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**THE EXISTENCE OF STRONG LIFTINGS FOR TOTALLY  
ORDERED MEASURE SPACES**

A. SAPOUNAKIS

## THE EXISTENCE OF STRONG LIFTINGS FOR TOTALLY ORDERED MEASURE SPACES

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**Let  $X$  be a totally ordered space,  $\mu$  a finite Borel measure on  $X$  with full support, and  $\mathcal{F}$  the  $\sigma$ -algebra of all  $\mu^*$ -measurable subsets of  $X$ . Then there exists a lifting  $\rho: \mathcal{F} \rightarrow \mathcal{F}$  which satisfies  $U \subset \rho(U)$  for every open subset  $U$  of  $X$ .**

Assume that  $X$  is a topological space and  $\mu$  a finite, Borel measure on  $X$  with full support. We are interested in finding conditions for the topology of  $X$ , which insure the existence of strong liftings for the associated topological measure space. In [8] Losert has given an example showing that this is not always possible even if  $X$  is compact. On the other hand Graf has proved that strong liftings always exist for measures on second countable spaces [6]. Other positive results on the existence of strong liftings are given in [2] and [4].

In this paper we show that every totally ordered measure space admits a strong lifting. Moreover we prove that if  $\mu$  is a Radon, non-atomic measure on a totally ordered space then every lifting of the associated measure space is almost strong, if and only if, the set of all two sided limit points of the support of  $\mu$  is  $\mu^*$ -measurable with full measure.

**1. Preliminaries and notation.** Throughout  $X$  will be a set and " $\leq$ " a total order on  $X$ . If  $x, y$  are two points of  $X$ , let  $x < y$  means that  $x \leq y$  and  $x \neq y$ . Let  $(-\infty, x) = \{z \in X: z < x\}$ ,  $(x, +\infty) = \{z \in X: x < z\}$  and  $(x, y) = (-\infty, y) \cap (x, +\infty)$ . Assume that  $Y \subset X$  and  $y \in Y$ . We say that  $y$  is a left limit point of  $Y$  if  $(-\infty, y) \cap Y \neq \emptyset$  and there is no  $z$  in  $Y$  such that  $z < y$  and  $(z, y) \cap Y = \emptyset$ . Analogous is the definition of the right limit point. A point  $y$  in  $Y$  is said to be a two sided limit point of  $Y$  if it is both a left and right limit point of  $Y$ .

We say that  $(X, \leq)$  is a totally ordered (topological) space if its topology is generated by all the intervals of the form  $(-\infty, x)$ ,  $(x, +\infty)$ . By a measure on  $X$  we mean a finite, non-negative, countably additive set function defined on the Borel sets of  $X$ . A measure  $\mu$  on  $X$  is said to be Radon if it is inner approximated by the compact sets of  $X$ . The support  $S_\mu$  of a measure  $\mu$  on  $X$  is defined by

$$S_\mu = \bigcap \{F: F \text{ closed subset of } X \text{ such that } \mu(F) = \mu(X)\}.$$

Clearly  $S_\mu$  is a closed subset of  $X$  and if  $\mu(X) = \mu(S_\mu)$ ,  $S_\mu$  satisfies the countable chain condition.

A totally ordered measure space is the quadruple  $(X, \leq, \mathcal{F}, \mu)$ , where  $(X, \leq)$  is a totally ordered space,  $\mu$  is a measure on  $X$  and  $\mathcal{F}$  is the completion of the Borel  $\sigma$ -algebra of  $X$  w.r.t.  $\mu$ .

A map  $\rho: \mathcal{F} \rightarrow \mathcal{F}$  is called lifting [7] if for all  $A, B \in \mathcal{F}$

- (i)  $\mu(A\Delta\rho(A)) = 0$ ,
- (ii)  $\mu(A\Delta B) = 0 \Rightarrow \rho(A) = \rho(B)$ ,
- (iii)  $\rho(\emptyset) = \emptyset, \rho(X) = X$ ,
- (iv)  $\rho(A \cup B) = \rho(A) \cup \rho(B)$ ,
- (v)  $\rho(A \cap B) = \rho(A) \cap \rho(B)$ .

If  $\rho$  satisfies only the properties (i), (ii), (iii), (iv) (resp. (i), (ii), (iii), (vi)) is called an upper (resp. lower) density.

A lifting  $\rho$  is said to be

- (a) strong, if  $U \subset \rho(U)$  for all open subsets  $U$  of  $X$ ,
- (b) almost strong if there is a set  $N \in \mathcal{F}$  with  $\mu(N) = 0$ , and  $U \subset N \cup \rho(U)$  for all open subsets  $U$  of  $X$ .

**2. The results.** We start with the main result of this paper, the proof of which is divided in three steps. The proof of the first step is based on an idea of [[7], Example 3, page 122], and on the arguments used in the proof of Theorem 3.2 in [10].

**THEOREM 2.1.** *Let  $(X, \leq, \mathcal{F}, \mu)$  be a totally ordered measure space such that  $X = S_\mu$ . Then there exists a strong lifting  $\rho: \mathcal{F} \rightarrow \mathcal{F}$ .*

*Proof.* Without loss of generality we may assume that  $\mu(X) = 1$ .

*Step I.* We prove the theorem in the case when  $S$  is compact and  $\mu$  a Radon, non-atomic measure.

Let  $f$  be the distribution function of  $\mu$  defined by  $f(x) = \mu((-\infty, x))$  for every  $x \in X$ . Then  $f$  is an ordered preserving, continuous function from  $X$  onto the unit interval  $[0, 1]$ , such that  $f^{-1}f(B) - B$  is at most countable for every Borel subset  $B$  of  $X$  (see proof of Theorem 3.2 in [10]). Moreover if  $\lambda$  denotes the Lebesgue measure on  $[0, 1]$  we can easily verify that  $\lambda(B) = \mu(f^{-1}(B))$  for every Borel subset  $B$  of  $[0, 1]$ .

We show that for every  $F \in \mathcal{F}$

$$(*) \quad f(F) \text{ is Lebesgue measurable and } \lambda(f(F)) = \mu(F).$$

Indeed since  $X$  is hereditary Lindelöf [[3], Theorem 2.2], every Borel subset  $F$  of  $X$  is  $\kappa$ -analytic and so  $f(F)$  is an absolutely measurable subset of  $[0, 1]$ . Thus  $f(F)$  is Lebesgue measurable and since  $f^{-1}f(F) - F$  is at most countable we deduce that  $\lambda(f(F)) = \mu(F)$ . This shows that (\*) is true for Borel subsets of  $X$  and can easily be extended for every  $F$  in  $\mathcal{F}$ .

Now let  $L_1$  (resp.  $L_2$ ) be the set of all left (resp. right) limit points of  $X$ . Clearly since  $X$  has no isolated points we have that  $X = L_1 \cup L_2$ . We consider each  $L_i$  ( $i = 1, 2$ ) with the order topology, given by the restriction of the ordering on  $L_i$ . Let  $f_i$  be the restriction of  $f$  on  $L_i$ . Then since  $f_1$  (resp.  $f_2$ ) is an ordered preserving, one-to-one map from  $L_1$  (resp.  $L_2$ ) on  $(0, 1]$  (resp.  $[0, 1)$ ), we may easily conclude that  $f_1$  (resp.  $f_2$ ) is an homeomorphism.

Let  $\lambda_i$ , be the measure on  $L_i$  ( $i = 1, 2$ ) defined by  $\lambda_i(E) = \lambda(f_i(E))$  for every Borel  $E \subset L_i$ , and  $\mathfrak{F}_i$  the completion of the Borel  $\sigma$ -algebra of  $L_i$  w.r.t.  $\lambda_i$ . Then using (\*) we can easily deduce that

(1) If  $F \in \mathfrak{F}$  then  $F \cap L_i \in \mathfrak{F}_i$  for  $i = 1, 2$ .

(2) For every  $A \subset X$ ,  $\mu^*(A) = 0$  if and only if  $\lambda_i^*(A \cap L_i) = 0$  for every  $i = 1, 2$ .

(3)  $\lambda_1^*(A) = \lambda_2^*(A)$  for every  $A \subset L_1 \cap L_2$ .

Now by [[7], Theorem 6, page 123] there are strong liftings  $\rho_1, \rho_2$  of  $\mathfrak{F}_1, \mathfrak{F}_2$  respectively, such that

$$A \subset \rho_1(A) \text{ for every left-open right-closed interval } A \text{ of } L_1$$

and

$$B \subset \rho_2(B) \text{ for every left-closed right-open interval } B \text{ of } L_2.$$

We define  $\theta$  on  $\mathfrak{F}$  by

$$\theta(F) = \rho_1(F \cap L_1) \cup \rho_2(F \cap L_2).$$

Then using (1), (2) and (3) we may easily check that  $\theta$  is an upper density of  $\mathfrak{F}$ . Let  $\theta'$  be the associated to  $\theta$  lower density of  $\mathfrak{F}$  defined by  $\theta'(F) = X - \theta(X - F)$ . Then

$$\theta'(F) = (\rho_1(F \cap L_1) \cup X - L_1) \cap (\rho_2(F \cap L_2) \cup X - L_2).$$

We show that  $U \subset \theta'(U)$ , where  $U$  is either of the form  $(-\infty, x)$  or  $(x, +\infty)$ . Indeed if  $U = (-\infty, x)$ , then  $U \cap L_1$  is either an open interval of  $L_1$  (if  $x \in L_1$ ) or a left-open right-closed interval of  $L_1$  (if  $x \in L_2$ ). It follows that  $U \cap L_1 \subset \rho_1(U \cap L_1)$ . On the other hand  $U \cap L_2$  is always a left-closed right-open interval of  $L_2$  and so  $U \cap L_2 \subset \rho_2(U \cap L_2)$ . Thus  $U \subset \theta'(U)$ . In the same way we show that  $(x, +\infty) \subset \theta'((x, +\infty))$ . By [[7], corollary page 58] there exists a lifting  $\rho$  of  $\mathfrak{F}$  such that  $\theta'(F) \subset \rho(F) \subset \theta(F)$  for every  $F \in \mathfrak{F}$ . It follows that  $\rho$  is a strong lifting of  $\mathfrak{F}$ .

*Step II.* Here we assume that  $X$  is compact and  $\mu$  Radon.

Set  $Y = \{x \in X: \mu(\{x\}) > 0\}$ . Clearly  $Y$  is at most countable. Let  $\nu$  be the Radon, non-atomic measure on  $X$  defined by  $\nu(B) = \mu(B \cap X - Y)$  for every Borel  $B \subset X$ . Since the support  $S_\nu$  of  $\nu$  is a compact totally order space, under the subspace topology, by step I there

exists a strong lifting for the restriction of  $\nu$  on  $S_\nu$ . Let  $Z = S_\nu \cap (X - Y)$ . Clearly  $\nu(Z) = \nu(S_\nu)$  and since  $\mu, \nu$  coincide on the Borel subsets of  $Z$ , there exists a strong lifting  $\rho'$  for the restriction of  $\mu$  on  $Z$ .

Let  $T = Z \cup Y$ . We define  $\rho$  on  $\mathcal{F}_T$ , the class of all  $\mu^*$ -measurable subsets of  $T$ , by

$$\rho(E) = \rho'(E \cap Z) \cup (E \cap Y).$$

It follows that  $\rho$  is a strong lifting for the restriction of  $\mu$  on  $T$  and since  $\mu(T) = \mu(X)$  using a standard argument ([7], Remark 1, page 127) we can find a strong lifting of  $\mathcal{F}$ .

*Step III. The general case.*

Let  $X^*$  be the totally ordered compactification of  $X$  (see [5], §6). We define a measure  $\bar{\mu}$  on  $X^*$  by  $\bar{\mu}(B) = \mu(B \cap X)$ , for every Borel  $B \subset X^*$ . We note that  $\bar{\mu}$  has full support in  $X^*$  and thus  $X^*$  is hereditary Lindelöf [3]. It follows easily that  $\bar{\mu}$  is a Radon measure on  $X^*$ .

Let  $\bar{\mathcal{F}}$  be the  $\sigma$ -algebra of all  $\bar{\mu}^*$ -measurable subsets of  $X^*$ . Then we have  $F \in \bar{\mathcal{F}}$  if and only if there is  $E \in \bar{\mathcal{F}}$  such that  $F = E \cap X$ . By Step II, there exists a strong lifting  $\bar{\rho}$  of  $\bar{\mathcal{F}}$ . We define  $\rho$  on  $\bar{\mathcal{F}}$  by

$$\rho(F) = \bar{\rho}(E) \cap X$$

where  $E \in \bar{\mathcal{F}}$  such that  $F = E \cap X$ . It follows that  $\rho$  is a strong lifting of  $\bar{\mathcal{F}}$  and hence the theorem is completely proved.

As measures on totally ordered spaces share a number of properties with measures on metric spaces [10], it is natural to ask whether every lifting of  $(X, \leq, \mathcal{F}, \mu)$  is almost strong. The answer is given by using the next theorem, the proof of which uses an argument of [[1], Theorem 3.1] and some already familiar techniques from Theorem 2.1.

**THEOREM 2.2.** *Let  $(X, \leq, \mathcal{F}, \mu)$  be a totally ordered measure space, where  $\mu$  is a Radon, non-atomic measure on  $X$ . Then the following are equivalent*

- (i) *Every lifting of  $\bar{\mathcal{F}}$  is almost strong.*
- (ii) *The set  $L$  of all two sided limit points of  $S_\mu$  is  $\mu^*$ -measurable with full measure.*

*Proof.* We first prove the theorem in the case when  $X$  is compact. Also in this case without loss of generality we may assume that  $\mu$  is a probability measure with full support.

(i)  $\Rightarrow$  (ii) As in the Step I of Theorem 2.1 we denote by  $f$  the distribution function of  $\mu$ . We define  $\rho$  on  $\bar{\mathcal{F}}$  by

$$\rho(E) = f^{-1}(\rho'(f(E)))$$

where  $\rho'$  is a lifting of the Lebesgue measure on  $[0, 1]$ . Clearly  $\rho$  is a lifting of  $\mathcal{F}$  and so there exists a  $G_\delta$  subset  $N$  of  $X$  such that  $\mu(N) = 0$  and  $U \subset \rho(U) \cup N$  for every open subset  $U$  of  $X$ .

We will prove that  $X - N$  contains at most countable many non-two sided limit points of  $X$ . Assume that this is not true, without loss of generality let  $\{x_i\}_{i \in I}$  be an uncountable set of non-left limit points of  $X - N$ . Let  $y_i \in X$  such that  $y_i < x_i$  and  $(y_i, x_i) = \emptyset$ . Then  $y_i \in N$  for each  $i$ , because otherwise  $y_i \in \rho((-\infty, x_i))$  and if  $t_i = f(x_i) = f(y_i)$  we have that

$$t_i \in \rho'(f(-\infty, x_i)) \cap \rho'(f(y_i, +\infty)),$$

while

$$\rho'(f(-\infty, x_i)) \cap \rho'(f(y_i, +\infty)) = \rho'(\{t_i\}) = \emptyset.$$

Now since  $N$  is a  $G_\delta$  subset of  $X$  we may find an open subset  $U$  of  $X$  such that  $N \subset U$  and  $x_i \notin U$  for uncountable many  $i$ . Further since  $X$  is hereditary Lindelöf there is a sequence  $\{(a_n, b_n)\}$  of open intervals in  $X$  such that  $U = \bigcup_{n=1}^\infty (a_n, b_n)$ . Then if we pick an  $i$  such that  $x_i \notin U$  and  $x_i \neq b_n$  for all  $n$ , we get that  $y_i \notin U$ , which is a contradiction. This shows that  $L \in \mathcal{F}$  and  $\mu(L) = \mu(X)$ .

(ii)  $\Rightarrow$  (i) Let  $\rho$  be a lifting of  $\mathcal{F}$ . We define an upper density  $\theta$  of the Lebesgue measure space by

$$\theta(E) = f(\rho(f^{-1}(E)))$$

where  $f$  again denotes the distribution function of  $\mu$ . Clearly since every lifting of the Lebesgue measure space is almost strong we can find a set  $N \subset [0, 1]$  such that  $\lambda(N) = 0$  and  $V \subset \theta(V) \cup N$  for every open  $V \subset [0, 1]$ . It follows easily that

$$U \subset \rho(U) \cup f^{-1}(N) \cup (X - L)$$

for every open subset  $U$  of  $X$ . This shows that  $\rho$  is almost strong.

We now consider the general case. Let  $X^*$  be the totally ordered compactification of  $X$  and  $\bar{\mu}$  the induced by  $\mu$  Radon measure on  $X^*$ . Clearly  $\bar{\mu}$  is non-atomic and  $S_\mu = S_{\bar{\mu}} \cap X$ .

Let  $\bar{L}$  be the set of all two sided limit points of  $S_\mu$  and  $\bar{\mathcal{F}}$  the completion of the Borel  $\sigma$ -algebra of  $X^*$  w.r.t.  $\bar{\mu}$ . Clearly since  $\mu$  is Radon,  $X$  is  $\bar{\mu}^*$ -measurable with full  $\bar{\mu}$ -measure and so every lifting of  $\bar{\mathcal{F}}$  is almost strong, if and only if, every lifting of  $\mathcal{F}$  is almost strong. It remains to prove that  $L$  is  $\mu^*$ -measurable with full measure, if and only if,  $\bar{L}$  is  $\bar{\mu}^*$ -measurable with full measure. Indeed assume that  $L \in \mathcal{F}$  and  $\mu(L) = \mu(X)$ . Then using similar arguments as before we can show that  $K \cap (X^* - \bar{L})$  is at most countable, for every compact  $K \subset \bar{L}$ . It follows by the

Radon property of  $\mu$  that  $\bar{L}$  is  $\bar{\mu}^*$ -measurable with full  $\bar{\mu}$ -measure. On the other hand if  $\bar{L}$  is  $\bar{\mu}^*$ -measurable and  $\bar{\mu}(\bar{L}) = \bar{\mu}(X^*)$  since  $\bar{L} \cap X \subset L$ , we deduce that  $L \in \mathcal{F}$  and  $\mu(L) = \mu(X)$ . Hence the proof of the theorem is complete.

REMARKS 2.3. (i) Since every measure  $\mu$  (with full support on a totally ordered space) has separable measure algebra, under the continuum hypothesis Theorem 2.1 follows also by Theorem 9 in [9].

(ii) Let  $X$  be the space we obtain from  $[0, 1]$  by replacing each point  $t \in (0, 1)$  by two points say  $(t, 0)$  and  $(t, 1)$ . We define an order " $<$ " on  $X$  by

$$(t, i) < (t', j) \quad \text{iff} \quad t < t' \text{ or } t = t' \text{ and } i = 0, j = 1.$$

Clearly  $(X, <)$  is a compact totally ordered space which supports non-atomic, Radon measures [[10, Example 4.4]. Clearly since  $X$  has no two sided limit points, by Theorem 2.2, every non-zero, non-atomic Radon measure on  $X$  provides an example of a totally ordered measure space which admits not almost strong liftings.

(iii) Theorem 2.2 is not valid for atomic measures. For example let  $X$  be a compact totally ordered space which supports a non-atomic Radon measure  $\nu$  such that the set  $L$  of all two sided limit points is  $\nu^*$ -measurable with full measure and  $X \neq L$ . Then if  $x \in X - L$  and  $\mu = \nu + \delta_x$ , we see that Theorem 2.2 is not applicable for  $\mu$ . (Here  $\delta_x$  denotes the Dirac measure at  $x$ .)

(iv) In Theorem 2.2 the Radon property of  $\mu$  cannot be omitted. To see this let  $Y$  be the unit interval with the topology generated by all left-closed right-open intervals. It is known that  $Y$  is a closed subspace of a totally ordered space  $X$  (see Example 2.2(b) and Theorem 2.9 in [9]). Moreover, since the Borel sets of  $Y$  and  $(0, 1)$  are the same, the Lebesgue measure  $\lambda$  is a Borel measure on  $Y$ . Let  $\mu$  be the extension of  $\lambda$  on  $X$ . Then  $\mu$  is a non-atomic measure on  $X$  and  $S_\mu = Y$ . Further every point of  $Y$  is a two sided limit point of  $Y$ , but  $\mu$  admits not almost strong liftings (see Example 10 in [2]).

(v) The proofs of Theorems 2.1 and 2.2 apply unchanged for measures on generalized ordered spaces [9]

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