra of type I may be studied by decomposing it into homogeneous algebras. In an analogous way, we propose to study semifinite algebras via their decompositions into uniform algebras.

DEFINITION [**2**, p. 242, Exer. 5]. An AW^*-algebra is said to be *uniform* if it contains an orthogonal family of equivalent finite projections with supremum 1. (The definition of homogeneous algebra is obtained by replacing "finite" by "abelian".)

LEMMA 1. *Every semifinite AW^*-algebra is the C^*-sum of a family of uniform algebras.*

Proof. Since finite algebras are trivially uniform, one can suppose the given algebra A to be properly infinite. Let $(e_i)_{i\in I}$ be a maximal orthogonal family of pairwise equivalent finite projections; since A is infinite, one can suppose the index set I to be infinite. Then there exist a nonzero central projection h of A and an orthogonal family of projections $(f_i)_{i\in I}$ such that $h = \sup f_i$ and $f_i \sim he_i$ for all $i \in I$ [**1**, p. 102, Prop. 2]. This shows that the algebra hA is uniform, and an exhaustion by Zorn's lemma completes the proof. □

2. Matrix units. A uniform von Neumann algebra A may be regarded as a tensor product $A = D \otimes L(H)$ with D finite and $L(H)$ the algebra of all bounded operators on a Hilbert space H [**2**, p. 25, Prop. 5]. There is no analogous theory of tensor product for AW^*-algebras, but an effective substitute is to pursue the discussion of "matrix units" in [**4**, §5].

Let A be an AW^*-algebra, with center Z, containing an orthogonal family $(e_i)_{i\in I}$ of pairwise equivalent projections with $\sup e_i = 1$. As in [**4**, §5] construct a family of elements $e_{ij} \in e_iAe_j$ $(i, j \in I)$ such that $e_{ii} = e_i$, $e_{ij}^* = e_{ji}$, $e_{ij}e_{jk} = e_{ik}$ and $e_{ij}e_{mk} = 0$ for $j \neq m$. In particular, $e_{ij}e_{ij}^* = e_i$

and $e_{ij}^*e_{ij} = e_j$, thus e_{ij} is a partial isometry effecting the equivalence $e_i \sim e_j$. Let

$$S = \{e_{ij} : i, j \in I\}, \qquad T = \{e_i : i \in I\}$$

and let

$$D = S', \qquad W = T''$$

be the commutant and bicommutant, respectively, of these sets in A; D and W are AW^*-subalgebras of A with $D = D''$, $W = W''$ [**1**, p. 23, Prop. 8]. Since T is a commutative set, W is a commutative algebra; from $W \subset W'$ we see that W' has center W, thus the e_i are orthogonal central projections in W' with supremum 1, consequently $W' = \bigoplus e_i W'$ [**1**, p. 53, Prop. 2]. If $x_i \in e_i W'$ for all $i \in I$ and $\sup \|x_i\| < \infty$, we write $\bigoplus x_i$ for the unique element $x \in W'$ such that $e_i x = x_i$ for all i. Since $T \subset S$, one has

$$D = S' \subset T' = T''' = W',$$

thus $Z \subset W \subset D'$. The center of D is $D \cap D' = Z$ [**4**, Lemma 14].

For each $i \in I$, the mapping $d \mapsto de_i$ is a $*$-isomorphism $D \to e_i A e_i$ [**4**, Lemma 12], consequently $\|de_i\| = \|d\|$ for all $d \in D$ and $i \in I$ [**3**, 1.3.8 and 1.8.1]. Moreover [**4**, Lemma 13],

$$e_i A e_j = D e_{ij} \qquad (i, j \in I);$$

the mapping $d \mapsto de_{ij}$ is an isomorphism of Banach spaces $D \to e_i A e_j$, since

$$\|de_{ij}\|^2 = \|(de_{ij})(de_{ij})^*\| = \|dd^*e_i\| = \|dd^*\| = \|d\|^2.$$

In particular, for each element $a \in A$ there exists a unique family (a_{ij}) of elements of D determined by the relations

$$e_i a e_j = a_{ij} e_{ij} \qquad (i, j \in I); \tag{1}$$

one calls (a_{ij}) the "matrix" of a relative to the matrix units e_{ij}. One has

$$\|a_{ij}\| \le \|a\| \qquad (i, j \in I) \tag{2}$$

because $\|a_{ij}\| = \|a_{ij}e_{ij}\| = \|e_i a e_j\|$.

From $D \subset W'$ we see that $e_i D \subset e_i W' = e_i W' e_i \subset e_i A e_i = e_i D$, thus $e_i W' = e_i D$; therefore $W' = \bigoplus e_i D = \bigoplus e_i A e_i$.

LEMMA 2. *With the preceding notations,*

(3) $$D' = \{a \in A : e_i a e_j \in Z e_{ij} \text{ for all } i, j\},$$

(4) $$W' = \bigoplus e_i W' = \bigoplus e_i D = \bigoplus e_i A e_i$$
$$= \{a \in A : e_i a e_j = 0 \text{ whenever } i \neq j\},$$

(5) $$W = \bigoplus e_i Z,$$

(6) $$W = D' \cap W',$$

(7) $$Z = D \cap W.$$

The algebra D' is homogeneous, with center Z.

Proof. Let $a \in A$ and write $e_i a e_j = a_{ij} e_{ij}$ as in (1).

(3) If $d \in D = S'$ then d commutes with every e_{ij}, therefore
$$e_i(ad - da)e_j = (a_{ij}d - da_{ij})e_{ij}.$$
This expression is 0 if and only if $a_{ij}d - da_{ij} = 0$; thus $a \in D'$ if and only if $a_{ij} \in D \cap D' = Z$ for all i, j.

(4) The formulas $W' = \bigoplus e_i W' = \bigoplus e_i D = \bigoplus e_i A e_i$ are noted above. For all i, j, k one has
$$e_i(ae_k - e_k a)e_j = \delta_{jk} a_{ik} e_{ik} - \delta_{ik} a_{kj} e_{kj}$$
$$= \delta_{jk} a_{ij} e_{ij} - \delta_{ik} a_{ij} e_{ij} = (\delta_{jk} - \delta_{ik}) a_{ij} e_{ij},$$
which is 0 whenever $i = j$. One has $a \in W' = T'$ if and only if this expression is 0 for all i, j, k. If $a \in W'$ and $i \neq j$ then $a_{ij} = 0$ (take $k = j$); on the other hand if $a_{ij} = 0$ whenever $i \neq j$, then the expression is 0 for all i, j, k, so $a \in W'$. Thus $W' = \{a \in A : e_i a e_j = 0 \text{ for } i \neq j\}$.

(5), (6) From (3) we have $e_i Z = e_i D' e_i$; since $e_i \in W \subset D'$, this shows that $e_i Z$ is an AW^*-algebra, and $Z \subset W$ yields $\bigoplus e_i Z \subset \bigoplus e_i W = W$. Obviously $W \subset D' \cap W'$. If $a \in D' \cap W'$ then $a = \bigoplus e_i a$ by (4), and $e_i a = e_i a e_i = e_i a_{ii}$ with $a_{ii} \in Z$ by (3), thus $a \in \bigoplus e_i Z$. Summarizing, we have $\bigoplus e_i Z \subset W \subset D' \cap W' \subset \bigoplus e_i Z$, whence equality throughout.

(7) Citing (6), $D \cap W = D \cap D' \cap W' = Z \cap W' = Z$.

Finally, $e_{ij} \in S \subset S'' = D'$ for all i, j; this shows that the projections e_i are equivalent in D'. By (3), $e_i D' e_i = Z e_i$ is commutative, so the e_i are abelian projections in D'. Thus D' is homogeneous, with center $D' \cap D'' = D' \cap D = Z$. □

3. Semifinite algebras. The foregoing results on matrix units yield a structure theorem for semifinite algebras; we first review some definitions needed for its statement.

Let A be an AW^*-algebra, A_p its projection lattice, A_h the ordered linear space of hermitian elements of A with the set of elements x^*x as positive cone; A is said to be *normal* [**15**] if A_p is monotonely embedded in A_h, that is, whenever (f_α) is an increasingly directed family of projections with supremum f in A_p, then f is also the supremum of the family in A_h (briefly, $f_\alpha \uparrow f$ in A_p implies $f_\alpha \uparrow f$ in A_h). Every finite AW^*-algebra is normal [**15**, Th. 4], as is every AW^*-algebra that acts faithfully on a separable Hilbert space [**16**, Cor. 3.4]. (It is not known if there exists a non-normal AW^*-algebra.) Every von Neumann algebra is normal, hence so is every W^*-algebra. A positive linear mapping $\varphi\colon A \to B$ between AW^*-algebras is said to be *normal* if $a_\alpha \uparrow a$ in A_h implies $\varphi(a_\alpha) \uparrow \varphi(a)$ in B_h, and *completely additive on projections* (CAP) if $f_\alpha \uparrow f$ in A_p implies $\varphi(f_\alpha) \uparrow \varphi(f)$ in B_h. If A is a normal algebra and φ is a normal mapping, then φ is CAP.

LEMMA 3 [**10**]. *If A is a normal AW^*-algebra, then for every element $x \in A$ the positive linear mapping $a \mapsto xax^*$ on A is* CAP.

Proof. Suppose $f_\alpha \uparrow f$ in A_p and $xf_\alpha x^* \le b \in A_h$ for all α; we are to show that $xfx^* \le b$. Let $\varepsilon > 0$ and let $c = (b + \varepsilon)^{-1/2}$. Then

$$cxf_\alpha x^*c \le cbc = b(b+\varepsilon)^{-1} \le 1,$$

thus $(cxf_\alpha)(cxf_\alpha)^* \le 1$; this means that $\|cxf_\alpha\| \le 1$, so $(cxf_\alpha)^*(cxf_\alpha) \le 1$, whence $f_\alpha(1 - x^*c^2x)f_\alpha \ge 0$ for all α. It follows from normality that $f(1 - x^*c^2x)f \ge 0$ [**10**, Lemma 3], whence $fx^*c^2xf \le f \le 1$, $\|cxf\| \le 1$, $cxfx^*c \le 1$, $xfx^* \le c^{-2} = b + \varepsilon$. Thus $xfx^* - b \le \varepsilon$ for all $\varepsilon > 0$, therefore $xfx^* - b \le 0$. □

THEOREM 1. *Let A be a semifinite AW^*-algebra with center Z. There exist AW^*-subalgebras D and W of A with the following properties*:

(i) *$D = D''$ and $W = W''$ in A*;

(ii) *D is finite, its center is Z, and D' is of type* I *with center Z; D is $*$-isomorphic to eAe, with e a faithful finite projection of A*;

(iii) *W is commutative, $W = D' \cap W'$ and $Z = D \cap W$*;

(iv) *there is a mapping $\sharp\colon A \to W'$ that is left and right W'-linear, positive, faithful, and leaves fixed the elements of W'; when A is a normal algebra, the mapping $\sharp$ is* CAP.

(v) *If Z is a W^*-algebra then so are D' and W; if D is a W^*-algebra, then so is W'.*

(vi) *If A is normal and D is a W^*-algebra, then A is a W^*-algebra.*

Proof. By Lemma 1 we are reduced to the case that A is uniform; we adopt the notations of Lemma 2, with the e_i finite projections of A. In particular, D is $*$-isomorphic to e_iAe_i, hence is finite; the rest of (i)–(iii) is clear from Lemma 2.

(v) The formula $W = D' \cap W'$ means that W coincides with its commutant in D' (thus is a maximal abelian subalgebra of D'); if Z is a W^*-algebra (that is, $*$-isomorphic to a von Neumann algebra) then so is the type I algebra D' with center Z [**4**, Th. 2], hence so is W. On the other hand, if D is a W^*-algebra, then so are the isomorphic algebras e_iD, hence so is W' by formula (4) of Lemma 2; in this case, the center Z of D is also a W^*-algebra, hence so are D' and W.

(iv), (vi) If $a \in A$ then $\|e_iae_i\| \leq \|a\|$ for all i, so by (4) of Lemma 2 we can define $a^\sharp = \bigoplus e_iae_i \in W'$. It is clear that $a \mapsto a^\sharp$ is a positive linear mapping $A \to W'$, leaving fixed the elements of W' hence having range W'. If $a \geq 0$ and $a^\sharp = 0$, then $(e_ia^{1/2})(e_ia^{1/2})^* = e_iae_i = 0$ for all i, whence $a = 0$; thus $\sharp$ is faithful.

If $c \in W' = T'$ and $a \in A$, then c commutes with every e_i, thus $e_icae_i = (e_ice_i)(e_iae_i)$ for all i; therefore $(ca)^\sharp = c^\sharp a^\sharp = ca^\sharp$, similarly $(ac)^\sharp = a^\sharp c$.

Finally, suppose A is a normal algebra and $f_\alpha \uparrow f$ in A_p. By Lemma 3, for each i one has $e_if_\alpha e_i \uparrow e_ife_i$ in A_h, hence in $(e_iAe_i)_h$; therefore $\bigoplus e_if_\alpha e_i \uparrow \bigoplus e_ife_i$ in $(\bigoplus e_iAe_i)_h$, that is, $f_\alpha^\sharp \uparrow f^\sharp$ in $(W')_h$. Thus $\sharp$ is CAP. If, in addition, D is a W^*-algebra, then by (v) so is W', therefore W' has a separating family of normal positive linear forms; since $\sharp$ is CAP, it follows that A has a separating family of positive linear forms that are CAP, therefore A is a W^*-algebra by a theorem of G. K. Pedersen [**7**]. □

4. Trace and expectations. Our next objective is to show that, in the notations of Theorem 1, a center-valued trace $\natural: D \to Z$ on the finite algebra D is extendible to a trace-like mapping $\natural: A \to W$ (more precisely, in the terminology of [**6**], an expectation of A onto W). If, in addition, the algebra A is normal, then the resulting expectation of A is a normal mapping. All of these hypotheses are fulfilled when A is a semifinite W^*-algebra. First, we review a result implicit in [**12**]:

LEMMA 4. *Let A be a finite AW^*-algebra with center Z, possessing a trace $\natural: A \to Z$. Then A is monotone complete and the mapping $\natural$ is normal.*

Proof. The hypothesis is that $\natural$ is a positive Z-linear mapping such that $1^\natural = 1$ and $(ab)^\natural = (ba)^\natural$ for all a, b in A. It follows that $z^\natural = z$ for all $z \in Z$. Moreover, $\natural$ is faithful: if $a \geq 0$ and $a^\natural = 0$ then $a = 0$ (because every nonzero positive element of A majorizes a positive scalar multiple of a simple projection [**1**, §26]).

Let D: $A_p \to Z$ be the dimension function A [**1**, p. 181, Th. 1]. By the uniqueness of D, $e^\natural = D(e)$ for all projections e; since D is completely additive, $\natural$ is CAP [**1**, p. 184, Exer. 4]. It follows that for every $x \in A$, the Z-linear mapping $a \mapsto (xax^*)^\natural$ is also CAP (cf. the Appendix), thus $\natural$ is continuous in the sense of [**12**, p. 316]. Since $\natural$ is faithful, it follows that there exists an AW^*-algebra B of type I, with center Z, such that A is an AW^*-subalgebra of B [**12**, Th. 3.1], indeed $A = A''$ in B [**12**, Th. 4.4]. Since B is monotone complete [**12**, Lemma 1.4] and $A = A''$ in B, it follows that A is monotone complete. (An AW^*-algebra A is said to be *monotone complete* if every increasingly directed family in A_h, majorized by an element of A_h, has a supremum in A_h.)

Suppose $a_\alpha \uparrow a$ in A_h; we are to show that $a_\alpha^\natural \uparrow a^\natural$ in Z_h. Passing to a cofinal set of indices, we can suppose that $\|a_\alpha\|$ is bounded. Viewing B as the algebra of bounded operators on an AW^*-module over Z [**5**, Th. 8], a_α is strongly convergent to a [**12**, Lemma 1.4], therefore $a^\natural = \liminf a_\alpha^\natural$ in Z_h [**12**, Lemma 4.3]; since the family $(a_\alpha^\natural)$ is increasing, $\liminf a_\alpha^\natural = \sup a_\alpha^\natural$, thus $a_\alpha^\natural \uparrow a^\natural$ in Z_h. □

In Theorem 2 it will be assumed that the finite algebra D of Theorem 1 has a trace, equivalently, that the isomorphic algebra eAe has a trace; the next two lemmas free this hypothesis from its reference to a particular faithful finite projection e.

LEMMA 5. *If the finite AW^*-algebra A has a trace, then so does every corner eAe of A and every matrix algebra $M_n(A)$ over A.*

Proof. If $\natural$: $A \to Z$ is the trace of A (Z the center of A) and if r is the relative inverse of $e^\natural$ in the regular ring of A [**1**, p. 235], then the trace $eAe \to eZ$ of eAe is given by the formula $x \mapsto erx^\natural$. Identifying the center of $M_n(A)$ with Z, the trace of a matrix is defined to be the average of the traces of its diagonal elements. □

LEMMA 6. *Let A be a semifinite AW^*-algebra containing a faithful finite projection f such that fAf has a trace. Then for every finite projection e of A, eAe has a trace.*

Proof. The first step of the proof is to find a nonzero central projection h of A such that $(he)A(he) = heAe$ has a trace. We can suppose $e \neq 0$; then $eAf \neq 0$ (because f is faithful), so there exist nonzero subprojections $e_1 \leq e, f_1 \leq f$ with $e_1 \sim f_1$. Passing to a subprojection of e_1, we can suppose that e_1 is a simple projection in eAe [**1**, §26]. The central cover of e_1 in eAe has the form he with h a central projection of A [**1**, p. 37, Prop. 4], thus $heAe = M_n(e_1Ae_1)$ for a suitable integer n (the "order" of e_1 in eAe). Since fAf has a trace, so does its corner f_1Af_1 (Lemma 5), hence so does the isomorphic algebra e_1Ae_1, hence so does the matrix algebra $heAe$ (Lemma 5).

Let (h_α) be a maximal orthogonal family of nonzero central projections of A such that every $h_\alpha eAe$ has a trace. Necessarily $\sup h_\alpha = 1$ (otherwise the preceding argument could be used to contradict maximality); thus $eAe = \oplus h_\alpha eAe$, $eZ = \oplus h_\alpha eZ$ (Z the center of A), and the traces of the $h_\alpha eAe$ may be combined to give a trace for eAe. □

THEOREM 2. *Let A be a semifinite AW^*-algebra with center Z, and adopt the notations of Theorem 1. Suppose, in addition, that the finite algebra D has a trace $\natural$: $D \to Z$ (as is the case when A is a W^*-algebra). Then the trace of D is extendible to a positive linear mapping $\natural$: $A \to W$ with the following properties*:

(i) $w^\natural = w$ *for all* $w \in W$;

(ii) $(wa)^\natural = wa^\natural = a^\natural w = (aw)^\natural$ *for all* $a \in A$, $w \in W$;

(iii) $a \geq 0$ *and* $a^\natural = 0$ *imply* $a = 0$;

(iv) $(ad)^\natural = (da)^\natural$ *for all* $a \in A$, $d \in D$; *equivalently*, $(uau^*)^\natural = a^\natural$ *for all* $a \in A$ *and all unitary* $u \in D$;

(v) *if A is a normal algebra, then the mapping $\natural$: $A \to W$ is normal and there exists a type* I *AW^*-algebra B with center Z such that $A = A''$ in B.*

Proof. By Lemma 6 and the proof of Theorem 1, we can suppose A to be uniform; we adopt the notations of Lemma 2, with the e_ι finite projections, and we write $\sharp$: $A \to W'$ for the mapping defined in the proof of Theorem 1.

Suppose, more generally, that φ: $D \to Z$ is any positive linear mapping. For each $i \in I$ let φ_ι: $e_iAe_i \to e_iZ$ be the unique (positive, linear) mapping such that $\varphi_\iota(e_id) = e_i\varphi(d)$ (recall that $d \mapsto e_id$ is a $*$-isomorphism $D \to e_iAe_i$); then

$$\|\varphi_i(e_id)\| \leq \|\varphi(d)\| \leq \|\varphi\|\,\|d\| = \|\varphi\|\,\|e_id\|,$$

so $\|\varphi_i\| \leq \|\varphi\|$ for all i. Define a mapping $\bar{\varphi}$: $W' \to W$ as follows. By (4) of Lemma 2, every $x \in W'$ has the form $x = \oplus x_i$ with $x_i \in e_\iota Ae_i$ and $\|x_\iota\|$ bounded; then $\|\varphi_i(x_\iota)\|$ is bounded and we can define

$$\bar{\varphi}(x) = \bigoplus \varphi_\iota(x_i) \in \bigoplus e_iZ = W$$

by (5) of Lemma 2. (So to speak, $\bar{\varphi} = \oplus\varphi_\iota$.)

Composing the positive linear mappings $\sharp\colon A \to W'$ and $\bar{\varphi}\colon W' \to W$, we obtain a positive linear mapping $\Phi\colon A \to W$, where $\Phi(a) = \bigoplus \varphi_i(e_i a e_i)$ for $a \in A$; thus if $e_i a e_j = a_{ij} e_{ij}$ as in (1), we have

$$\Phi(a) = \bigoplus e_i \varphi(a_{ii}). \tag{8}$$

Φ extends φ. {*Proof*: If $a \in D$ then $e_i a e_i = a e_i$ shows that $a_{ii} = a$ for all i, whence $\Phi(a) = \bigoplus e_i \varphi(a) = \varphi(a)$.}

If φ is faithful then so is Φ. {*Proof*: If φ is faithful then so is every φ_i, therefore so is $\bar{\varphi}$; since $\sharp$ is also faithful, so is $\Phi = \bar{\varphi} \circ \sharp$.}

If φ is Z-linear, then each of the mappings $a \mapsto \varphi(a_{ii})$ is Z-linear and Φ is both left and right W-linear. {*Proof*: Clearly every φ_i is $e_i Z$-linear, therefore $\bar{\varphi}$ is both left and right $\bigoplus e_i Z$-linear, that is, W-linear. If $z \in Z$ then za has matrix (za_{ij}), whence the Z-linearity of the mappings $a \mapsto \varphi(a_{ii})$.}

If φ is normal then so is $\bar{\varphi}$; if, moreover, A is a normal algebra, then the mappings Φ and $a \mapsto \varphi(a_{ii})$ on A are CAP. {*Proof*: If φ is normal then so is every φ_i, hence so is $\bar{\varphi} = \bigoplus \varphi_i$. Suppose in addition that A is normal. If $f_\alpha \uparrow f$ in A_p, then $f_\alpha^\sharp \uparrow f^\sharp$ in $(W')_h$ by (iv) of Theorem 1, therefore $\bar{\varphi}(f_\alpha^\sharp) \uparrow \bar{\varphi}(f^\sharp)$ in W_h, that is, $\Phi(f_\alpha) \uparrow \Phi(f)$; thus Φ is CAP. Also, for each i the mapping $a \mapsto e_i a e_i = e_i a_{ii}$ is CAP (Lemma 3); by virtue of the $*$-isomorphism $e_i D \to D$ and the normality of φ, it follows that the mapping $a \mapsto \varphi(a_{ii})$ is also CAP.}

Assume now that there exists a trace $\natural\colon D \to Z$ and let $\natural$ play the role of φ. By the foregoing remarks, the mapping $\natural\colon A \to W$ defined by the formula

$$a^\natural = \bigoplus e_i a_{ii}^\natural$$

is left and right W-linear, positive, faithful, and extends the trace of D; thus the properties (ii), (iii) are verified, hence so is (i) (because $1^\natural = 1$). If $a \in A$ has matrix (a_{ij}) and if $u \in D$ is unitary, then uau^* has matrix $(ua_{ij}u^*)$, therefore

$$(uau^*)^\natural = \bigoplus e_i (ua_{ii}u^*)^\natural = \bigoplus e_i a_{ii}^\natural = a^\natural.$$

This is equivalent to the identity $(ad)^\natural = (da)^\natural$ since every $d \in D$ is a linear combination of unitary elements of D [**2**, p. 4, Prop. 3].

The trace of D is normal (Lemma 4); if, moreover, A is a normal algebra, the above remarks show that the mappings $\natural\colon A \to W$ and $a \mapsto a_{ii}^\natural$ on A are CAP; in particular, A has a family of Z-linear mappings $A \to Z$ that are CAP and separating (for, if $a \geq 0$ and $a_{ii}^\natural = 0$ for all i, then

$a^\natural = 0$, therefore $a = 0$). It then follows from K. Saitô's embedding theorem [**9**, Th. 2] that there exists a type I AW^*-algebra B with center Z, such that $A = A''$ in B. By the arguments in the proof of Lemma 4, A is monotone complete and the above-mentioned Z-linear mappings $A \to Z$ are normal, therefore so is the mapping $\natural: A \to W$. □

The following corollary is due in essence to H. Widom [**11**, Th. 6.3]:

COROLLARY 1. *If A is a normal, semifinite AW^*-algebra containing a faithful finite projection f such that fAf has a trace, then A may be embedded as a bicommutant in a type* I *algebra with the same center.*

Proof. With notation as in Theorem 1, it follows from Lemma 6 that eAe has a trace, hence so does the isomorphic algebra D; thus all of the hypotheses of Theorem 2 are fulfilled. □

{We remark that the result in [**11**, Th. 6.3] is stated without assuming normality, but normality figures in the proof [**11**, p. 55, line 4] via an appeal to the property in Lemma 3 above. The countability hypothesis in [**11**, Th. 6.3] can be omitted by virtue of Saitô's embedding theory [**9**, Th. 1].}

COROLLARY 2. *If, under the hypotheses of Corollary* 1, *the center of A is a W^*-algebra, then A is also a W^*-algebra.*

Proof. The type I algebra given by Corollary 1 is also W^* [**4**, Th. 2], hence so is its subalgebra A. □

It is an open question whether every AW^*-factor of type II_1 has a trace; if the answer is yes, then Corollary 2 would imply that every normal AW^*-factor of type II_∞ is a W^*-algebra.

COROLLARY 3 [**13**, p. 445, Cor.]. *Let A be a normal, semifinite AW^*-algebra whose center Z is a W^*-algebra. If A has a faithful positive linear form then it is a W^*-algebra.*

Proof. With notations as in Theorem 1, the finite algebra D also has center Z and has a faithful positive linear form, hence is a W^*-algebra [**14**, p. 437, Cor. 7]; therefore D has a trace and Corollary 2 applies. □

5. Appendix. The following proposition (stated without proof in [**8**]) is implicit in the proof of Saitô's embedding theorem [**9**, Th. 2]; the brief proof given here was communicated to me by Professor Saitô.

PROPOSITION [**8**, 1.1.2]. *If A is an AW^*-algebra, B is a commutative AW^*-algebra, and $\varphi: A \to B$ is a positive linear mapping that is* CAP, *then for every $x \in A$ the mapping $a \mapsto \varphi(xax^*)$ is also* CAP.

Proof. Assuming $f_\alpha \downarrow 0$ in A_p, it will suffice to show that $\varphi(xf_\alpha x^*) \downarrow 0$ in B_h. This is clear if x is unitary, for then $xf_\alpha x^* \downarrow 0$ in A_p. In general, x is a linear combination of four unitaries, say $x = \sum_{i=1}^4 \lambda_i u_i$. Then

$$\varphi(xf_\alpha x^*) = \sum_{i,j} \lambda_i \bar{\lambda}_j \varphi(u_i f_\alpha u_j^*).$$

Writing $|b| = (b^*b)^{1/2}$ for $b \in B$, the Cauchy-Schwarz inequality [cf. **5**, p. 840] yields

$$\begin{aligned}|\varphi(u_i f_\alpha u_j^*)|^2 &= |\varphi(u_i(u_j f_\alpha)^*)|^2 \\ &\leq \varphi(u_i u_i^*)\varphi(u_j f_\alpha u_j^*) = \varphi(1)\varphi(u_j f_\alpha u_j^*);\end{aligned}$$

writing $M = \max |\lambda_i \lambda_j|$, we thus have

$$\varphi(xf_\alpha x^*) \leq 4M\varphi(1)^{1/2} \sum_{j=1}^4 \varphi(u_j f_\alpha u_j^*)^{1/2},$$

where $\varphi(u_j f_\alpha u_j^*)^{1/2} \downarrow 0$ in B_h for each j, therefore also $\varphi(xf_\alpha x^*) \downarrow 0$. □

We remark that for the CAP mappings occurring in Lemmas 3 and 4 (hence in Theorems 1 and 2), the conclusion of the Proposition can be seen directly: in the case of Lemma 3, one notes that $y(xf_\alpha x^*)y^* = (yx)f_\alpha(yx)^*$; in the case of Lemma 4, $(xf_\alpha x^*)^\natural = (f_\alpha x^* x f_\alpha)^\natural \leq \|x\|^2 f_\alpha^\natural$.

PROBLEMS. 1. Is every semifinite AW^*-algebra normal?

2. In the notations of Lemma 2, does every $*$-automorphism of D extend to a $*$-automorphism of A?

3. If A is an AW^*-algebra containing a faithful projection e such that eAe is a W^*-algebra, does it follow that A is a W^*-algebra? (The answer is yes if A is normal.)

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UNIVERSITY OF TEXAS
AUSTIN, TX 78712

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