THE ROOT SUBGROUPS FOR MAXIMAL TORI IN FINITE GROUPS OF LIE TYPE

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Let $G_1$ be a finite group of Lie type defined over a field of characteristic $p$. The results of this paper represent an attempt to achieve a better understanding of the subgroup structure of $G_1$. It is somewhat surprising how limited our knowledge is, in this regard. For example, while centralizers of semisimple elements (i.e. $p'$-elements) of $G_1$ have been studied in detail and are fairly well understood, very little has been written about subgroups of $G_1$ generated by such centralizers. Even in explicit examples the analysis of such subgroups can be very difficult, the difficulty stemming from an inability to relate the generated group to the Lie structure of $G_1$. To deal with these situations and others we set up a framework that allows us to effectively study a fairly large class of subgroups of $G_1$ (those containing a maximal torus), by studying subgroups of the corresponding algebraic group. Essential to the development is a theory of root subgroups for arbitrary maximal tori of $G_1$.

1. Introduction. The theorems we establish have as their origin Lemma 3 of [22], which was later extended in [7] to show that if $q \geq 5$ and if $H$ is a Cartan subgroup of $G_1$ normalizing the $p$-group $V$, then $V$ is the product of root subgroups of $H$. This result is quite useful and provided the starting point for the result in [21] which showed that with further field restrictions one could determine all $H$-invariant subgroups of $G_1$. For example, if $H \leq L \leq G_1$, then it was shown that $L$ could be generated by $N_p(H)$ together with certain of the root subgroups of $H$. Hence, $L$ is determined by a subset of the root system of $G_1$ together with a subgroup of the Weyl group of $G_1$. One wants to extend these results to cover the case of an arbitrary maximal torus, not just a Cartan subgroup. Therefore, one would like to develop a theory of root subgroups that makes sense for an arbitrary maximal torus and then establish results like those above. The present paper carries out this program.

The group $G_1$ satisfies $O^{p'}(\overline{G}_\sigma) \trianglelefteq G_1 \trianglelefteq \overline{G}_\sigma$, where $\overline{G}$ is a connected simple algebraic group over the closure of $\mathbb{F}_p$, and $\sigma$ is an endomorphism of $\overline{G}$ whose fixed point set, $\overline{G}_\sigma$, is a finite group. Set $G = \overline{G}_\sigma$ and $G_0 = O^{p'}(\overline{G}_\sigma)$. A maximal torus of $G_1$ is a group of the form $T \cap G_1$, where $T = \overline{T}_\sigma$ and $\overline{T}$ is a $\sigma$-invariant maximal torus of $\overline{G}$. The group $\overline{G}$ has a root system, $\overline{\Sigma}$, and for each root $\alpha \in \overline{\Sigma}$, there is a $\overline{T}$-root subgroup $\overline{U}_\alpha$ of $\overline{G}$. These root subgroups are permuted by $\sigma$. If $\Delta$ is a $\langle \sigma \rangle$-orbit of such root subgroups, let $X = O^{p'}(\langle \Delta \rangle_{\sigma})$, a subgroup of $G_1$. Such a group is
called a *T-root subgroup* of $\bar{G}_\sigma$, and these groups are the groups we wish to consider. The $T$-root subgroups of $G$ are either $p$-groups or themselves groups of Lie type, and even when they are $p$-groups their structure can be complicated. Nevertheless, the situation is manageable, as we indicate in the sample results below.

Write $G_1 = G_1(q)$, where $q$ is a power of $p$ and fix a maximal torus $T$ of $G = \bar{G}_\sigma$. Set $T_0 = T \cap G_0$.

**Theorem (6.1).** Suppose $q > 1$, $T$ is a maximal torus of $\bar{G}_\sigma$ and $T_0 \leq Y \leq \bar{G}_\sigma$ with $Y$ solvable. Then $Y = O_p(Y) N_Y(T_0)$ and $O_p(Y)$ is a product of $T$-root subgroups of $\bar{G}_\sigma$.

As a consequence of (6.1) we see that for $q > 7$ any $T_0$-invariant $p$-subgroup is a product of $T$-root subgroups; the exact analogue of the result in [7]. When one considers arbitrary subgroups invariant under (or containing) a maximal torus, additional field restrictions must be made. In addition our proofs depend on the classification of finite simple groups.

In each of the following results assume that $p > 3$ and $q > 11$.

**Theorem (12.1).** The map $X \rightarrow X_\sigma$ is a bijection between the collection of all closed, connected, $\sigma$-invariant subgroups of $G$ containing a maximal torus of $\bar{G}$, and the collection of all subgroups of $G$ generated by maximal tori of $G$.

The inverse of the map $X \rightarrow X_\sigma$ is given in §12; it involves the $T$-root groups described above. We remark that a group of the form $X_\sigma$ has known structure (see (2.5)), so by (12.1) we can describe the structure of any subgroup of $G$ generated by maximal tori.

The next result contains parts of (10.1) and (10.2), and concerns those subgroups of $G$ containing a maximal torus of $G_0$. The result establishes part of the conjecture in [24]; the full conjecture follows from (10.1).

**Theorem.** Let $T_0 \leq Y \leq G$. Then

(i) The normal closure, $\langle T_0^Y \rangle$, of $T$ in $Y_0$ is generated by $T_0$ and those $T$-root subgroups of $G$ that are contained in $Y$.

(ii) If $T_1$ is any maximal torus of $G_0$ with $T_1 \leq Y$, then $\langle T_0^Y \rangle = \langle T_1^Y \rangle$.

(iii) $T_1$ can be chosen so that $Y = \langle T_1^Y \rangle N_Y(T_1)$.

(iv) $O_p(Y) \leq \langle T_0^Y \rangle$ and $O^p(\langle T_0^Y \rangle) / O_p(Y) = E(Y / O_p(Y))$ (the product of all quasisimple subnormal subgroups of $Y / O_p(Y)$).
In view of the above results it is clear that subgroups of $G$ generated by $T$-root subgroups are of particular importance. The next result indicates such subgroups can be studied by studying $\Gamma$ and subsets of $\Sigma$.

**Theorem (12.9).** Let $T$ be a maximal torus of $G$ and $X_1, \ldots, X_k$ $T$-root subgroups of $G$ corresponding to $\langle \sigma \rangle$-orbits $\Delta_1, \ldots, \Delta_k$ of $\tilde{T}$-root subgroups of $\tilde{G}$. Then $\langle X_1, \ldots, X_k \rangle = O^\sigma(\langle \Delta_1, \ldots, \Delta_k \rangle)$.\[\Box\]

The following are applications of some of the above results. The second theorem should be compared with the main results in [23].

**Theorem (12.10)(ii).** Assume $\tilde{G}$ is simply connected and let $S$ be an arbitrary set of $p'$-elements of $G$. Then $G_1 = \langle C_{G_1}(s) : s \in S \rangle$ if and only if $\tilde{G} = \langle C_{\tilde{G}}(s) : s \in S \rangle$.\[\Box\]

**Theorem (12.12).** Let $T_1$ be a maximal torus of $G_1$ and $R \leq T_1$. Then $G_1 = \langle E(C_{G_1}(R_1)) : R_1 \leq R$ and $R/R_1$ cyclic $\rangle$.\[\Box\]

The paper is organized into three chapters, each containing several sections. The first chapter is the basic development of $T$-root subgroups. In the second chapter we begin the consideration of subgroups of $G$ invariant under a maximal torus, although the classification of finite simple groups does not enter in. The last chapter contains the proofs of several of the main results and it is here where we apply the classification theorem.

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**Notation.** Throughout the paper $\tilde{G}$ will denote a connected simple algebraic group over the algebraic closure, $\bar{K}$, of the prime field $\mathbb{F}_p$. As before, $\sigma$ is a surjective endomorphism of $\tilde{G}$ with $\tilde{G}_\sigma$ finite. Then $G = \tilde{G}_\sigma = \tilde{G}_\sigma(q)$, where $q$ is a power of $p$. If $\tilde{T}$ is a maximal torus of $\tilde{G}$, let $U_\alpha$ denote the $\tilde{T}$-root subgroup corresponding to the root $\alpha \in \Sigma$, the root system of $\tilde{G}$. Let $\tilde{W} = N_\tilde{G}(\tilde{T})/\tilde{T}$, the Weyl group of $\tilde{G}$.

If $X$ is a finite group, $\text{Fit}(X)$ denotes the unique largest normal nilpotent subgroup of $X$ and $F^*(X)$ is the product of $\text{Fit}(X)$ and $E(X)$, where $E(X)$ is the (commuting) product of all subnormal quasisimple subgroups of $X$. $O_p(X)$ denotes the largest normal $p$-subgroup of $X$ and $O^p(X)$ is the normal subgroup of $X$ generated by all $p$-elements of $X$. If $d$
is a positive integer, then let $\Phi_d(x)$ be the corresponding cyclotomic polynomial of degree $\varphi(d)$. Some additional notation is given at the beginning of §§2, 3, 5, 9, and 10.

We label Dynkin diagrams as follows

$$
\begin{align*}
A_n & \quad \bullet \cdots \bullet \\
1 & \quad 2 \quad \cdots \quad n-1 \quad n \\
B_n & \quad \Rightarrow \leftarrow \\
1 & \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n \\
C_n & \quad \Rightarrow \leftarrow \\
1 & \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n \\
D_n & \quad 1 \quad \bullet \\
2 & \quad 3 \quad 4 \quad \cdots \quad n-1 \quad n, \quad n \geq 4 \\
E_n & \quad 1 \quad 3 \quad 4 \quad 5 \quad \cdots \quad n-1 \quad n \quad , \quad n = 6, 7, 8 \\
F_4 & \quad \Rightarrow \leftarrow \\
1 & \quad 2 \quad 3 \quad 4 \\
G_2 & \quad \Rightarrow \\
1 & \quad 2
\end{align*}
$$

I. T-Root Subgroups

2. Preliminaries. In this section we establish a number of basic results concerning maximal tori. In addition there are results on subgroups of algebraic groups generated by root subgroups and a somewhat curious number theoretical result.

The group $\overline{G}$ is as in §1 with root system $\overline{\Sigma}$ and Weyl group $\overline{W}$. We assume that $\overline{\Sigma}$ is indecomposable, so that $\overline{G}$ can be regarded as a Chevalley group over $K$. $\sigma$ is a surjective endomorphism of $\overline{G}$ and $G = \overline{G}_q$ is finite. Then $G$ is of Lie type and associated with a field $F_q$ of characteristic $p$. The number $q$ will be specified below; in nearly every case it is the order of the center of a root subgroup of $G$ for a long root. Write $G = G(q)$. Usually we will regard $\sigma$ as an element of the semidirect product $\overline{G} \langle \sigma \rangle$; hence $\sigma$ acts on $\overline{G}$ by conjugation.

By (10.10) of [26] we may choose a $\sigma$-stable maximal torus, $\overline{H}$, of $\overline{G}$ contained in a $\sigma$-stable Borel subgroup of $G$. Let $\overline{T}$ be a fixed $\sigma$-stable maximal torus. Then $\overline{T} = \overline{H}^g$ for some $g \in \overline{G}$. Therefore, $\overline{H}^{g\sigma} = \overline{H}^g$ so
\(H^{g_\sigma g^{-1}} = \overline{H}, g_\sigma g^{-1} \in N_G(\overline{H}),\) and we write \(g_\sigma = n_\sigma g\) for \(n \in N.\) This shows that the diagram

\[
\begin{array}{ccc}
\overline{H} & \xrightarrow{\ n_\sigma\ } & \overline{H} \\
g & \downarrow & \downarrow g \\
\overline{T} & \xrightarrow{\ \sigma\ } & \overline{T}
\end{array}
\]

commutes. Hence we will identify the action of \(\sigma\) on \(\overline{T}\) and on the character group \(X = X(\overline{T})\) with the action of \(n_\sigma\) on \(\overline{H}\) and on \(X(\overline{H})\), the identification being made via conjugation by \(g\). Now, \(n\) induces an element \(w \in \overline{W}\) on \(X(\overline{H})\) and, except for the Ree and Suzuki groups, \(\sigma\) induces \(q_\gamma\) on \(X(\overline{H})\), where \(\gamma\) is a graph automorphism of \(\overline{\Sigma}\). If \(G\) is a Ree or Suzuki group, then setting \(q_i = \sqrt{q}\), \(\sigma\) induces \(q_i\gamma\) on \(R \otimes X(\overline{H})\) and \(\gamma\) is an isometry (which interchanges long and short roots). So \(\sigma\) induces \(q_\tau\) or \(q_i\tau\) on \(R \otimes X(\overline{H})\), where \(\tau = w_\gamma\) is an isometry of \(R \otimes X(\overline{H})\) of finite order. We now carry this over to \(\overline{T}\) and \(X\), regarding \(w \in W \cong N_G(\overline{T})/\overline{T}\) and \(w, \gamma\) acting on \(X\). We then have

(2.1)(i) If \(G\) is not a Suzuki or Ree group, then \(\sigma\) induces \(q_\tau = qw_\gamma\) on \(X\).

(ii) If \(G\) is a Suzuki or Ree group, then \(\sigma\) acts on \(X\) and induces \(q_1\tau = q_iw_\gamma\) on \(R \otimes X\).

(iii) \(\overline{T}_\sigma \cong X/X(\sigma - 1)\).

(iv) \(|\overline{T}_\sigma| = |f(q)|\cdot(\cdot|f(q_i)|\) in the Suzuki or Ree groups), where \(f(x)\) is the characteristic polynomial of \(w_\gamma\) on \(R \otimes X\).

**Proof.** This follows from the above identification and (1.7) of [25].

The following lemma explains (2.1)(iv) and can be used to obtain the structure of \(T = \overline{T}_\sigma\) in certain cases.

(2.2) Let \(Y\) be a free \(Z\)-module and \(\theta\) an endomorphism of \(Y\). Suppose that \(R \otimes Y\) is a Euclidean space and \(\theta\) induces \(q_\varphi\) on \(R \otimes Y\), where \(\varphi\) is an isometry of finite order and \(|q_1| \geq 1\). If \(q_1 = \pm 1\), assume that \(C_\gamma(\theta) = 0\). Then

(i) \(\text{rank}_Z(Y) = \text{rank}_Z(Y_0),\) where \(Y_0 = Y(\theta - 1)\).

(ii) \(|Y/Y_0| = |f(q_1)|,\) where \(f(x)\) is the characteristic polynomial of \(\varphi\) on \(R \otimes Y\).

(iii) Suppose that \(q_1\) is an integer, \((Y)\varphi = Y\) and \(Y\) has a free basis in which the matrix of \(\varphi\) is in rational form. Then \(Y/Y_0 \cong Z_{f_1(q_1)} \times \cdots \times Z_{f_k(q_1)},\) where \(f_1(x) | \cdots | f_k(x)\) are the invariant factors of \(\varphi\) on \(R \otimes Y\).
Proof. Let $V = \mathbb{R} \otimes Y$. If $q_1 = \pm 1$, then we are assuming that $C_y(\theta) = 0$. If $|q_1| > 1$, then use the fact that $\varphi$ has finite order to conclude $C_y(\theta) = 0$. In either case $C_y(\theta) = 0$ and $\theta - 1$ is injective. This proves (i).

For (ii), choose a basis $\{v_1, \ldots, v_n\}$ of $Y$ and positive integers $z_1, \ldots, z_n$ such that $z_1 | \cdots | z_n$ and $\{z_1 v_1, \ldots, z_n v_n\}$ is a $\mathbb{Z}$-basis of $Y_0 = Y(\theta - 1)$. For $i = 1, \ldots, n$ let $(v_i)(\theta - 1) = \sum a_{ij} z_j v_j$. There is an integral matrix $(b_{ij})$ such that

$$
(b_{ij})(a_{ij}) = \begin{pmatrix}
z_1 \\
\vdots \\
z_n
\end{pmatrix} = \begin{pmatrix}
z_1 \\
\vdots \\
z_n
\end{pmatrix}.
$$

Then $(b_{ij})(a_{ij}) = 1$, $\det(b_{ij}) = \pm 1$, and $\det(\theta - 1) = \pm z_1 \cdots z_n$. Passing to $V$, we have $z_1 \cdots z_n = |\det(\theta - 1)| = |\det(q_1 \varphi - 1)|$. Since $\varphi$ is an isometry of $V$, $\varphi$ and $\varphi^{-1}$ have the same eigenvalues with equal multiplicities, hence $\varphi$ and $\varphi^{-1}$ have the same characteristic polynomial, say $f(x)$. Since $\det(\varphi) = \det(\varphi^{-1}) = \pm 1$, we have $|\det(q_1 \varphi - 1)| = |\det(q_1 - \varphi^{-1})| = |f(q_1)|$, proving (ii).

For (iii), suppose $Y = Y_1 \oplus \cdots \oplus Y_k$, where $Y_i = (Y_i)\varphi = \langle \beta_i \rangle$ and $\beta_i$ is a $\mathbb{Z}$-basis of $Y_i$ in which the matrix of $\varphi$ is the companion matrix of $f_i(x)$. Here, $f_i(x)$ are the invariant factors of $\varphi$. Fix $1 \leq i \leq k$ and let $\beta_i = \{\beta_1, \ldots, \beta_i\}$. Write $f_i(x) = a_1 + a_2 x + \cdots + a_l x^{l-1} + x^l$. Then

$Y_i = \langle \beta_i \rangle \oplus \langle q_1 \beta_1 - \beta_{i-1} \rangle \oplus \cdots \oplus \langle q_1 \beta_2 - \beta_1 \rangle$.

Also, $q_1 \beta_i - \beta_{i-1} = \beta_{i-1}(q_1 \varphi - 1)$ for $i = 1, \ldots, l - 1$. Thus,

$Y_i(q_1 \varphi - 1) = \langle q_1 \beta_2 - \beta_1, \ldots, q_1 \beta_i - \beta_{i-1} \rangle + \langle \beta_i(q_1 \varphi - 1) \rangle$.

Now,

$\beta_i(q_1 \varphi - 1) = -a_1 q_1 \beta_1 - \cdots - a_{i-1} q_1 \beta_{i-1} - (a_i q_1 + 1) \beta_i$

$= a_1 q_1 (q_1 \beta_2 - \beta_1) + (a_1 q_1^2 + a_2 q_1)(q_1 \beta_3 - \beta_2) + \cdots + (-1)(a_i q_1 + \cdots + a_i q_1 + 1) \beta_i$.

Therefore, $Y_i/Y_i(q_1 \varphi - 1) \cong \langle \beta_i \rangle / \langle g(q_1) \beta_i \rangle$, where $g(x) = a_1 x' + \cdots + a_l x + 1$. Now $g(x) = x'f_i(1/x)$. Since $|\varphi| < \infty$, $f_i(x)$ is a product of cyclotomic polynomials, hence the roots of $f_i(x)$ and $g(x)$ are equal and also $a_i = \pm 1$. Thus, $g(q_1) = \pm f_i(q_1)$ and $Y_i/Y_i(q_1 \varphi - 1) \cong Z_{f_i(q_1)}$. From here (iii) is immediate.

(2.3) Let $\pi: \tilde{G} \to \tilde{G}$ be the natural surjection, where $\tilde{G}$ is the universal covering group of $G$, and let $\tilde{T}$ be the preimage of $T$. Then $\sigma$ can be
viewed as an endomorphism of \( \tilde{G} \). Also,

(i) \( (\tilde{G}_o)^H = O^{p'}(\tilde{G}_o) = O^{p'}(G) \).

(ii) \( \tilde{T}_o/(\tilde{T}_o)^\pi \cong G/O^{p'}(G) \).

**Proof.** The first fact is standard (see (12.6) of [26]). The second assertion is proved as in (2.12) of [23] (or see 5.10.1 in [10]).

(2.4) Let \( \tilde{G} \) be as in (2.3), \( |Z(\tilde{G}_o)| = d \) and \( |Z(O^{p'}(\tilde{G}_o))| = d_1 \). Then

(i) \( |\tilde{G}_o : O^{p'}(\tilde{G}_o)| = d/d_1 \).

(ii) If \( T_0 = T \cap O^{p'}(\tilde{G}_o) \), then \( |T_0| = (d_1/d) |f(q)| (f(q_1) \text{ in the Suzuki and Ree groups}) \), where \( f(x) \) is as in (2.1)(iv).

(iii) If \( \tilde{G} \) has Lie rank \( r \), then

\[
(q - 1)^r \leq |f(q)| \leq (q + 1)^r
\]

\[
((q_1 - 1)^r \leq |f(q_1)| \leq (q_1 + 1)^r \text{ for Suzuki and Ree groups}).
\]

**Proof.** With notation as in (2.3), \( (Z(\tilde{G}_o))^\pi = Z(O^{p'}(\tilde{G}_o)) \). By (2.1)(iv) \( |\tilde{T}_o| = |\tilde{T}_o| \), so \( |f(q)| = |\tilde{T}_o| = (d/d_1) |(\tilde{T}_o)^\pi| = (d/d_1) |T_0| \) (similar equations in the Suzuki and Ree cases). Then (i) and (ii) follow. For (iii) use the triangle inequality and the fact that the roots of \( f(x) \) are roots of unity.

(2.5) Let \( \tilde{D} = \tilde{D}_o \) be a closed connected subgroup of \( \tilde{G} \) with \( \tilde{T} \leq \tilde{D} \).

(i) \( \tilde{D} = R_u(\tilde{D})\tilde{L} \), where \( \tilde{L} = \tilde{L}_o \) is reductive and \( \tilde{T} \leq \tilde{L} \).

(ii) \( \tilde{L} = \tilde{L}'\tilde{T} \) and \( \tilde{L}' = [\tilde{L}, \tilde{L}] \) is semisimple.

(iii) \( R_u(\tilde{D}) \) is a product of \( \tilde{T} \)-root subgroups of \( \tilde{G} \) and \( \tilde{L}' \) is generated by \( \tilde{T} \)-root subgroups, corresponding to a subsystem of \( \tilde{\Sigma} \).

(iv) \( \tilde{D}_o = O^{p'}(\tilde{D}_o)^\pi \tilde{T}_o = R_u(\tilde{D})_o O^{p'}(\tilde{L}'_o)^\pi \tilde{T}_o \).

(v) \( O^{p'}(\tilde{L}'_o) \) is a commuting product of groups of Lie type and \( \tilde{T} \) contains a maximal torus of each factor.

**Proof.** Set \( \bar{Q} = R_u(\bar{D}) \) and let \( \bar{A} \) be a Borel subgroup of \( \bar{D} \) with \( \bar{A} \supseteq \bar{T} \). Embedding \( A \) in a Borel subgroup of \( \tilde{G} \) we see that \( R_u(A) \) and \( \bar{Q} \) are both products of \( \tilde{T} \)-root subgroups of \( \tilde{G} \) (one can modify the argument of Lemma 3 of [22] to establish this). Let \( \Delta_1 \) be those roots \( \alpha \in \tilde{\Sigma} \) with \( \tilde{U}_\alpha \leq \bar{Q} \) and let \( \Delta \) be all roots \( \alpha \in \tilde{\Sigma} \) such that \( \tilde{U}_\alpha \leq \bar{A} \) and \( \tilde{U}_{-\alpha} \leq \bar{D} \). We then have \( \bar{Q} = \prod_{\alpha \in \Delta_1} \tilde{U}_\alpha \) and \( (\bar{D}/\bar{Q})' = \langle \tilde{U}_{\pm \alpha} \mid \alpha \in \Delta \rangle \bar{Q}/\bar{Q} \) (from the structure theory of reductive groups).

Let \( \bar{E} = \langle \tilde{U}_{\pm \alpha} \mid \alpha \in \Delta \rangle \). From the Bruhat decomposition and the fact that \( \Delta \) is a subsystem of \( \tilde{\Sigma} \) we conclude that \( \tilde{L} = E\tilde{T} \) is reductive, and then \( E \) is semisimple. Then \( \tilde{L}' = \bar{E} \), and since \( \bar{D}/\bar{Q} = \bar{E}TQ/\bar{Q} \), we have established (i), (ii), and (iii).
Since \( \bar{Q} \) is connected, Lang's theorem implies that \( (D/\bar{Q})_\sigma = \bar{D}_\sigma \bar{Q}/\bar{Q} \).

As \( \bar{D} \) is the semidirect product of \( \bar{Q} \) and \( \bar{L} \), (iv) will be proved once we know that \( \bar{L}_\sigma = O^\nu(\bar{L}'_\sigma) \bar{T}_\sigma \). We first note that \( \bar{L}_\sigma = E_\sigma \bar{T}_\sigma \). To see this let \( \bar{J} = \bar{E} \cap \bar{T} \), a maximal torus of \( \bar{E} \), normalized by \( \sigma \). Suppose \( e \in \bar{E}, t \in \bar{T} \) and \( (et)^\sigma = et \). Since \( \bar{E} \) and \( \bar{T} \) are both \( \sigma \)-stable, we have \( e^\sigma = ej \) and \( t^\sigma = jy^{-1} \) for some \( j \in \bar{J} \). From Lang's theorem ((10.1) of [26]) there is an element \( j_\sigma E \) with \( y^\sigma = j_\sigma j^\sigma \). Then \( e = (ej_\sigma)(j_\sigma^{-1} j) \) represents \( e \) as an element in \( \bar{E}_\sigma \bar{T}_\sigma \). The proof of (iv) has now been reduced to the semisimple group \( \bar{E} \), where the result follows from (2.12) of [23] (that result concerned a simple adjoint group, but these conditions were never used).

For (v), apply (11.7) of [26] to get the structure of \( O^\nu(\bar{L}'_\sigma) \). The remaining part of (v) is obtained by considering orbits of \( \langle \sigma \rangle \) on the simple factors of \( \bar{L}' \).

(2.6) Let \( T_0 = T \cap O^\nu(G) \) and assume \( q \geq 4 \). Then \( C_G(T_0)^0 = \bar{T} \).

**Proof.** From the Bruhat decomposition of \( \bar{G} \) (with respect to a Borel subgroup \( \bar{B} \supset \bar{T} \)) we see that the result holds unless \( T_0 \leq C(\bar{U}_\alpha) \) for some root subgroup \( \bar{U}_\alpha \) of \( \bar{G} \). Let \( \Delta = \{ \alpha \in \Sigma | [T_0, \bar{U}_\alpha] = 1 \} \). Then \( \Delta \) is closed under taking negatives and we set \( D = \langle \bar{U}_\alpha | \alpha \in \Delta \rangle \).

We have \( D^\sigma = D \) and by (2.5) \( D = D_1 \cdots D_k \), a commuting product of quasisimple groups \( D_i \), where each \( D_i \) is generated by certain of the root subgroups, \( \bar{U}_\alpha \), for \( \alpha \in \Delta \). Each \( D_i \) is a Chevalley group with indecomposable root system. Reorder so that \( \{ D_1, \ldots, D_k \} \) is a \( \langle \sigma \rangle \)-orbit. Then \( \sigma^k \) normalizes each \( D_i \), \( 1 \leq i \leq k \), and \( (D_i)_{\sigma^k} = D_i(q^k) \), a group of Lie type associated with \( F_{q^k} \) (see (11.6) of [26] and the proof of (2.6) of [23]). Also, \( (D_1 \cdots D_k)_{\sigma} \) is obtained as a diagonal copy of \( D_1(q^k) \) (except for amalgamation of centers). Let \( T_1 = \bar{T} \cap O^\nu((D_1 \cdots D_k)_{\sigma}) \). Then \( T_1 \leq T_0 \leq C(\bar{U}_\alpha) \) for each \( \alpha \in \Delta \), and projecting to \( D_1 \), we see that \( T_2 = \bar{T} \cap O^\nu((D_1)_{\sigma^k}) \leq Z(O^\nu((D_1)_{\sigma^k})) \).

By (2.4)(ii) and (2.4)(iii) \( |T_2| = (e_1/e) |f(q^k)| \geq (e_1/e)(q^k - 1)^r \) (replace \( q \) by \( q_1 \) if \( (D_1)_{\sigma^k} \) is a Suzuki or Ree group), where \( e_1 = |Z(O^\nu((D_1)_{\sigma^k}))| \), \( e \) is the order of the center of the universal covering group of \( (D_1)_{\sigma^k} \), and \( r \) is the Lie rank of \( D_1 \). But \( |T_2| \leq e_1 \) (which is 1 in the Suzuki and Ree cases), whereas \( q^k - 1 \geq 3 (\sqrt{3} \) in the Suzuki and Ree cases). Then \( 3^r \leq e (or (\sqrt{3})^r \leq 1) \), a contradiction.

An immediate consequence of (2.6) is

(2.7) Let \( q \geq 4 \) and set \( T_0 = T \cap O^\nu(G) \). Then \( N_G(T_0) \leq N_G(\bar{T}) \).

For \( q > 5 \), we also have control over \( C_G(T_0) \).

(2.8) Let \( q > 5 \) and set \( G_0 = O^\nu(G), T_0 = T \cap G_0 \). Assume that \( \bar{G} \) is adjoint. Then
(i) $C_G(T_0) = T$.
(ii) If $G_0 \neq \text{PSL}(2, 9), \text{Sz}(8), \text{or } \text{Sz}(8)$, then $C_{\text{Aut}(G_0)}(T_0) = T$.

**Proof.** $G$ is adjoint, so $Z(G) = Z(G_0) = 1$. Let $a \in \text{Aut}(G_0)$ and suppose $a \in C(T_0) - T$. Replacing $a$ by a power of $a$ we may assume that $a^s \in T$ for some prime $s$. Extend $a$ to an endomorphism of $\overline{G}$ commuting with $\sigma$. By (2.6) $\overline{T}$ is $a$-invariant, so $a$ acts on $X$. $C_X(a)$ is $G$-invariant and both $C_X(a)$ and $\overline{X} = X/C_X(a)$ are free $Z$-modules.

Let $X_0$ denote the annihilator in $X$ of $T_0$. Since $T_0 \leq C_T(\sigma) \cap C_T(a)$ we have $[X, a] \leq X_0 \geq [X, \sigma]$. Also $|X_0 : [X, \sigma]| = |T : T_0| = d$, where $d = |G : G_0|$ (see (2.3)ii). Both $[X, a]/([X, a] \cap [X, \sigma])$ and $[\overline{X}, a]/([\overline{X}, a] \cap [\overline{X}, \sigma])$ are isomorphic to sections of $X_0/[X, \sigma]$. In particular, each has order a divisor of $d$ and exponent dividing that of $X_0/[X, \sigma]$.

Write $A = \text{Aut}(G)$ and let $A_0$ be the subgroup of $A$ generated by all inner, diagonal, and graph automorphisms of $G_0$. The elements of $A_0$ are precisely the automorphisms of $G_0$ that can be extended to automorphisms of the algebraic group $\overline{G}$ (elements of $A - A_0$ can be extended to surjective endomorphisms of $\overline{G}$).

Let $\overline{Y}$ be the subgroup of $\text{Aut}(\overline{G})$ normalizing $\overline{T}$ and let $F$ be the Frobenius morphism with respect to $\overline{T}$. Then $\sigma$ and $F$ commute in their action on $X$, so $\sigma F$ and $F \sigma$ differ in their action on $\overline{G}$ by an inner automorphism induced from an element of $\overline{T}$. Using Lang’s theorem, we modify $F$ so that it commutes with $\sigma$. For convenience we postpone discussion of the cases where $\overline{G}$ has type $C_2$, $G_2$, $F_4$ and $p = 2, 3, 2$, respectively. Then there is a power $n$ such that $\sigma \in \overline{Y} F^n$. Consequently, $q = p^n$ and $\sigma$ induces $t \sigma^n$ on $X$, where $t$ is an isomorphism of $R \otimes X$. Similarly, there is an isommetry $\epsilon$ and power $p^m, m \geq 0$, such that $a$ induces $\epsilon \sigma^m$ on $X$.

Let $f(x), \tilde{f}(x)$ be the characteristic polynomials of $t$ on $R \otimes X, R \otimes \overline{X}$, respectively. Similarly, let $g(x), \tilde{g}(x)$ be the corresponding polynomials of $\epsilon$.

Assume $a \notin A_0$. The group $A/A_0$ is cyclic of order $n$ and generated by $\gamma A_0$, where $\gamma = F|_{G_0}$. Replacing $a$ by a suitable power, we may assume that $a \in A_0 \cdot \gamma^{n/s}$. That is, $a \in \overline{Y} \cdot F^{n/s}$ and so $a$ induces $\epsilon \sigma^{n/s}$ on $X$. By (2.2)(ii) $|f(p^n)| = |X : [X, \sigma]|$ which divides $|X : [X, \sigma] \cap [X, a]|$. Another application of (2.2)(ii) and previous remarks show that the latter number divides $d |g(p^{n/s})|$. So (2.4)(iii) yields the inequality $(p^n - 1)^r \leq d(p^{n/s} + 1)^r$. Using this together with the inequality $d \leq r + 1$ we calculate and obtain $r = 1, p^n = 9, s = 2$. But then $G_0 \cong \text{PSL}(2, 9)$, which is excluded in (ii). So $a \in A_0$ and $a$ induces $\epsilon$ on $X$. 


One checks that $C_X(a) = 0$, so by (2.2)(ii) we argue as above that $|\tilde{f}(q)| \leq d_0|\tilde{g}(1)| \leq d|\tilde{g}(1)|$, where $d_0$ is the order of $[\tilde{X}, a]/([\tilde{X}, a] \cap [\tilde{X}, \sigma]) = D$.

Then (2.4)(iii) yields $(q - 1)\tilde{r} \leq d_02\tilde{r} \leq d2\tilde{r}$. Since $d \leq q + 1$, we are led to $\tilde{r} = 1$, $\tilde{f}(x) = x \pm 1$, $\tilde{g}(x) = x + 1$, and $6 \leq q - 1 \leq 2d_0 \leq 2d$.

If $\overline{G} = D_r$, then $T/T_0$ is a subgroup of $Z_2 \times Z_2$, so $D$ is isomorphic to a cyclic (as $r = 1$) subgroup of $X_0/[X, \sigma] = T/T_0$. Hence $d_0 \leq 2$, against the above inequality. Suppose $a$ is an inner automorphism. Then $\tilde{X}$ centralizes $T/T_0$ and hence $a$ centralizes $D$. However, $a$ inverts $\tilde{X}$ (since $\tilde{g}(x) = x + 1$) and $\tilde{X}$ is cyclic. We conclude $|D| \leq 2$, so again $d_0 \leq 2$, giving a contradiction. At this point we have $a$ in the coset of an involutory graph automorphism of $\overline{G}$, and the inequality of the previous paragraph shows that $\overline{G}$ is either of type $E_6$ (and $q = 7$) or of type $A_r$.

Write $\overline{W} = \overline{N}/\overline{T}$ and consider the action of $a$ on $\overline{W}$. Then $\overline{N}(a) = \overline{N}(b)$, where $\overline{N}(b)/\overline{T} = \overline{W} \times \langle b\overline{T} \rangle$, $b$ sends each root to its negative and inverts $X$. $a$ acts as $w_1b$ for $w_1 \in \overline{W}$, and the eigenspace of $w_1$ for eigenvalue $-1$ has codimension 1 in $R \otimes X$. If $\overline{W} \cong S_{r+1}$, use the cycle decomposition of $w_1$ to see that no such involution exists for $r > 3$. If $r = 2$, then $G_0 \cong PSL(3, 7)$, while if $r = 3$, then $G_0 \cong PSL(4, 9)$ or $PSU(4, 7)$. If $G_0 = PSL(3, 7)$ or $PSL(4, 9)$ we can take $w_1$ to be $s_3$, $s_1s_3$, respectively. If $G_0 = PSU(4, 7)$, take $w_1$ to be $s_1s_2$, $(s_1s_3s_2)^2$, or $s_2s_1s_3s_2$. In all cases we can explicitly compute $[X, a]$, $[X, \sigma]$, and contradict the earlier observation that $[X, a]/[X, a] \cap [X, \sigma]$ has order dividing $d$.

Suppose $\overline{W} \cong E_6$. Then $w_1$ has determinant $-1$. Now $\overline{W} \cong Aut(PSU(4, 2))$ and by (19.5) of [1] $\overline{W} - \overline{W}'$ has two classes of involutions, represented in $\overline{W}$ by reflections and the product of 3 commuting reflections. This contradicts the condition on eigenvalues.

At this point we have proved the lemma for all cases except $\overline{G}$ of type $C_2$, $G_2$, $F_4$ and $p = 2, 3, 2$, respectively. We indicate the necessary adjustments in the previous arguments. First note that $d = 1$ in all cases. As in (2.1) $\sigma$ acts on $R \otimes X$ as $t\hat{q}$, with $t$ an isometry and $\hat{q} = q$ or $\sqrt{q}$. If $a \in A_0$, proceed as before to get the inequality $|\tilde{f}(\hat{q})| \leq d|\tilde{g}(1)|$. By (2.4) we then obtain $(\hat{q} - 1) \tilde{r} \leq 2\tilde{r}$, a contradiction. So $a \not\in A_0$.

Let $\hat{F}$ be the endomorphism of $\overline{G}$ such that $\hat{F}^2 = F$ is the Frobenius map (with respect to $\overline{T}$). If $\hat{q} = \sqrt{q} = p^{r+1/2}$, then $A/A_0$ is cyclic of order $2l + 1$, with quotient generated by a field automorphism. Since $F |_{A_0}$ induces a field automorphism of order $2l + 1$, we argue as before that $a$ is an element of $T$ one can assume $a \in PO^+(2r, q)$ and argue within the linear group.

*Added in proof. If $r$ is odd, $T/T_0$ could be $Z_4$ with $q = 7, 9$. In this case multiplying $a$ by an element of $T$ one can assume $a \in PO^+(2r, q)$ and argue within the linear group.
can be taken in $\overline{YF^n/s}$. Now argue as before to get a contradiction (recall that $G_0 F^\gamma_{2s}$ has order $2n$ and $A/A_0 = \langle A_0 \gamma \rangle$). We may then assume that $a$ induces $e p^{n/2s}$ on $X$, for $\epsilon$ an isometry ($n/2s$ need not be an integer). The usual inequalities give a contradiction and complete the proof of (2.8).

(2.9) Let $x \in T$. Then $C_G(x)$ contains normal subgroups $Y_0$ and $Y$ such that

(i) $Y = Y_0 T$.

(ii) $Y_0 = D_1 \cdots D_k X$, a commuting product, where for each $i = 1, \ldots, k$ there exists a power, $q_i$, of $q$ such that $D_i = D_i(q_i)$ is a quasisimple group of Lie type defined over $F_q$. Also $Z(Y_0) \geq X \leq T$.

(iii) If each $q_i \geq 4$, then $D_1, \ldots, D_k$ are the components of $E(C_G(x))$ and $X = C_Y(E(C_G(x))) \leq T$.

(iv) $C_G(x)/Y$ is isomorphic to a subgroup of the center of the universal covering group of $G_0$.

Proof. Since $x \in T \leq \overline{T}$, $C_G(x)$ can be computed from the Bruhat decomposition of $\overline{G}$ (with respect to the root subgroups of $T$). We have $T \leq C_G(x)^0$ and $C_G(x)^0 = Y_1 \cdots Y_i Z$, where the product is a commuting product, each $Y_i$ is a Chevalley group defined with respect to an indecomposable subsystem of $\Sigma$, and $Z \leq \overline{T}$. Let $Y = (C_G(x)^0)_o$. Since $Y_1 \cdots Y_i$ is connected, Lang’s theorem (see (10.1) of [26]) implies that $Y = (Y_1 \cdots Y_i)_o \overline{T}_o$.

Let $R = Y_1 \cdots Y_i$. The argument in (2.13) of [23] shows that $R_o = O^{\phi}(R_o)_o(\overline{T} \cap R)_o$. Moreover, the proof of (2.6)(ii) of [23] shows that $O^{\phi}(R_o) = D_1 \cdots D_k$ has the required structure. So setting $Y_0 = O^{\phi}(R_o)Z_o$, we have (i), (ii), and (iii) holding. For (iv) see (4.4) of Springer-Steinberg [25].

The following number theoretical result will be needed in §7.

(2.10) Let $p > 3$ be prime, $x = \prod_{i=1}^m \Phi_d_i(p)$, and $y = \prod_{i=1}^n \Phi_f_j(p)$. Suppose that

(a) $x \mid y$;

(b) $d_1 < \cdots < d_m$; and

(c) $\sum \varphi(d_i) \geq \sum \varphi(f_j)$.

Then $m = n$ and $\{d_i \mid 1 \leq i \leq m\} = \{f_j \mid 1 \leq j \leq n\}$. In particular, $x = y$.

Proof. Suppose false. Factoring out common factors we may assume $d_i \neq f_j$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. Suppose it is not the case that $d_i = 2$ and $p$ a Mersenne prime. Then by Zsigmondy [28], for each $1 \leq i \leq m$ there is a prime $r_i$ such that $r_i \mid p^{d_i} - 1$, but $r_j \mid p^d - 1$ for $d < d_i$. We call these primes primitive divisors. In the exceptional case,
\( \Phi_d(p) = p + 1 \) is a power of 2 and we set \( r_i = 2 \). If this case occurs and \( d_{i'} = 1 \) for some \( i' \), then \( d_{i'} = d_1 \) and \( \Phi_{d_{i'}}(p) \) is divisible by some odd prime, which we may take to be \( r_1 \). Therefore, \( r_i \neq r_{i'} \) for \( i \neq i' \). Choose \( r_1 \neq 2 \), if possible.

In the primitive divisor situation, \( d_i \) is the order of \( p \) modulo \( r_i \). So \( d_i | r_i - 1 \) and \( \varphi(d_i) \leq d_i \leq r_i - 1 \). Fix \( 1 \leq i \leq m \). There exists \( j \in \{1, \ldots, n\} \) such that \( r_i | \Phi_j(p) \). Then \( d_i | f_j \). Set \( g(t) = (t^{d_i} - 1)/(t^{d_i} - 1) \) and expand \( g(t) \) in powers of \( t^{d_i} \). Letting \( t = p \) and using the congruence \( p^{d_i} \equiv 1 \) (mod \( r_i \)) we have \( r_i | (f_j/d_i) | f_j \).

We estimate \( \sum \varphi(f_j) \) as follows. If \( 1 \leq j \leq m \) is fixed and \( r_{i_1}, \ldots, r_{i_k} \) are the primes satisfying \( r_i | \Phi_j(p) \), then, by the above, \( \varphi(f_j) \) is divisible by \( (r_{i_1} - 1) \cdots (r_{i_k} - 1) \). If \( c > 1 \) and \( d > 1 \) are integers, then \( cd \geq c + d \). Suppose that no \( r_i = 2 \). Using these facts we have \( \sum \varphi(r_i - 1) \leq \sum \varphi(f_j) \). This inequality combined with our hypothesis and the remarks of the previous paragraph yield

\[
(*) \quad \sum \varphi(f_j) \leq \sum \varphi(d_i) \leq \sum d_i \leq \sum (r_i - 1) \leq \sum \varphi(f_j).
\]

Therefore all inequalities are equalities. In particular, \( \varphi(d_i) = d_i \) for \( i = 1, \ldots, m \). But this forces \( d_i = 1 \), and so \( 1 \neq d_i = r_i - 1 \), against our supposition. Therefore, either \( d_1 = 1 \) and \( r_1 = 2 \) (\( p \) a Fermat prime) or \( d_1 = 2 = r_i \), for \( i = 1 \) or 2 (\( p \) a Mersenne prime).

To deal with these cases we slightly modify the above argument. Say \( d_i = 2 = r_i \). Choose \( j \) with \( r_i | \Phi_j(p) \) and \( f_j \neq 1 \). Then \( r_i | f_j/d_i \), so \( 4 | f_j \). If \( r_{i_1}, \ldots, r_{i_k} \) are the other primes among \( r_1, \ldots, r_m \) that are factors of \( f_j \), then \( \varphi(f_j) \geq 2(r_{i_1} - 1) \cdots (r_{i_k} - 1) \). So we again have the inequality \( \sum (r_i - 1) \leq \sum \varphi(f_j) \). So we will again obtain \( (*) \) provided \( \sum d_i \leq \sum (r_i - 1) \). Suppose this fails and let \( 1 \leq k \leq m, k \neq i \). Since \( d_i = (r_i - 1) + 1 \) and \( d_k | r_k - 1 \) we necessarily have \( d_k = r_k - 1 \). So \( d_k \) is even and \( \varphi(d_k) \leq \frac{1}{2}d_k \). On the other hand, if we add 1 to each of the last two terms in \( (*) \), the resulting inequalities hold. Thus \( \varphi(d_k) \geq d_k - 1 \), which is impossible. We conclude that no such \( k \) exists, \( m = 1 \), \( d_i = d_1 \), and \( \sum \varphi(f_j) \leq \sum \varphi(d_i) = 1 \). So \( n = 1, f_1 = 1, \) and \( p + 1 | p + 1 \) (as \( x \) divides \( y \)). This is absurd. Therefore \( (*) \) holds and \( \sum \varphi(d_i) = \sum d_i \). This is a contradiction.

Finally, suppose \( d_1 = 1 \) and \( r_1 = 2 \). Then \( d_i | r_i - 1 \) for each \( i = 1, \ldots, m \). If \( (*) \) holds, then \( m = 1 \) and \( \sum \varphi(f_j) \leq 1 \). Therefore, \( n = 1, f_1 = 2, \) and \( p - 1 | p + 1 \), a contradiction. So we suppose \( (*) \) to be false. Then the inequality \( \sum (r_i - 1) \leq \sum \varphi(f_j) \) must fail to hold. Let notation be as in the previous paragraph. Then \( 2r_{i_1} \cdots r_{i_k} \) is a factor of \( f_j \) and \( \varphi(f_j) \) is divisible by \( (r_{i_1} - 1) \cdots (r_{i_k} - 1) \). So \( \varphi(f_j) \geq (r_{i_1} - 1) + \cdots + (r_{i_k} - 1) \). Combining this with the other values, \( \varphi(f_1), \ldots, \varphi(f_n) \), we do have
\[ \Sigma(r_i - 1) \leq \Sigma \varphi(f_j) + 1, \text{ where the 1 corresponds to } r_1 - 1. \] In view of our assumption, equality must hold. The previously used inequality, \( cd \geq c + d \), is strict unless \( c = d = 2 \), and this forces \( f_j = 2r_i \) for some \( r_i \), while for \( k \neq j \), \( f_k = r_{i(k)} \) or \( 2r_{i(k)} \) with \( r_{i(k)} \in \{r_2, \ldots, r_m\} \). However, we have seen that \( r_{i(k)} \mid (f_k/d_{i(k)}) \). So the only possibilities are \( m = 1 \) or \( m = 2 = d_2 \). That is \( x = p - 1 \) or \( x = (p - 1)(p + 1) \). Considering the possibilities for \( y \), we have a contradiction.

The next several lemmas concern subgroups of \( \widetilde{G} \) generated by \( \Gamma \)-root subgroups.

(2.11) Let \( S \subset \Sigma \), \( X = \langle \widetilde{U}_\alpha \mid \alpha \in S \rangle \), and \( \Delta = \{ \delta \in \Sigma \mid \widetilde{U}_\delta \not\leq X \} \). Suppose \( \Delta \cap -\Delta = \emptyset \). Then \( X = \prod_{\delta \in \Delta} \widetilde{U}_\delta \) and \( X \) is unipotent.

**Proof.** Let \( S \), \( X \), \( \Delta \) be as in the statement. It will be convenient to exclude the case \( \Sigma \) of type \( G_2 \). This case can be handled by a direct check. For a fixed ordering on \( \Sigma \), let \( \Delta^+ = \Delta \cap \Sigma^+ \), \( \Delta^- = \Delta \cap \Sigma^- \), \( X^+ = \langle \widetilde{U}_\alpha \mid \alpha \in \Delta^+ \rangle \), and \( X^- = \langle \widetilde{U}_\alpha \mid \alpha \in \Delta^- \rangle \). Then \( X^+ \leq \widetilde{U}^+ \) and \( X^- \leq \widetilde{U}^- \), where \( \widetilde{U}^- \) is opposite to \( \widetilde{U} \). We then have \( X^+ = \prod_{\delta \in \Delta^+} \widetilde{U}_\delta \) and \( X^- = \prod_{\delta \in \Delta^-} \widetilde{U}_\delta \).

If \( \Sigma \) has two root lengths, let \( \Sigma_0 \) be the subsystem of long roots. Then \( \widetilde{G}_0 = \langle \widetilde{U}_\alpha \mid \alpha \in \Sigma_0 \rangle \) is proper in \( \widetilde{G} \), so by induction (on \( |\Sigma| \)) we have \( X_0 = \langle \widetilde{U}_\alpha \mid \alpha \in \Delta \cap \Sigma_0 \rangle \) unipotent. So, in this case, we may use a different ordering, if necessary, so that \( \Delta \cap \Sigma_0 \subset \Sigma^- \). That is, \( X_0 \leq X^- \).

We claim that \( X = X^+ X^- = X^- X^+ \). It will suffice to show \( X^- X^+ \subseteq X^+ X^- \). Let \( \alpha \in \Delta^+ \) and \( \beta \in \Delta^- \). The Chevalley commutator relations give \( [\widetilde{U}_\beta, \widetilde{U}_\alpha] \leq \prod_{i,j>0} \widetilde{U}_{i\alpha+j\beta} \). If \( \widetilde{U}_{i\alpha+j\beta} \) is contained in the commutator, then \( i\alpha + j\beta \in \Delta \). Suppose this occurs. If \( i > 1 \), then since \( \Sigma \) is not of type \( G_2 \), \( \alpha \) is short and \( i\alpha + j\beta \) long. By our convention this gives \( i\alpha + j\beta \in \Delta^- \) and \( \widetilde{U}_{i\alpha+j\beta} \leq X^- \). If \( i = 1 \), either \( U_{i\alpha+j\beta} \leq X^- \) or \( \alpha + j\beta = i\alpha + j\beta \in \Delta^+ \), but \( \text{ht}(i\alpha + j\beta) < \text{ht}(\alpha) \). From these remarks we conclude that for \( u_\alpha \in \widetilde{U}_\alpha \) and \( u_\beta \in \widetilde{U}_\beta \), \( \text{ht}(\gamma) \geq \text{ht}(\alpha) \). To prove the claim, let \( \alpha \in \Delta^+ \) and show \( X^- \widetilde{U}_\alpha \subseteq X^+ X^- \) by induction on \( \text{ht}(\alpha) \). Therefore, \( X = X^+ X^- \).

Let \( \tau \) be a field automorphism of \( \widetilde{G} \) with respect to \( \Sigma \), \( \widetilde{T} \), \( \widetilde{U} \), and such that \( \widetilde{G}_\tau = G(q_0) \), where \( q_0 > 4 \). Repeat the above to show that \( Y = \langle (\widetilde{U}_\alpha), \mid \alpha \in \Delta \rangle = \prod_{\delta \in \Delta} (\widetilde{U}_\delta)_\tau \). Therefore, \( Y \) is a \( p \)-subgroup of \( \widetilde{G}_\tau \) normalized by the split torus \( H = \widetilde{T}_\tau \). By (3.12) of [4] we embed \( YH \) in a proper parabolic subgroup \( P \) of \( \widetilde{G}_\tau \) such that \( Y \leq O_p(P) \). Embedding \( YH \) in a Borel subgroup of \( P \), we see that in some new ordering of \( \Sigma \), each root \( \delta \in \Delta \) is positive. In particular, \( X \) is unipotent, proving the result.
(2.12) Let $S \subseteq \Sigma$ and $X = \langle \overline{U}_\alpha | \alpha \in S \rangle$. Let $\Delta = \{ \alpha \in \Sigma | \overline{U}_\alpha \leq X \}$. Set $\Delta_1 = \{ \alpha \in \Delta | -\alpha \notin \Delta \}$ and $\Delta_2 = \{ \alpha \in \Delta | -\alpha \in \Delta \}$. Then

(i) $X_1 = \langle \overline{U}_\alpha | \alpha \in \Delta_1 \rangle = \prod_{\alpha \in \Delta_1} \overline{U}_\alpha$ is unipotent.

(ii) $X_2 = \langle \overline{U}_\alpha | \alpha \in \Delta_2 \rangle$ is semisimple.

(iii) $X = X_1X_2$ with $X_1 \leq X$.

Proof. (i) follows immediately from (2.11). Consider the set $\Delta_2$. Let $\Delta_+ = \Delta_2 \cap \Sigma^+$ and $X_2^+ = \langle \overline{U}_\alpha | \alpha \in \Delta_2^+ \rangle$. Then $X_2^+ \leq \overline{U}$, so $X_2^+$ is unipotent and $X_2^- = \prod_{\alpha \in \Delta_2^-} \overline{U}_\alpha$. It is easy to verify that $\Delta_2$ is a root system. It follows from the Bruhat decomposition that $X_2T$ is a group with a $(B, N)$-pair and $X_2T$ is a Borel subgroup of $X_2T$. This implies (ii).

For (iii) it will suffice to show $[\overline{U}_\alpha, \overline{U}_\beta] \leq X_1$, whenever $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. Suppose $Y = [\overline{U}_\alpha, \overline{U}_\beta]$ and $Y \neq 1$. If $\alpha, \beta$ are long roots, then $Y = \overline{U}_{\alpha + \beta}$ and $\overline{U}^-_{-\alpha} = [\overline{U}^-_{-(\alpha + \beta)}, \overline{U}_\beta]$. So if $\alpha + \beta \in \Delta_2$ we have $\alpha \in \Delta_2$, which is not the case. Now consider the general case, but exclude $\Sigma$ of type $G_2$. Then $\overline{U}_{\alpha + \beta} \leq Y$ and the only possible difficulty is when $[\overline{U}^-_{-(\alpha + \beta)}, \overline{U}_\beta] = 1$. This forces $\alpha + \beta$ and $\beta$ to be short, $K$ of characteristic 2 and $\alpha, \beta$ fundamental roots in a system of type $B_2$. But here $X \geq \langle \overline{U}_{\pm \beta}, \overline{U}_{\pm (\alpha + \beta)}, \overline{U}_\alpha \rangle$ and a direct shows the latter group contains $\overline{U}^-_{-\alpha}$. Again we have a contradiction. Similar arguments work if $\Sigma$ has type $G_2$, and we omit the details.

(2.13) Let $\delta_1, \ldots, \delta_k \in \Sigma^+$ and assume that for each $i \neq j$, $(Z\delta_i + Z\delta_j) \cap \Sigma$ is a root system with $\{\delta_i, \delta_j\}$ as a fundamental set of roots. Assume that the corresponding graph (with vertices $\delta_1, \ldots, \delta_k$) is a Dynkin diagram. Then $\overline{X} = \langle \overline{U}_{\pm \delta_1}, \ldots, \overline{U}_{\pm \delta_k} \rangle$ is a Chevalley group associated with the same Dynkin diagram. Moreover, $\{\delta_1, \ldots, \delta_k\}$ is a fundamental system for the root system of $\overline{X}$.

Proof. For each $1 \leq i \leq k$ let $t_i$ denote the reflection associated with $\delta_i$, and let $W_0 = \langle t_1, \ldots, t_k \rangle$. The roots $\delta_1, \ldots, \delta_k$ are pairwise obtuse, so a standard argument implies that they are linearly independent. Comparing the action of $W_0$ on the $Z$-span of $\{\delta_1, \ldots, \delta_k\}$ with the action of the appropriate Weyl group on the underlying lattice we see that $\Delta = \{\delta_i^{W_0} : i = 1, \ldots, k\}$ is a root system with $\delta_1, \ldots, \delta_k$ as fundamental system.

Since each $t_i$ can be realized by conjugation by an element of $\langle U_{\pm \delta_i} \rangle$, we have $\overline{X} = \langle \overline{U}_{\pm \delta} : \delta \in \Delta \rangle$. We claim that $\Delta$ is a closed system of roots. We have $\Delta$ locally closed in the sense that the root subsystem of $\Delta$ spanned by $\pm \delta_i, \pm \delta_j$ is closed in $\Sigma$ for all $i, j$. On the other hand, any pair of roots in $\Delta$ can be conjugated by an element of $W_0$ into such a local
system. This proves the claim. At this stage the result follows from the Bruhat decomposition and the classification of reductive groups.

In view of the claim we see that $\Gamma$ is a root system and $\{\delta_1, \ldots, \delta_k\}$ a fundamental set for $\Gamma$. This completes the proof.

The following lemma shows that the maximal tori of $G$ are defined unambiguously and will be used implicitly throughout the paper.

(2.14) Let $L$ be of Lie type over $\mathbb{F}_q$. Suppose $L_1, L_2$ are semisimple algebraic groups over $\mathbb{F}_q$ and $\tau_1, \tau_2$ are surjective endomorphisms of $L_1, L_2$, respectively, such that $L = O^p((L_i)_{\tau_i})$ for $i = 1, 2$. For $i = 1, 2$ let $J_i$ denote the set of maximal tori of $L$ defined with respect to $\tau_i$-invariant maximal tori of $L_i$. Then $J_1 = J_2$.

Proof. Write $L_1 = L_{i_1} \cdots L_{i_n}$, where the product is a commuting product of simple algebraic groups. Then $\langle \tau_i \rangle$ acts transitively on $\{L_{i_1}, \ldots, L_{i_n}\}$ for $i = 1, 2$. Let $\bar{X}_1 = \langle \prod_{l=1}^{n_i} l_{i_l}^{-1} : l \in L_{i_1} \rangle$. Then $\bar{X}_1, \bar{X}_2$ are $\tau_i$-invariant images of $L_{i_1}, L_{21}$, respectively, so are connected simple algebraic groups. Moreover, $L \leq \bar{X}_i Z(L_i)$ for $i = 1, 2$. On the other hand $L = O^p(L)$, so $L \leq \bar{X}_i$ for $i = 1, 2$ and we may now replace $L_1, L_2$ by $\bar{X}_1, \bar{X}_2$.

Next argue that we may assume $\bar{X}_1, \bar{X}_2$ are simply connected. It then follows that there is a surjective endomorphism $\gamma$ from $\bar{X}_1$ to $\bar{X}_2$ satisfying $\tau_1 \gamma = \gamma \tau_2$. Then $\gamma$ induces a bijection between the set of $\tau_1$-invariant maximal tori of $\bar{X}_1$ and the set of $\tau_2$-invariant maximal tori of $\bar{X}_2$. It follows that $\gamma |_L$ is an isomorphism with $J_1^\gamma = J_2$. On the other hand, any isomorphism of $L$ can be lifted to an endomorphism of $L_1$ commuting with $\tau_1$. It follows that $J_1^\gamma = J_1$, proving the result.

(2.15) Let $A$ be an abelian $p'$-group acting on a commuting product $Y_1 \cdots Y_k$, where for $1 \leq i \leq k$, $Y_i$ is a group of Lie type over a field of order $p^e$. Assume that $p \geq 5$. Then $A$ normalizes a maximal torus of $Y_1 \cdots Y_k$ (the product of maximal tori, one from each $Y_i$).

Proof. Argue by induction on $|Y_1 \cdots Y_k| | A |$. Clearly we may assume that $A$ is transitive on $\{Y_1, \ldots, Y_k\}$ and that $Z(Y_1 \cdots Y_k) = 1$. If $A_1 = N_A(Y_1)$ and $A_1 < A$, then $A_1 = N_A(Y_i)$ for $1 \leq i \leq k$. Inductively, there is a maximal torus $T_i$ of $Y_i$ with $T_i^{A_1} = T_i$. Then $A$ normalizes $T = \langle T_i^a : a \in A \rangle$ and $T$ is a maximal torus of $Y_1 \cdots Y_k$. So we now assume $k = 1$. 

Regard \( A/C(A(Y_1)) \) as a subgroup of \( \operatorname{Aut}(Y_1) \) and let \( A_0/C_A(Y_1) = A/C_A(Y_1) \cap \tilde{Y}_1 \), where \( \tilde{Y}_1 \) denotes the subgroup of \( \operatorname{Aut}(Y_1) \) generated by inner and diagonal automorphisms. Assume \( A_0 > C_A(Y_1) \) and choose a subgroup \( B \) of \( A_0 \) with \( B \leq A \) and \( |B/C_A(Y_1)| \) a prime. Then \( B \) centralizes maximal tori of \( Y_1 \) and we let \( D \) be the subgroup of \( Y_1 \) generated by all such maximal tori. By (2.9) we have \( D = D_1 \cdot \cdots \cdot D_l I \), where the \( D_i \) are commuting groups of Lie type over extension fields of \( F_{p^e} \), and \( I \) can be taken as any maximal torus centralizing \( B \).

By induction we may assume \( A \leq N(J_1 \cdots J_i) \), where for \( 1 \leq i \leq l \), \( J_i \) is a maximal torus of \( D_i \). Write \( Y_1 = O^p(\tilde{Y}_1) \), where \( \tilde{Y} \) is the corresponding adjoint algebraic group and \( \tau \) an endomorphism of \( \tilde{Y} \). Then \( A \) can be extended to a group of endomorphisms of \( \tilde{Y} \) (automorphisms of the abstract group \( \tilde{Y} \)). Hence \( A \) normalizes the group \( \tilde{C} = C_{\tilde{Y}}(B)^0 \). Write \( \tilde{C} = \tilde{L}_1 \cdots \tilde{L}_s \tilde{I} \), where the \( \tilde{L}_i \) are commuting quasisimple algebraic groups and \( \tilde{I} \) is the \( \tau \)-invariant maximal torus of \( \tilde{Y} \) containing \( I \). We may assume \( J_i \leq \tilde{I} \) for \( i = 1, \ldots, l \). Each of the groups \( D_1, \ldots, D_l \) is the group generated by all \( p \)-elements fixed by \( \tau \) in a particular orbit product of \( \langle \tau \rangle \) on \( \{ \tilde{L}_1, \ldots, \tilde{L}_s \} \). Using (2.6) we see that \( \tilde{I} = C_{\tilde{C}}(J_1 \cdot \cdots \cdot J_l)^0 \). Hence, \( A \leq N(\tilde{I}) \) and so \( A \leq N(\tilde{I} \cap Y_1) = N(I) \). Consequently, we may now assume \( A_0 = C_A(Y_1) \).

Now \( A/C_A(Y_1) = \langle a \rangle \times \langle b \rangle \), where no element of \( \langle a \rangle \) is in the coset of a nontrivial graph automorphism of \( Y_1 \) and \( |b| = s^k \) for \( s = 2 \) or \( 3 \). If \( k > 0 \) then \( \langle b \rangle \) contains the coset of a graph automorphism (either \( a \) or \( b \) could be trivial). By Lang’s theorem ((10.1) of [26]) \( a \) induces a field or graph-field automorphism of \( Y_1 \). It follows from the fact \( p \geq 5 \) that \( A \) centralizes an element \( c \) of \( Y_1 \) with \( |c| = 2 \) or \( 3 \). Consequently, \( A \) normalizes \( C_{\gamma_1}(c) \) and we can argue as in the preceding paragraph, replacing \( B \) by \( \langle c \rangle \). This completes the proof of (2.14).

3. Basic properties. In this section we begin the discussion of \( T \)-root groups. We maintain the notation of §2 and introduce additional notation and terminology as follows. Set \( G_0 = O^p(G) \) and \( T_0 = T \cap G_0 \). If \( G_0 \leq G_1 \leq G \), then a group of the form \( T \cap G_1 \) is called a maximal torus of \( G_1 \). Let \( \Delta = \{ \tilde{U}_\alpha \mid \alpha \in \tilde{\Sigma} \} \), the root subgroups of \( \tilde{G} \) with respect to \( \tilde{T} \). Set \( \tilde{U} = \langle \tilde{U}_\alpha \mid \alpha \in \tilde{\Sigma}^+ \rangle \) and \( \tilde{B} = \tilde{U} \cdot \tilde{T} \), a Borel subgroup of \( \tilde{G} \) (not necessarily \( \sigma \)-invariant).

For \( \alpha \in \tilde{\Sigma} \), regard \( \tilde{U}_\alpha \) as a 1-dimensional \( K \) representation of \( \tilde{T} \). Let \( \varphi_\alpha \) denote this representation (\( \varphi_\alpha \) equals \( \alpha \) if we regard \( \alpha \) as a character of \( \tilde{T} \)).

Since \( \Delta \) is the set of minimal \( \tilde{T} \)-invariant unipotent subgroups of \( \tilde{G} \), \( \Delta = \Delta^a \), so \( \Delta = \Delta_1 \cup \cdots \cup \Delta_6 \), a union of \( \langle \sigma \rangle \)-orbits. Correspondingly,
we set $\Sigma_i = \{\alpha \mid U_\alpha \in \Delta_i\}$. For $i = 1, \ldots, v$ we set $X_i = \langle \Delta_i \rangle$ and $X_i = O^p((X_i)_\sigma)$. The groups $X_1, \ldots, X_v$ are the $T$-root subgroups of $G$. We note that for $i = 1, \ldots, v$, $X_i \leq O^p(G) = G_0$. For $1 \leq i \leq v$, there is a unique $j \in \{1, \ldots, v\}$ such that $\Sigma_j = -\Sigma_i$. We set $X_j = X_j^*$, $X_i = X_i^*$, and $\Delta_j = \Delta_i^*$.

The first result is that a $T$-root subgroup is either a $p$-group or a group of Lie type defined over an extension field of $F_q$.

(3.1) Fix $i = 1, \ldots, v$. The group $X_i$ is either unipotent or semisimple. Correspondingly, $X_i$ is either a $p$-group or $X_i$ is a group of Lie type defined over an extension field of $F_q$.

**Proof.** Consider the group $X_i$. We may assume that $X_i$ is not unipotent. But $X_i = X_i^0$ and $X_i$ is $T$-invariant. Let $\overline{X_i}T = \overline{Q} \cdot \overline{L}$, where $\overline{Q} = R_u(\overline{X_i})$ and $\overline{L}$ is the product of $\overline{X}$ with those root subgroups $U_\alpha$ such that $\overline{U}_\alpha$ and $\overline{U}_-\alpha$ are both contained in $\overline{X}$ (see (2.12)). We have $\overline{L} \cap \overline{Q} = 1$. Also, each root subgroup of $X_i$ is contained in either $\overline{L}$ or $\overline{Q}$. Since $\sigma$ normalizes each of $\overline{L}$ and $\overline{Q}$, we conclude that $\langle \sigma \rangle$ is transitive on $\{\overline{L}_1, \ldots, \overline{L}_k\}$ and $X_i/Z(X_i) \cong O^p((\overline{L}_1)_\sigma)/Z(O^p((\overline{L}_1)_\sigma))$. The argument in (2.6) of [23] shows that $X_i/Z(X_i)$ is associated with $F_{q^k}$, completing the proof of (3.1).

Order the $T$-root groups so that $X_1, \ldots, X_t$ are $p$-groups and $X_{t+1}, \ldots, X_v$ are groups of Lie type. We note that if $\overline{T}$ is contained in a $\sigma$-stable Borel subgroup of $\overline{G}$, then $t = v$ and $\{X_1, \ldots, X_v\}$ are the usual root subgroups of $G$. We also point out that there may be containments among the $X_i$'s. This even occurs when $T$ is a Cartan subgroup of $G$. For example, in the case of $PSU(n, q)$ with $n$ odd, there is a non-abelian root subgroup $E$ of order $q^3$ and $Z(E)$ is also a root group.

The next result gives bounds on the nilpotence class of the groups $X_1, \ldots, X_t$. First, we require the following (temporary) notation. Let $\overline{H} = \overline{H}_\sigma$ be a maximal torus of $\overline{G}$ with $\overline{H} \leq \overline{B}_1 = \overline{B}_1^\sigma$, where $\overline{B}_1$ is a Borel subgroup of $\overline{G}$. Let $\hat{\Sigma}$ be the root system of $\overline{G}$, with respect to $\overline{H}$, and $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_n\}$ a fundamental set of roots. For $\alpha \in \hat{\Sigma}$, write $\hat{\alpha} = \Sigma n_i \hat{\alpha}_i$ and let $c(i, \hat{\alpha}) = \Sigma n_j$, the sum ranging over those $j$ with $\alpha_j \in \alpha_i^{(\sigma)}$. Let $c = \max\{c(i, \hat{\alpha}) : \hat{\alpha} \in \hat{\Sigma}, 1 \leq i \leq n\}$.

(3.2) If $1 \leq i \leq t$, then $X_i$ has nilpotence class at most $c$.

**Proof.** By either (3.9) of [3] or by [4] there exist a parabolic subgroup $\overline{P}_1 \leq \overline{G}$ such that $\overline{X}_i \leq R_u(\overline{P}_1)$ and $N_{\overline{G}}(\overline{X}_i) \leq \overline{P}_1$. Moreover, $\overline{P}_1$ is obtained
canonically from $X_i = \bar{X}_i^\sigma$, so $P_\lambda = \bar{P}_1^\sigma$. Therefore, we may assume that $\bar{B}_1 \leq \bar{P}_1$.

Let $\bar{P}_1 \leq \bar{P}_0$, where $\bar{P}_0$ is a maximal parabolic subgroup of $\bar{G}$. Let $\bar{P} = \cap_i P_0^\sigma$. Notice that $\bar{B}_1 \leq \bar{P}$ and $P_0^{(\sigma)}$ has size 1, 2, or 3, with the latter case possible only for $\bar{G} = D_4(K)$. Now $R_u(\bar{P}) \leq R_u(P_1)$. Indeed, $R_u(\bar{P}) = \prod R_u(P_0^\sigma)$. $R_u(\bar{P}_1)$ is a product of root subgroups for $\bar{T}$. Choose $P_0$ such that some element $\bar{U}_\alpha$ of $\Delta$ is such that $\alpha$ has non-zero coefficient of the fundamental root defining $\bar{P}_0$. It follows that $\bar{U}_\alpha \leq R_u(\bar{P})$, and hence $X_i \leq R_u(\bar{P})$.

We now know that $X_i$ has nilpotence class bounded by that of $R_u(\bar{P})$. $R_u(\bar{P})$ is also the product of root groups of $\bar{G}$ with respect to the maximal torus $\bar{H}$. Viewing $R_u(\bar{P})$ in this way and using the Chevalley commutator relations, the result follows.

The table below gives bounds on the nilpotence class of the groups $X_i$, $1 \leq i \leq t$, for the various groups $G$. This bound is $c$ except for the cases of $G = Sz(q)$, $^2G_2(q)$, and $^2E_6(q)$. In the latter cases the bounds are less than $c$. This is because the characteristic restrictions needed to define $G$ force the appropriate parabolic subgroups of the above proof to have unipotent radicals of smaller nilpotence class. In these cases direct computations give the bounds. Otherwise, the number $c$ is computed easily once the root system is given; for $\sigma$ induces a (possibly trivial) graph automorphism on the root system for $\bar{H}$.

<table>
<thead>
<tr>
<th>$G_0/Z(G_0)$</th>
<th>bound on class $X_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{PSL}(n,q)$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{Psp}(n,q)$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{PSU}(n,q)$</td>
<td>2</td>
</tr>
<tr>
<td>$PO^\pm(n,q)$</td>
<td>2</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>3</td>
</tr>
<tr>
<td>$E_7(q)$</td>
<td>4</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>6</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>3</td>
</tr>
<tr>
<td>$F_4(q)$</td>
<td>4</td>
</tr>
<tr>
<td>$^3D_4(q)$</td>
<td>3</td>
</tr>
<tr>
<td>$^2E_6(q)$</td>
<td>4</td>
</tr>
<tr>
<td>$Sz(q)$</td>
<td>2</td>
</tr>
<tr>
<td>$^2G_2(q)$</td>
<td>3</td>
</tr>
<tr>
<td>$^2F_4(q)$</td>
<td>5</td>
</tr>
</tbody>
</table>
It will be a consequence of later work that the above bounds are best possible. Also, we will discuss the embedding of $T$-root groups in $G$, and for the classical groups we describe the action of $T$-root groups on the natural module.

Our next two results concern the embedding of $X_i$ in $G$ and the embedding of $\overline{X}_i$ in $\overline{G}$, for $i = 1, \ldots, t$.

(3.3) Let $1 \neq C$ be a unipotent group generated by a subset of $\{\overline{X}_1, \ldots, \overline{X}_t\}$, and let $C$ be the subgroup of $G$ generated by the corresponding subset of $\{X_1, \ldots, X_t\}$. Then $C$ is a $p$-group. There is a parabolic subgroup $\overline{P} = P^\sigma \geq \overline{T}$ of $\overline{G}$, such that $C \leq R_u(\overline{P})$. The group $\overline{P} = P$ is a parabolic subgroup of $G$ satisfying $C \leq O_p(P)$, and $T$ is contained in a conjugate of a Levi factor of $P$.

**Proof.** Since $\overline{C}$ is unipotent, there is a canonical parabolic subgroup $\overline{P} \leq \overline{G}$ with $\overline{C} \leq R_u(\overline{P})$ (Borel-Tits, (3.9) of [3]). Then $\overline{P} = P^\sigma$ and $\overline{T} \leq N_{\overline{C}}(\overline{P}) = \overline{P}$. To see that $P = \overline{P}^\sigma$ is a parabolic subgroup of $G$ first use Lang's theorem to get a Borel subgroup $\overline{J}$ of $\overline{P}$ (hence of $\overline{G}$) stabilized by $\sigma$ and then use (2.12) of [25] to conclude $\overline{J} = \sigma$ is a Borel subgroup of $G$. This forces $P$ to be a parabolic subgroup of $G$. Clearly $C \leq R_u(\overline{P}) \leq O_p(P)$. Choose $x \in \overline{P}$ such that $\overline{T} = \overline{J}$ is a product of some of the root subgroups $\overline{U}_\alpha$ for $\alpha \in \Sigma$ and the Levi factor of $\overline{P}$ is generated by $\overline{T}$ together with those $\overline{U}_\alpha \leq \overline{P}$ such that $\overline{U}_\pm \leq \overline{P}$. So $\sigma$ stabilizes the Levi factor and the result follows.

(3.4) Let $i \in \{1, \ldots, t\}$, let $\overline{X} = \overline{X}_i$, and $X = \overline{X}_\sigma$. Choose $j \in \{1, \ldots, t\}$ so that $\overline{\Sigma}_j = -\overline{\Sigma}_i$, and set $X^* = (\overline{X}_j)^\sigma$. Then

(i) $\overline{Y} = \langle \overline{X}_i, \overline{X}_j \rangle = \overline{D}_1 \cdots \overline{D}_k$, a commuting product of $\langle \sigma \rangle$-conjugate, semisimple groups, each generated by certain root subgroups of $\overline{T}$.

(ii) $Y = \langle \overline{X}_i, \overline{X}_j \rangle = Y(q^k)$, a group of Lie type defined over $F_{q^k}$.

(iii) There exists a unique $\overline{T} \langle \sigma \rangle$-stable parabolic subgroup $\overline{P}_0$ of $\overline{Y}$ such that $\overline{X} \leq R_u(\overline{P}_0)$. Also, $\overline{T} \leq N_{\overline{C}}(\overline{P}_0)$.

(iv) $\overline{P}_0$ is the intersection of an $\langle \sigma \rangle$-orbit of maximal parabolic subgroups of $\overline{Y}$. If $\sigma$ induces a field automorphism of $\overline{G}$, then $\overline{P}_0$ is a maximal parabolic subgroup of $\overline{Y}$.

(v) Suppose $q \geq 4$. Then $P_0 = (\overline{P}_0) = $ the unique parabolic subgroup of $Y$ normalized by $T$ and satisfying $X \leq O_p(P_0)$.

(vi) Suppose $q \geq 4$. Then $T \cap Y$ is a maximal torus in $Y$, $T \cap Y/Z(Y)$ is cyclic and there is a Levi factor of $P_0$ in which $T \cap Y$ is a minisotropic torus.

**Proof.** Let $\overline{Y} = \langle \overline{X}_i, \overline{X}_j \rangle$. Then $\overline{Y}$ is normalized by $\overline{T}$. Let $\Gamma$ be the collection of all roots $\alpha \in \Sigma$ such that $\overline{U}_\pm \leq \overline{Y}$. Then $\Gamma^\sigma = \Gamma$ and by
(2.12) \( \langle \overline{U}_a | \alpha \in \Gamma \rangle \) is a semisimple subgroup of \( \overline{Y} \). Since \( \Sigma_j, \Sigma_j \leq \Gamma \) we have \( \overline{Y} = \langle \overline{U}_a | \alpha \in \Gamma \rangle \) and \( \overline{Y} \) can be expressed \( \overline{Y} = \overline{D}_1 \cdots \overline{D}_k \), with each \( \overline{D}_i \) quasisimple and generated by certain of the root subgroups in \( \Gamma \). As \( \overline{T} \) is connected, \( \overline{T} \leq N_G(\overline{D}_i) \) for \( i = 1, \ldots , k \). Also, each \( \gamma \in \Sigma_i \) satisfies \( \overline{U}_y \leq \overline{D}_i \) as well, we conclude that \( \langle \sigma \rangle \) is transitive on \( \{ \overline{D}_1, \ldots , \overline{D}_k \} \). This gives (i). The argument in (2.6) of [23] shows that (ii) holds.

We may write \( \overline{Y} \cdot \overline{T} = \overline{D}_1 \cdots \overline{D}_k \overline{Z} \), where \( \overline{Z} \) is a subtorus of \( \overline{T} \) and \( \overline{Z} = Z(\overline{Y} \cdot \overline{T})^0 \). Now \( \sigma^k \) stabilizes each of \( \overline{D}_1, \ldots , \overline{D}_k \) and we observe that for the purpose of proving the remaining parts of (3.4) we may replace \( (G, T, \sigma) \) by \( (\overline{D}_1 / Z(\overline{D}_1), \overline{T} \cap \overline{D}_1 / Z(\overline{D}_1), \sigma^k) \). Therefore, we now assume that \( \overline{Y} = \overline{G} \). In particular, \( C_{\overline{T}}(\overline{X}) = 1 \).

Let \( \overline{P} \) be a parabolic subgroup of \( \overline{G} \) such that \( \overline{X} \leq R_{\overline{u}}(\overline{P}) \leq \overline{P} = \overline{P}^w \geq \overline{T} \) (see (3.3)). Conjugating, if necessary, we may assume that \( \overline{X} \leq \overline{B} \leq \overline{P} \), where \( \overline{B} \) is the Borel subgroup \( \overline{T} \langle \overline{U}_a | \alpha \in \overline{\Sigma}^+ \rangle \). So \( \overline{P} = \langle \overline{B}, \overline{U}_a \rangle \) and \( S = \{ \alpha_i, \ldots , \alpha_n \} \) is a subset of \( \Pi = \{ \alpha_1, \ldots , \alpha_n \} \). Now we may write \( \sigma = w_\gamma \), where \( w \in \overline{W} = \overline{N}/\overline{T} \) and \( \gamma \) is a field or graph-field automorphism of \( \overline{G} \), defined with respect to \( \overline{B} \). Then \( \overline{P}^w = \overline{P}^w = \overline{P}^w \gamma^{-1} \). Since \( \overline{B} \gamma^{-1} = \overline{B} \), \( \overline{P} \gamma^{-1} \) is also a standard parabolic subgroup for \( \overline{B} \). Thus \( \overline{P}, \overline{P}^w \) are both parabolics containing \( \overline{B} \) and this forces \( \overline{P} = \overline{P}^w \), whence \( w \in \overline{P} \). Also, \( \gamma \in N(\overline{P}) \). We claim that the permutation of \( \overline{\Sigma} \) associated with \( \gamma \) is transitive on \( S \).

If \( G \) is a Suzuki or Ree group, then this is clear from \( \overline{P} = \overline{P}^\gamma \), unless \( \overline{G} \) is of type \( F_4(K) \) and \( \overline{P} \) a Borel subgroup. But in this case \( X \) is an ordinary root group of \( G \), contradicting \( \langle \overline{X}_i, \overline{X}_i \rangle = G \). We now exclude the Ree and Suzuki groups for purposes of establishing the claim. For \( \alpha \in \overline{\Sigma}_i \), \( \overline{U}_a \leq R_u(\overline{P}) \) and we let \( (\alpha) \) be the coefficient of \( \alpha_j \) in \( \alpha \). Since \( w \in \overline{P} \), \( (\alpha)_i = (\alpha^w)_i \) for each \( \alpha \in \overline{\Sigma}_i \).

Write \( S = S_1 \cup \cdots \cup S_m \), where the union is disjoint, each \( S_i \) is a \( \langle \gamma \rangle \)-orbit, and suppose \( m > 1 \). For \( \alpha \in \overline{\Sigma}_i \), let \( \alpha(k) = \Sigma_{\alpha_j \in S_k} (\alpha)_j \). Then \( \alpha(k) = \beta(k) \) for each \( \alpha, \beta \in \overline{\Sigma}_i \). Choose \( \chi(\alpha_j) = 1 \) for each \( \alpha_j \in S_1 \cup S_2 \), \( \chi(\alpha_j) = \varphi \) for each \( \alpha_j \in S_1 \), and \( \chi(\alpha_j) = \eta \) for each \( \alpha_j \in S_2 \). Consideration of the numbers \( \alpha(1) \) and \( \alpha(2) \), for \( \alpha \in \overline{\Delta}_1 \), shows that it is possible to choose \( \chi(1) \neq \chi \) such that \( \chi(\alpha) = 1 \) for each \( \alpha \in \overline{\Sigma}_i \). But then \( h(\chi) \in C_{\overline{T}}(\overline{X}) = 1 \), a contradiction. This proves the claim.

The claim gives (iv), once (iii) is checked. So suppose \( \overline{P}_1 \) is another \( \overline{T}(\sigma) \)-stable parabolic subgroup of \( \overline{G} \) with \( \overline{X} \leq R_u(\overline{P}_1) \). Then \( \overline{P}_1 = \overline{P}_2 \) where \( \overline{B} \leq \overline{P}_2 \). So \( \overline{T}, \overline{T}^w \leq \overline{P}_2 \) and, conjugating, we may assume that \( g \in N_G(\overline{T}) = \overline{N} \). So \( \overline{P}_1 = \overline{P}_2^w \) for some \( w \in \overline{W} \). But now consider \( \overline{P}_1 \cap \overline{P}_2^w \) of this group is \( \overline{T}(\sigma) \)-invariant and we apply the results in §2 of [8]. Write
\( \overline{P} = \overline{P}_1 \) and \( \overline{P}_2 = \overline{P}_{J_2} \) where \( J_1, J_2 \subseteq \Pi \), and \( w = w_2w'w_1 \), where \( w_2 \in W_{J_2}, w_1 \in W_{J_1} \), and \( w' \in W_{J_2,J_1} \), the distinguished set of double coset representatives. Then \( \overline{P} \cap \overline{P}_2^w = (\overline{P}_1 \cap \overline{P}_2^w)^w \), and we consider the group \( P_{J_1} \cap P_{J_2}^w \). Let \( K = J_1 \cap J_2^w \). By (2.4) of [8], \( \overline{P}_K = (\overline{P}_1 \cap \overline{P}_2^w)R_u(\overline{P}_J) \), so \( \overline{X} \leq \overline{P} \cap \overline{P}_2^w = (\overline{P}_1 \cap \overline{P}_2^w)^w \leq \overline{P}_K^w \) and \( \overline{P}_K \) is a parabolic subgroup of \( \overline{P}_1 = \overline{P} \). Since \( \overline{P}_K^w \) is \( T(\sigma) \)-stable we apply the above claim to conclude that \( \overline{P}_K^w = \overline{P}_1^w = \overline{P} \). Thus, \( w_1 \in \overline{P}_K, \overline{P}_K = \overline{P}_{J_1}, \) and \( J_1 \subseteq J_2^w \). By (2.6) of [8], \( L_{J_1} \leq L_{J_2}^w \), where \( L_{J_1} \) and \( L_{J_2}^w \) are the standard Levi factors of \( \overline{P}_{J_1} \) and \( \overline{P}_{J_2}^w \), respectively. Reversing the roles of \( \overline{P} \) and \( \overline{P}_2^w \) we have \( L_{J_1} = L_{J_2}^w = L_K \) and \( J_1 = J_2^w \).

Fix \( \alpha_i \in \pi - J_1 \). By (2.4) of [9] we see that \( W_K \) is transitive on the set of roots, \( \alpha \), such that \( \alpha \) has coefficient equal to 1 and \( \alpha_j \) coefficient 0 for each \( \alpha_j \neq \alpha_i \in \pi - J_1 \). Using this together with the fact that \( \pi - J_1 \) is a \( \langle \sigma \rangle \)-orbit we have \( W_K \langle \sigma \rangle \) transitive on those roots \( \beta \in \Sigma \) satisfying \( \Sigma_{\alpha_i \in J_1}(\beta)_j = 1 \). For the remainder of the proof, if \( \beta \in \Sigma \), let \( (\beta)_j \) denote the integer \( \Sigma_{\alpha_i \in J_1}(\beta)_j \). Let \( o = \{ \beta | (\beta)_j = 1 \} \), a \( W_K \langle \sigma \rangle \)-orbit of roots.

We claim that \( \Sigma_i \subseteq o \). For suppose \( \alpha \in \Sigma_i \) and \( (\alpha)_j = c > 1 \). Let \( \Gamma \) be the collection of all \( \gamma \in \Sigma \) with \( (\gamma)_j \) a multiple of \( c \). It is easily checked that \( \Gamma \) is a closed root system of \( \Sigma \) and this contradicts the fact that \( G = \langle \overline{X}_i, \overline{X}_j \rangle \). So \( \Sigma_i \subseteq o \). Now, \( \overline{P}_2^w = \overline{P}_2^w = \overline{P}_2^{w_1} = \overline{P}_2^{w_2} \), since \( w_1 \in L_K = L_{J_2}^w \). So, by symmetry, there is a \( W_K \langle \sigma \rangle \)-orbit \( o' \) such that \( \Sigma_i \subseteq O' \) and \( (\pi - J_2)^w \subseteq o' \). But then, \( o = o' \) and we have \( \pi \langle o = o \rangle = J_2^w \) and \( (\pi - J_2)^w \subseteq J_1 \subseteq o \subseteq \Sigma^+ \). Therefore, \( (\Sigma^+)^w = \Sigma^+ \) and \( w' = 1 \). We now have \( \overline{P}_1 = \overline{P}_2 = \overline{P}_2^w = \overline{P}_2^{w_1} = \overline{P}_2^{w_2} = \overline{P}_2^{w_3} = \overline{P}_1 \). This proves (iii).

Suppose \( X \leq O_p(P_1) \) and \( q \geq 4 \), where \( P_1 \) is a \( T \)-stable parabolic subgroup of \( G \). Then \( P_1 = N_G(O_p(P_1)) \), so by (3.9) of [3], there is a canonical parabolic subgroup \( \overline{P}_1 \) of \( G \) such that \( P_1 \leq \overline{P}_1 \) and \( O_p(P_1) = R_u(\overline{P}_1) \). Then \( T \leq \overline{P}_1 \) and we claim \( \overline{T} \leq \overline{P}_1 \). To see this first note that if \( h(\chi) \in T \), then \( \chi^a = \chi \) and hence given \( \alpha, \beta \in \Delta_i, \phi_\alpha = \phi_\beta^a \) for some integer \( k \) (a slight change is required for the Suzuki and Ree groups). Therefore, \( C_T(\overline{U}_\alpha) = C_T(\overline{U}_\beta) \) for each \( \alpha, \beta \in \Sigma_i \), and since \( C_T(\overline{X}) = 1 \), we necessarily have \( T \) cyclic. Write \( T = \langle t \rangle \). Then \( t \) is semisimple and is contained in a maximal torus \( \overline{T}_1 \) of \( \overline{P}_1 \). Then \( \overline{T}_1 \leq C_{\overline{G}}(T)^0 = \overline{T} \) by (2.7). So \( \overline{T} = \overline{T}_1 \) and \( \overline{P}_1 = \overline{P}_0 \) by (iv). Therefore, \( P_1 \leq (P_0)_a = (\overline{P}_0)_a = P_0 \). If equality failed to hold, then \( O_p(P_1) > O_p(P_0) \). However, \( O_p(P_1) \leq R_u(\overline{P}_1) = R_u(\overline{P}_0) \), so this is impossible. This proves (v) and (vi) follows from (v) and the above argument.

In §4 we will describe the \( T \)-root subgroups of the classical groups and also the groups \( Y \) and \( P_0 \) of (3.4). One additional result of interest is the following.
(3.5) The set \( \{ \Sigma_1, \ldots, \Sigma_t \} \) can be partitioned as \( \{ \Sigma_{i_1}, \ldots, \Sigma_{i_k} \} \cup \{ \Sigma_{j_1}, \ldots, \Sigma_{j_k} \} \) in such a way that each of \( \langle X_{i_1}, \ldots, X_{i_k} \rangle \) and \( \langle X_{j_1}, \ldots, X_{j_k} \rangle \) is unipotent.

Proof. Let \( \overline{P}_1 \) be the canonical parabolic subgroup of \( \overline{G} \) such that \( \overline{X}_i \leq R_u(\overline{P}_1) \) and \( N_{\overline{G}}(\overline{X}_i) \leq \overline{P}_1 \) (see (3.9) of [4]). Then \( \overline{P}_1 = \overline{P}_1^\sigma \) and \( T \leq \overline{P}_1 \).

For each \( \alpha \in \Sigma \), either \( U_{\alpha} \) or \( U_{-\alpha} \) is a subgroup of \( \overline{P}_1 \). So for each \( 1 \leq i \leq t \), either \( \overline{X}_i \) or \( \overline{X}_i^* \) is a subgroup of \( R_u(\overline{P}_1) \) or both of \( \overline{X}_i \) and \( \overline{X}_i^* \) are contained in the Levi factor, \( \overline{L}_1 \), of \( \overline{P}_1 \). Inductively, we can partition the roots in the root system for \( \overline{L}_1 \) so that the result holds in \( \overline{L}_1 \). Now we obtain \( \{ \Sigma_{i_1}, \ldots, \Sigma_{i_k} \} \) by taking the \( \langle \sigma \rangle \)-orbits in one of the partitioning sets for the root system of \( \overline{L}_1 \), together with those \( \Sigma_\ast \) such that \( \overline{X}_i \leq R_u(\overline{P}_1) \). Passing to the opposite parabolic subgroup of \( \overline{P}_1 \), we see that the result holds.

In the notation of (3.5), the group \( \langle \overline{X}_{i_1}, \ldots, \overline{X}_{i_k} \rangle \sigma T \) can be regarded as a replacement for a Borel subgroup. However, there may be several ways to obtain partitions as in (3.5).

(3.6) Assume \( q \geq 4 \) and that \( Y \) is a group acting on \( G_0 \) and normalizing \( T_0 \). Then \( Y \) permutes the set of \( T \)-root subgroups of \( G \).

Proof. We may assume \( Z(G_0) = 1 \), \( Y \) acts faithfully on \( G_0 \), and \( Y = \langle y \rangle \) for some automorphism \( y \) of \( G_0 \). There is an endomorphism \( \tau \) of \( \overline{G} \) such that \( [\tau, \sigma] \) and \( \tau |_{G_0} = y \). So \( \tau \) normalizes \( T_0 \) and (2.6) implies that \( \tau \) normalizes \( \overline{T} \). Consequently, \( \tau \) permutes the set of \( \langle \sigma \rangle \)-orbits of \( \overline{T} \)-root subgroups. The result follows.

4. Classical groups. In this section we determine the \( T \)-root subgroups \( X_1, \ldots, X_t \), when \( G \) is a classical group. Choose notation as in §3 and fix \( 1 \leq i \leq t \). Set \( \Delta = \Delta_i \), \( \overline{X} = \overline{X}_i \), \( X = X_i \), \( X^* = X_i^* \). In addition, we set \( \overline{Y} = \langle \overline{X}, \overline{X}^* \rangle \), \( Y = \overline{Y}_\sigma \), and \( P, P \) the parabolic subgroups of \( \overline{Y} \) and \( Y \), respectively, as described in (3.4).

To make the statements and proofs less cumbersome we will assume throughout the section that \( G \) and \( \overline{G} \) are the appropriate linear groups. However, it is easy to pass from these results to those for other forms of \( G \) and \( \overline{G} \).

(4.1) If \( G_0 = \text{SL}(n, q) \), then there exist positive integers \( r, s, y \) such that the following hold:

(i) \( (r, s) = 1 \).
(ii) $X$ is elementary abelian of order $q^{(r-s)sy}$.

(iii) $\langle X, X^* \rangle \cong \SL(r, q^y)$.

(iv) $\langle \bar{X}, \bar{X}^* \rangle = \bar{D}_1 \cdots \bar{D}_y$ a commuting product of copies of $\SL(r, K)$, permuted transitively by $\langle \sigma \rangle$ and each generated by $T$-root subgroups of $\bar{G}$.

(v) $X = O_p(P)$ and $P$ is the stabilizer of an $s$-space of the usual module for $\SL(r, q^y)$.

(vi) The projection of $\bar{X}$ to $\bar{D}_1$ is the unipotent radical of a parabolic subgroup of $\bar{D}_1$ obtained by deleting the $(s)$th node of the Dynkin diagram for $\bar{D}_1$ (which has type $A_{r-1}$).

Proof. Write $\langle \bar{X}, \bar{X}^* \rangle = \bar{D}_1 \cdots \bar{D}_y$ with $\langle \sigma \rangle$ transitive on $\{ \bar{D}_1, \ldots, \bar{D}_y \}$ (see (3.5)(i)). Let $\gamma = \sigma^y$, so that $\gamma$ stabilizes each of $\bar{D}_1, \ldots, \bar{D}_y$. Since the root system of each $\bar{D}_j$ is a subsystem of $\bar{\Sigma}$, and since $\bar{\Sigma}$ has type $A_{n-1}$, there is an integer $r$ such that $\bar{D}_j \cong \SL(r, K)$, for $j = 1, \ldots, t$. So (iv) holds.

We first observe that $\gamma$ induces a field automorphism on $\bar{D}_1, \ldots, \bar{D}_y$. To see this write $\sigma = \tau q$ as in (2.1). Since $\sigma$ is a field automorphism of $\bar{G}$, $\tau = w \in \bar{W}$, so $\gamma = w^y q^y$. Now if we set $W_j = \bar{W} \cap \bar{D}_j$, then $N_{\bar{W}}(W_j) = \bar{W}/C_{\bar{W}}(W_j)$. It follows that $w^y$ induces an inner automorphism on $\bar{D}_j$; hence $(\bar{D}_j)_\gamma \cong \SL(r, q^y)$. At this point (iii) will follow from (v) and (2.3) of [23].

If we can verify (i) and (vi), then the remaining parts of (4.1) will follow, by projection, from the known structure of parabolic subgroups of type $A_{r-1}$ and the connection between parabolic subgroups of $\bar{D}_1$ and $(\bar{D}_1)_\gamma$. Hence we have reduced the problem to the case of $G = \SL(r, K)$, $G_\sigma = \SL(r, q^y)$, and $\langle \bar{X}, \bar{X}^* \rangle = \bar{G}$.

By (3.4)(iii), $\bar{X} \leq R_u(\bar{P})$. Regard $\bar{P}$ as the stablizer in $\bar{G}$ of an $s$-space of the usual module for $\SL(n, K)$ (here we are using (3.5)(iv)). Then $\bar{P}/R_u(\bar{P})$ is a central product of $\SL(s, K), \SL(r-s, K)$, and a 1-dimensional torus. Each of these groups is stabilized by $\sigma$. From the uniqueness of $\bar{P}$ we conclude that $T$ is a minisotropic torus of $P/O_p(P)$, so $T$ contains the central product of cyclic groups of order $(q^{ys} - 1)/(q^y - 1), (q^{y(r-s)} - 1)/(q^y - 1)$, and $q^y - 1$ (see Carter [5]).

Let $\alpha \in \Delta_i$. Then $T$ induces a cyclic group on $U_\alpha$ and induces algebraic conjugates of $\phi_\alpha$ on the other root subgroups in $\Delta_i$. Since $R_u(\bar{P})$ is abelian, $\bar{X}$ is the direct product of the groups $U_\alpha, \alpha \in \bar{\Sigma}_i$ and $T$ induces a cyclic group on $\bar{X}$. Then $C_{\bar{T}}(\bar{G}) = Z(\bar{G})$ implies $C_{\bar{T}}(\bar{X}) = Z(\bar{G})$ (as $\bar{G} = \langle \bar{X}, \bar{X}^* \rangle$ and $C_{\bar{T}}(\bar{X}^*) = C_{\bar{T}}(\bar{X})$) and so $T/Z(G)$ is cyclic. From this and the above description of $T$ we have $(s, r-s) = 1$. For if $(s, r-s) = d > 1$, then $Z_a \times Z_a \leq T/Z(G)$, where $a = (q^{jd} - 1)/(q^d - 1)$. This proves (i).
The group $R_u(P)$ is the product of $s(r - s)$ $\bar{T}$-root subgroups of $\bar{G}$. Since $T$ is a minisotropic torus of $P$, $\sigma$ acts as $w_1w_2q$ where for $i = 1, 2$, $w_i$ is a Coxeter element of the corresponding component, $\text{SL}(s, K)$ or $\text{SL}(r - s, K)$, of the Levi factor of $P$. Thus $|w_1| = s$, $|w_2| = r - s$ and $\langle w_1, w_2 \rangle$ is transitive on the set of $\bar{T}$-root subgroups in $R_u(P)$. By (i), $\langle w_1w_2 \rangle = \langle w_1w_2 \rangle$ so $\bar{X} = R_u(P)$ and this proves (vi). This completes the proof of (4.1).

A result quite similar to (4.1) holds when $G$ is replaced by an arbitrary classical group, although there does exist one ambiguity (which is cleared up in §12, but under additional hypotheses).

(4.2) Suppose that $G$ is one of the groups: $\text{Sp}(n, q)$, $\text{SU}(n, q)$, or $\text{SO}^\pm(n, q)'$. There exist positive integers $r$, $s$, $y$, such that the following hold:

(i) Either $(r, s) = 1$ or $r = 2s$.

(ii) If $r \neq 2s$, then $X/X'$ is elementary abelian of order $q^{ys(r - 2s)}$ ($q^{2ys(r - 2s)}$ if $Y$ is a unitary group).

(iii) $\langle \bar{X}, \bar{X}^* \rangle = \bar{D}_1 \cdots \bar{D}_y$, a commuting product of copies of one of the groups $\text{SL}(r, K)$, $\text{Sp}(r, K)$, or $\text{O}^\pm(r, K)'$. Also, $\langle \sigma \rangle$ is transitive on $\{\bar{D}_1, \ldots, \bar{D}_y\}$ and each $\bar{D}_i$ is generated by $\bar{T}$-root subgroups of $\bar{G}$.

(iv) If $r \neq 2s$, the $\langle X, X^* \rangle \cong \text{SL}(r, q^y)$, $\text{SU}(r, q^y)$, $\text{Sp}(r, q^y)$, or $\text{O}^\pm(r, q^y)$.

(v) If $r \neq 2s$, then $X = O_p(P)$ and $P$ is the stabilizer of a totally isotropic (singular) $s$-space of the usual module for $\langle X, X^* \rangle$.

(vi) If $r = 2s$ and $q \geq 4$, then $X$ is elementary abelian of order $q^{ys}$ ($q^{2ys}$ if $Y$ is a unitary group), $P$ is the stabilizer of a totally isotropic (singular) $s$-space of the usual module for $Y$, and $X \leq O_p(P)$, equality only if $s = 2$ and $\langle X, X^* \rangle$ a unitary group.

Proof. We first make reductions as in the proof of (4.1). Let $\langle \bar{X}, \bar{X}^* \rangle = \bar{D}_1 \cdots \bar{D}_y$, a central product. From (3.5)(i) we get (iii). Then each $\bar{D}_i$ is a classical group, although perhaps of a different type than that of $\bar{G}$. As before, the element $\sigma^y$ stabilizes each $\bar{D}_i$, but it need not be the case that $\sigma^t$ induces a field automorphism on each $\bar{D}_i$. Possibly $\sigma^t$ induces a graph-field automorphism on $\bar{D}_i$, $i = 1, \ldots, y$. In any case, we now project to $\bar{D}_1$, as before. That is we assume $\langle \bar{X}, \bar{X}^* \rangle = \bar{G}$ and $\bar{G}$ is defined over $F_q$. If $G = \text{SL}(r, q^y)$, then we are done by (4.1). So suppose this is not the case. Then $G = \text{Sp}(r, q^y)$, $\text{SU}(r, q^y)$, or $\text{O}^\pm(r, q^y)'$. Consider the group $\bar{G} = \text{Sp}(r, K)$, $\text{SL}(r, K)$, or $\text{O}^+(r, K)'$. Then $\bar{P}$ is the stabilizer in $\bar{G}$ of an $s$-space, $V_1$, and $(r - s)$-space, $V_2$, of the natural
module $V$ for $\overline{G}$, satisfying $V_1 \leq V_2$. In the symplectic and orthogonal cases $V_1$ is totally singular with $V_2 = V_1^\perp$. Let $\overline{Q} = R_\sigma(P)$, so that $Q = \overline{Q}_a = O_\sigma(P)$.

First suppose that $r = 2s$. One checks that $\overline{Q}$ is abelian and that the Levi factor of $G$ contains $SL(s, q^s)$ ($SL(s, q^{2s})$ if $G$ is a unitary group). It follows from (3.5)(v) and $q \geq 4$ that $T$ contains a cyclic group of order $q^{ys} - 1/q^y - 1$ ($q^{2ys} - 1/q^y + 1$ if $G$ is unitary). Since $\overline{X} = \bigoplus_{\alpha \in \Delta} U_\alpha$, $C_T(X) \leq C_T(\overline{X}) = C_T(\overline{X}^*)$. So $C_T(X) \leq Z(\overline{G})$ and $|X| \geq q^{ys}$ ($q^{2ys}$ in the unitary case). On the other hand, $\sigma = w_\tau q^y$, where $w$ is an $s$-cycle in the Weyl group, $S_s$, of a Levi factor of $\overline{P}$ and $\tau$ is a graph automorphism. It is easily checked that $T = 1$ unless $G$ is unitary, in which case $|T| = 2$ (for this use the fact that $G \neq O^-(r, q^y)$, since $V_1$ is singular). Therefore $|\Delta| \leq s$ ($2s$ in the unitary case) and so $|X| \geq q^{ys}$ (resp. $q^{2ys}$). Therefore (vi) holds.

From now on assume $r \neq 2s$. Let $\overline{Z}$ be the subgroup of $\overline{Q}$ that is trivial on $V_2$ and on $V/V_1$. Then $\overline{Z} \leq \overline{Q}$. If $\overline{G}$ is symplectic or orthogonal then $\overline{Q}$ is the product of those root subgroups $U_\beta$ having positive coefficient of $\alpha_s$ (temporarily we label the Dynkin diagram starting at the stalk of the diagram of type $A_1$) and $\overline{Z}$ is the product of those root subgroups $U_\beta$ such that $\beta$ has $\alpha_s$-coefficient equal to 2. If $G$ is unitary, $\overline{P}$ is the intersection of two maximal parabolics (conjugate under the graph automorphism of $\overline{G}$) and $\overline{Q}$ is the product of root subgroups corresponding to roots having positive coefficient of $\alpha_s$ or $\alpha_t = \alpha^*_s$. Moreover, $\overline{Z}$ is the product of root subgroups for roots having both $\alpha_s$ and $\alpha_t$ coefficient positive.

Now $\overline{X} \leq \overline{Q}$ and since $\overline{G} = \langle \overline{X}, \overline{X}^* \rangle$ we cannot have $\overline{X} \leq \overline{Z}$. Another observation is that $\overline{G} \neq Sp(r, K)$ with $\text{char}(K) = 2$. Otherwise, $\overline{X}$ is generated by roots with $\alpha_s$-coefficient equal to 1, these being short roots in the root system of type $C_{r/2}$. But $\text{char}(K) = 2$ implies that the collection of all root subgroups for short roots generates a proper subgroup of $\overline{G}$ having type $D_{r/2}$. Consequently, from the description of $\overline{Z}$ and $\overline{Q}$ we can conclude (using the commutator relations) that $\overline{X} = \overline{Q'} = Z(\overline{Q})$.

We claim that $\overline{X} = \overline{Q}$. In view of the above it will suffice to show that $\overline{XZ} = \overline{Q}$. Let $\Gamma$ denote the set of root subgroups $U_\beta \leq \overline{Q}$ such that $U_\beta \notin \overline{Z}$. Then $\overline{Q}/\overline{Z} \cong \bigoplus_{\beta \in \Gamma} U_\beta$. Similarly $\overline{XZ}/\overline{Z} \cong \bigoplus_{\beta \in \Delta} U_\beta$, so we must show $\Delta = \Gamma$. That is, we require that $\langle w_\tau \rangle$ be transitive on $\Gamma$. We illustrate the method with $\overline{G} = Sp(r, K)$; the other cases follow the same argument with only minor changes. The Levi factor $\overline{L}$ of $\overline{P}$ satisfies $\overline{L'} = \overline{L}_1 \times \overline{L}_2$ with $\overline{L}_1 \cong SL(s, K)$ and $\overline{L}_2 \cong Sp(r - 2s, K)$. Write $w = w_1 w_2$ with $w_i \in W_i$, the Weyl group of $\overline{L}_i$, for $i = 1, 2$. As $\overline{T}$ is minisotropic
in a Levi factor of $\tilde{P}$ (with respect to $\sigma = w_1 w_2 q$), we necessarily have $w_1$ an $s$-cycle in $W_1 \cong S_s$. Let $j = |w_2|$, $d = (s, j)$, $s = ds_1$, and $j = dj_1$.

Let $\Gamma_1, \ldots, \Gamma_l$ be the orbits of $\Gamma$ under $\langle w_1 \rangle$. Each orbit has the form $\Gamma_i = \{ \beta_i, \beta_i + \alpha_{s-1}, \ldots, \beta_i + \alpha_{s-1} + \cdots + \alpha_1 \}$, where $\beta_i$ is the element of $\Gamma_i$ having minimal height, and we order so that $ht(\beta_1) \leq \cdots \leq ht(\beta_l)$. Then $\beta_1 = \alpha_s$, $\beta_2 = \alpha_s + \alpha_{s+1}$, etc. and $l = r - 2s$. Let $M$ be the $(l \times s)$-matrix with rows $\Gamma_1, \ldots, \Gamma_l$, and let $C_1, \ldots, C_s$ denote the columns of $M$. The direct sum of the root subgroups in a given row or column affords the usual representation of $\tilde{L}_1, \tilde{L}_2$, respectively. For the rows this is easy. For a column $C_j$ one checks that for each $\beta \in C_j$ there is a unique $\gamma \in C_j$ such that $\beta + \gamma = \delta$ is a root. Moreover, $\delta$ depends only on $j$. Hence, we obtain the natural module for $\tilde{L}_2$ by letting the root subgroups $\tilde{U}_\beta$ be singular 1-spaces and realize the form via commutators. (If $\tilde{G}$ is an orthogonal group then root subgroups corresponding to a given column commute. However, by taking two adjacent columns we obtain a nondegenerate symplectic form, via commutation, and since $\tilde{L}_2$ is represented equivalently in the two column spaces we see that $\tilde{L}_2$ necessarily preserves an orthogonal form on each.)

Let $E = \bigoplus_{j=1}^l \tilde{U}_{\beta_j}$, viewed as the natural module for $\tilde{L}_2$, and let $E_0$ be the subspace spanned by those $\tilde{U}_{\beta_j}$ such that $\Delta \cap \Gamma_j \neq \emptyset$. Then $E_0$ is $\tilde{T} \langle w_2 \rangle$-invariant. As $\tilde{T}_2 = \tilde{T} \cap \tilde{L}_2$ is minisotropic (with respect to $w_2$), we conclude $\text{rad}(E_0) = 1$. Suppose $E_0 < E$. Then $C_{\tilde{T}_2}(E_0)$ has positive dimension, which implies $C_{\tilde{T}_2}(\tilde{X})$ has positive dimension. However, $C_{\tilde{T}_2}(\tilde{X}) = C_{\tilde{T}_2}(\tilde{X}^*)$ and $\tilde{G} = \langle \tilde{X}, \tilde{X}^* \rangle$. This is impossible, hence $E_0 = E$ and $j = l = r - 2s$.

If $j_1$ is odd then $(\tilde{T}_2)_\sigma$ has order divisible by $q^{d/2} + 1$, as does $(\tilde{T}_1)_\sigma$. Hence $|T/Z(G)|$ is not cyclic. It follows that there exists $a \in T - Z(G)$ and $\beta \in \Delta$ such that $a \in C(\tilde{U}_\beta)$. But then $a \in C_{\tilde{T}}(\tilde{X}) = C_{\tilde{T}}(\tilde{X}^*) = C_{\tilde{T}}(\tilde{G}) = Z(\tilde{G})$, a contradiction. So $j_1$ is even and $w_2^{j_1/2} \in \langle w_2^d \rangle$. On the other hand, $\langle w^* \rangle = \langle w_2^{d_1} \rangle = \langle w_2^d \rangle$, hence $w_2^{j_1/2} \in \langle w^* \rangle$ the latter group leaving each $C_i$ invariant. The element $w_2^{j_1/2}$ sends each $\beta \in C_i$ to the unique root $\gamma \in C_i$ with $\beta + \gamma$ a root.

Let $\Delta_1, \ldots, \Delta_d$ be the orbits of $\Delta$ under $\langle w^d \rangle = \langle w_2^d w_2^* \rangle = \langle w_2^d, w_2^* \rangle$ (a group of order $s_1 j_1$). Fix $\beta \in \Delta_m$ and suppose $\{ \beta \} = \Gamma_i \cap C_j$. Let $\Gamma_k$ be the image of $\Gamma_i$ under $w_2^{j_1/2}$. One checks that if $\gamma \in \Delta$ and $\beta \pm \gamma$ is a root, then $\gamma \in \Gamma_i \cup C_j \cup \Gamma_k$. It follows that $\gamma \in \Delta_m$. Setting $\tilde{G}_i = \langle \tilde{U}_{\pm \beta}: \beta \in \Delta_i \rangle$ we conclude that the groups $\tilde{G}_1, \ldots, \tilde{G}_d$ commute and generate $\tilde{G}$. This forces $d = 1$, $(s, l) = 1$, and $\langle w \rangle$ transitive on $\Gamma$, as required ($|\Gamma| = sl = sj$).

As $\tilde{X} = \tilde{Q}$ we have $X = \tilde{X}_\sigma = \tilde{Q}_\sigma = Q = O_p(P)$, proving (v). (iv) follows from this and (2.3) of [23]. Also $1 = (s, l) = (s, r - 2s) = (s, r)$,
proving (i). Finally, one checks that $Q' = \mathbb{Z}_a$, so $X/X' = Q_\sigma/\mathbb{Z}_a = (Q/\mathbb{Z})_\sigma$ which has order given in (ii). This completes the proof of (4.2).

5. The action of $T$ on root subgroups. In this section we are concerned with the action of $T$ on the nilpotent $T$-root subgroups $X_i, \ldots, X_t$. The results are fundamental to the rest of the paper and are in the spirit of Lemma 3 of [22].

We adopt the following notation. For $1 < i < t$, let $\overline{Y}_i = \langle \overline{X}_i, \overline{X}^* \rangle$, $Y_i = (\overline{Y}_i)_\sigma$, $P_i$ the $\overline{T}(\sigma)$-stable parabolic subgroup of $Y_i$ satisfying $X_i \leq R_u(P_i)$ (see (3.4)), and $P_i = (P_i)_\sigma$. In addition, set $\overline{V}_i = X_i R_u(P_i)'/R_u(P_i)'$ and $V_i = X_i R_u(P_i)'/R_u(P_i)$. Except for the cases where $\Sigma_i$ has roots of different lengths ($G$ a Suzuki or Ree group), we see from (2.1) that $V_i$ can be regarded as an $F_q$-module of dimension $|\Sigma_i|$.

(5.1) Let $1 \leq i \leq t$.

(i) $\overline{V}_i$ is $\overline{T}(\sigma)$-isomorphic to the external direct sum of the root subgroups $\overline{X}_a$ for $\alpha \in \Sigma_i$. Also, $V_i = (\overline{V}_i)_\sigma$.

(ii) If $G$ is not a Suzuki or Ree group, then the representation that $T$ induces on $K \otimes_{F_q} V_i$ is the direct sum of the representations $\varphi_\alpha |_T$, $\alpha \in \Sigma_i$.

(iii) If $q > 3$, the $T_0$ acts irreducibly on the elementary abelian $p$-group $V_i$.

Proof. Let $\overline{\Sigma}_i = \{\gamma_1, \ldots, \gamma_k\}$. The group $R_u(P_i)' = \overline{D}_i$ is the product of certain of the root subgroups for $\overline{T}$ and the proof of (3.4)(iii) shows that $\overline{U}_j \neq \overline{D}_i$ for $j = 1, \ldots, k$. Therefore, $\overline{V}_i \cong \overline{X}_{\gamma_1} \times \cdots \times \overline{X}_{\gamma_k}$. Let $\overline{J}_i = \overline{X}_i \cap \overline{D}_i$. Then $\overline{J}_i$ is a product of $\overline{T}$-root subgroups, so $\overline{J}_i$ is connected and Lang's theorem implies that $(\overline{X}_i/\overline{J}_i)_\sigma = \overline{X}_i \overline{J}_i/\overline{J}_i$. So (i) holds. From here we have (ii). For the Suzuki and Ree groups one can obtain (iii) from a direct check of the possible configurations. So we now exclude these cases.

Let $\hat{\cdot}$ denote images in $R_u(P_i)'/\overline{D}_i$. So $\hat{\overline{V}}_i = \hat{\overline{X}}_{\gamma_1} \times \cdots \times \hat{\overline{X}}_{\gamma_k}$. As $\Delta_i$ is a $\langle \sigma \rangle$-orbit, $\sigma^k$ stabilizes each of the groups $\hat{\overline{X}}_{\gamma_j}, j = 1, \ldots, k$, and $\sigma = \tau q$, where $\tau$ is an automorphism of $\overline{G}$. Then $\tau^k$ induces scalar multiplication and we see that $(\hat{\overline{X}}_{\gamma_j})_\sigma^k$ is elementary abelian of order $q^k$. By (3.5)(i), $\overline{V}_i$ is the commuting product of a $\langle \sigma \rangle$-orbit of quasisimple groups, so taking projections we may assume that $\overline{V}_i$ is quasisimple. By induction on $\dim(\overline{G})$ we may assume $\overline{G} = \overline{Y}_i$. Also, we may assume $Z(\overline{G}) = 1$, so $C_{\overline{T}}(\overline{X}_i) \leq C_{\tau}(\overline{Y}_i) = 1$, and $\overline{T}$ acts faithfully on $\overline{X}_i$. On the other hand, the representations of $T$ on $\hat{\overline{X}}_{\gamma_1}, \ldots, \hat{\overline{X}}_{\gamma_k}$ are algebraic conjugates of each other. We conclude that $T$ acts, faithfully, as a cyclic group on $\hat{\overline{X}}_{\gamma_i}$, for $i = 1, \ldots, k$. Taking projections it is clear that it will suffice to show that (iii) holds for
the action of $T_0$ on $(\hat{X}_{\eta})_{\alpha} = V$. We have $|V| = q^k$, $|T| = d$, $|T_0| = \Phi(n_1(q) \cdots \Phi(n_s(q))$, where $\Phi(n_i(q))$ is the cyclotomic polynomial of degree $\phi(n_i)$ and $d$ is the order of the center of the universal covering group of $G_0$. In particular, $d | q \pm 1$. We may assume $\tilde{G}$ is an adjoint group.

First suppose $T_0 = T$. By (2.1) $\Phi(n_i(x)) \cdots \Phi(n_s(x))$ is the characteristic polynomial of $\tau$ in its action on $R \otimes X(T)$. So $|\tau| = l$, where $l = \text{l.c.m.}\{n_1, \ldots, n_s\}$. In particular, $k | l$. By Zsigmondy [28], for $j = 1, \ldots, s$, $\varphi(n_j(q))$ has as a factor a primitive divisor of $q^{n_j} - 1$, unless $\varphi(n_j(q)) = p + 1 = 2^c$ or 9. We claim that $n_j | k$ for all $j$. In view of the above, this is clear unless there is a unique $j$ with $\varphi(n_j(q)) = \varphi_2(q) = 2^c$ or 9. But $\varphi_2(q) | q^k - 1$ forces $k$ even. So the claim holds. This implies that $l | k$; hence $l = k$. As $T$ acts faithfully on $V$ we also have $T$ acting irreducibly on $V$ (viewed as an $F_p$-space). So (iii) holds.

For the general case we first note that an easy check gives the result if $|T| = \varphi(n_i(q)) = q \pm 1$. If $n_j \neq 1, 2$, then each primitive divisor of $q^{n_j} - 1$ divides $|T_0|$. Setting $l_0 = \text{l.c.m.}\{n_j | n_j \neq 2\}$ we see that $T_0$ cannot act on $V_0 < V$ with $|V_0| < q^{l_0}$. So supposing $T_0$ reducible on $V$, we have $l_0$ odd, $l = 2l_0$, $V = V_0 + V_1$, $T_0$ acts irreducibly on $V_0$ and $V_1$, and $|V_0| = |V_1| = q^{l_0}$. Since $\varphi_{2}(q) | |T|$, $d | q + 1$ and any primitive divisor of $q + 1$ (if such exists) divides $d$. If $G_0$ is a unitary group, then $|T|$ is a product of terms $q^c - (1)^c$ (see Carter [6]). So $l_0$ odd forces $l = 2$, and since $T$ is cyclic $|T|$ divides $q^2 - 1$, and we contradict the fact that $q > 3$. From now on we have $d \leq 4$. Since $T$ is contained in a proper parabolic subgroup of $G$ we necessarily have $q - 1 | |T|$ and so $|T_0|$ is divisible by $\frac{1}{2}(q - 1)(q + 1)$. However, $|T_0|$ divides $q^6 - 1 = (q - 1)x$ with $(x, q + 1) = 1$. This forces $d = q + 1 = 4$, a contradiction.

The next two lemmas were communicated to the author by R. Steinberg and lead to a much shorter proof of (5.5) than our original one.

(5.2) If $\alpha \in \Sigma$ and if $\omega$ is a nonzero weight of $\Sigma$, then $|\alpha| \leq 2 |\omega|$, with equality precisely when $\Sigma$ has type $C_n$, $\alpha$ is a long root, and $\omega$ is $W$ conjugate to $\frac{1}{2} \alpha$.

Proof. Since $W$ acts irreducibly on $Q \otimes X(T)$ and preserves the form, we may assume $(\omega, \alpha) > 0$. Combining the fact that $\omega$ is a weight with the triangle inequality, we have

$$1 \leq 2(\omega, \alpha)/(\alpha, \alpha) \leq 2 |\omega|/|\alpha|.$$
This gives the desired inequality. If $|\alpha| = 2|\omega|$, then the second inequality implies $\omega = c\alpha$ with $c > 0$, while the first inequality shows that $c = \frac{1}{2}$. Finally, for $\frac{1}{2}\alpha$ to be a weight we must have $\alpha$ long and $\Sigma$ of type $C_n$.

(5.3) Let $\alpha \neq \beta \in \widetilde{\Sigma}$, $q > 5$, and assume $G \neq Sz(8)$ or $^2F_4(8)$. Then $\varphi_\alpha|_{T_0} \neq \varphi_\beta|_{T_0}$. This holds for $q = 5$ unless $\Sigma$ is of type $C_n$, $\beta = -\alpha$ a long root, and $\alpha^\sigma = 5\alpha$.

Proof. We may assume $\widetilde{G}$ is simply connected. Then $X = X(\overline{T})$ can be identified with the lattice of weights of $\overline{\Sigma}$. Let $q_1 = \sqrt{q}$ if $G$ is a Suzuki or Ree group; otherwise set $q_1 = q$. As in §2 $\sigma$ acts on $X$, inducing $q_1\tau$ on $\mathbf{R} \otimes X$, where $\tau$ is an isometry. Also the argument of (1.7) of [25] shows that $X(q_1\tau - 1)$ is the annihilator of $T_0 = \overline{T}_0$. Then $\alpha - \beta = \omega(q_1\tau - 1)$ for some $0 \neq \omega \in X$. Then

\begin{align*}
|\omega(q_1\tau - 1)| &= |\alpha - \beta| \\
(1) &\leq |\alpha| + |\beta| \text{ (triangle inequality)} \\
(2) &\leq 4|\omega| \text{ (by (5.2))} \\
(3) &\leq (q_1 - 1)|\omega| \quad (q_1 \geq 5) \\
(4) &\leq |\omega q_1\tau| - |\omega| \quad (\tau \text{ is an isometry}) \\
(5) &\leq |\omega(q_1\tau - 1)| \text{ (triangle inequality}).
\end{align*}

Therefore, we have equality at each stage. From equality in (1) we have $\alpha$, $\beta$ dependent. Hence $\beta = -\alpha$. From (2) and (5.2) we conclude $\overline{\Sigma}$ has type $C_n$ with $\alpha$ a long root. Equality in (3) yields $q_1 = q = 5$, while equality in (5) gives $\omega q_1\tau = c\omega$ with $c > 0$. As $\tau$ is an isometry, $c = q$ and $\omega \tau = \omega$. The equation $\alpha - \beta = \omega(q_1\tau - 1)$ now gives $\alpha = 2\omega$, so $\alpha^\sigma = 5\alpha$ and the proof of (5.3) is complete.

(5.4) Assume $G \neq Sz(q)$, $^2F_4(q)$, or $^2G_2(q)$ and assume $q \neq 2, 3, 4, 9$. If $\alpha, \beta \in \overline{\Sigma}$ with $\alpha^p|_{T_0} = \beta|_{T_0}$ for some $1 \leq p^i < q$, then $\alpha|_{T_0} = \beta|_{T_0}$. This also holds for $q = 9$, unless $p^i = 3$, $G$ is of type $C_n$, and $\beta = -\alpha$, a long root.

Proof. As in (5.3) we may take $\widetilde{G}$ to be simply connected and we may write $p^i\alpha - \beta = \omega(q\tau - 1)$ for $0 \neq \omega \in X$ and $\tau$ an isometry of $X$. Set $q = p^j$, so that $j > i$. Then $\omega = \beta + p^i(p^{-i}\omega \tau - \alpha)$, so write $\omega = \beta + p^i\delta$
with \( \delta \in X \). As \( j > i \) we have \( \delta \neq 0 \). Replacing \( \omega \) by \( \beta + p' \delta \) in the equation \( p' \alpha - \beta = \omega(q\tau - 1) \), we obtain \( \alpha + \delta = (q\delta + p'^{-i} \beta) \tau \). Then \( |\alpha| + |\delta| \geq |q\delta + p'^{-i} \beta| \geq q |\delta| - p'^{-i} |\beta| \), which together with (5.2) yields \( (p' - 1) |\delta| \leq 2(1 + p'^{-i}) |\delta| \). We conclude that \( p'^{-i}(p' - 2) \leq 3 \).

For \( p \geq 5 \), this is impossible. Suppose \( p = 3 \). Here the only possibility is \( q = 9 \) and \( i = 1 \). Moreover, all inequalities must by equalities. Using this and (5.2) one checks that \( 2 \delta = \alpha = -\beta \) a long root and \( G \) is of type \( C_n \). Finally, assume \( p = 2 \). Here \( i = 1 \) and \( (q - 1) |\omega| = |q\omega \tau| - |\omega| \leq |\omega(q\tau - 1)| = |2\alpha - \beta| \leq 2 |\alpha| + |\beta| \leq 6 |\omega| \) the last equality by (5.2). Hence \( q \leq 4 \), completing the proof of (5.4).

**Theorem** (5.5). Let \( 1 \leq i < j \leq t \) and assume \( q > 5 \).

(i) \( V_i \) and \( V_j \) are inequivalent irreducible \( F_p[T_0] \)-modules unless \( G = Sz(8), 2F_4(8), \) or \( q = 9 \) and \( G \) is of type \( C_n \).

(ii) If \( \langle X_i, X_j \rangle \) is a p-group, then \( V_i \) and \( V_j \) are inequivalent irreducible \( F_p[T_0] \)-modules.

**Proof.** We may assume \( G \) is simply connected. Write \( q = p^a \). By (5.1)(iv) each of \( V_i \) and \( V_j \) is an irreducible \( F_p[T_0] \)-module. For the moment exclude Suzuki and Ree groups. Then \( F_q \otimes_{F_p} V_i = V_i + V_i^p + \cdots + V_i^{p^{a-1}} \), the direct sum of the Galois conjugates of \( V_i \) (which is regarded as an \( F_q[T_0] \)-module on the right side of the equation). Similarly for \( V_j \). Assume that \( V_i \) and \( V_j \) are equivalent \( F_p[T_0] \)-modules and tensor the equations with \( K \). Then (5.1)(ii) implies that there exist \( 1 \leq k < p^a \) and roots \( \alpha \in \bar{\Sigma}, \beta \in \bar{\Sigma} \) such that \( \alpha^{p^k} |T_0 = \beta^{p'} |T_0 \). By (5.4) and (5.3) \( q = 9 \), \( G \) has type \( C_n \) and \( \beta = -\alpha \) is a long root of \( \bar{\Sigma} \). So (i) holds in this situation.

Suppose \( q = 9 \) with \( \beta = -\alpha \) a long root and \( \bar{\Sigma} \) of type \( C_n \). Then \( \langle \bar{X}_i, \bar{X}_j \rangle = \langle \bar{X}_i, \bar{X}_i^* \rangle = \prod_{\gamma \in \Sigma} \langle \bar{U}_\gamma, U_{-\gamma} \rangle \). It is easy to see that \( \langle \bar{X}_i, \bar{X}_j \rangle \approx SL(2, q^i) \), where \( s = |\bar{\Sigma}| \) and that \( \langle X_i, X_j \rangle = \langle \bar{X}_i, \bar{X}_j \rangle \approx SL(2, q^i) \). Therefore, we have proved (5.5) for all but the Suzuki and Ree groups.

The Suzuki and Ree groups are handled by direct calculation, which we leave to the reader. We observe that for \( G = Sz(q), 2G_2(q), T = T_0 \) is necessarily a Cartan subgroup of \( G \). If \( G = 2F_4(q) \) with \( T \) a Cartan subgroup, then as in Lemma 3 of [22] \( C_T(V_j) \neq C_T(V_j) \) unless \( X_j = X_i^* \). In this case \( \langle X_i, X_j \rangle = L_2(q) \) or \( Sz(q) \) and we are reduced to the above. Assuming \( T \) not a Cartan subgroup it follows that \( T \) is necessarily the direct product of \( Z_{q-1} \) with a minisotropic torus of \( L_2(q) \) or \( Sz(q) \).

We remark that the exceptions in (5.5)(i) are real. If \( G = Sz(8) \) and if \( T \) is a Cartan subgroup, then \( T \) has equivalent representations on \( U/Z(U) \)
and on $Z(V)$, where $U$, $V$ are the unique Sylow 2-subgroups normalized by $T$. This example carries over to $^{2}F_{4}(8)$. Similarly, $SL(2,9)$ is an exception, which carries over to $Sp(2n,9)$ for all $n \geq 1$.

We conclude this section with the following result.

(5.6) Assume that $q \geq 5$, $G \neq Sz(q)$ or $^{2}F_{4}(q)$, $T$ is a Cartan subgroup of $G$, and that $X_i \neq X_j$ are nilpotent $T$-root subgroups of $G$. Then $V_i$ and $V_j$ are inequivalent $F_p[T_0]$-modules unless $q = 5$ or $9$, $G$ is of type $C_n$ and $X_i, X_j$ are opposite long root subgroups.

Proof. The proof is just as in the first paragraph of the proof of (5.5).

II. $T_0$-Invariant Subgroups

This chapter will be concerned with general results concerning $T_0$-invariant subgroups of $G$.

6. $T_0$-invariant solvable groups. In this section we consider $T_0$-invariant solvable subgroups of $G$ and show that for $q > 7$ each such group is the product of a normal $T$-invariant $p$-group and part of the normalizer in $G$ of $T$. Moreover, we show that each $T_0$-stable $p$-subgroup of $G$ is a product of a set of $T$-root subgroups of $G$.

We maintain the notation in §2. So $T = T_0$, $G_0 = O^{P'}(G_0)$, and $T_0 = T \cap G_0$. The main result of this section is the following theorem, although there are several other results that will be useful in other sections.

THEOREM (6.1). Suppose $q > 7$ and $T_0 \leq S \leq G$, with $S$ solvable. Then
(i) $S = O_p(S)N_S(T_0)$;
(ii) $O_p(S)$ is the product of $T$-root subgroups of $G$.

This theorem will follow from the other results of this section, several of which are of independent interest.

(6.2) Suppose $q > 7$ and $A$ is a $T_0$-invariant, abelian, $p'$-subgroup of $Aut(G_0)$. Then $A \leq N(T)$.

Proof. Suppose false and take a counterexample so that $|A| \cdot |G_0|$ is minimal. Then $A$ is an $r$-group for some prime $r \neq p$, $A > N_A(T_0)$, (by (2.7)), and $A/N_A(T_0)$ is an irreducible $F_r[T_0]$-module. Also, $Z(G_0) = 1$. 
Suppose \( C_T(A) \neq 1 \) and let \( 1 \neq t \in C_T(A) \). We consider the groups \( C_G(t) \) and \( C_G(t) \). Since \( \bar{T} \leq C_G(t)^0 \), we have \( T \leq (C_G(t)^0)_t = Y \leq C_G(t) \). Let \( Y_0 \leq Y \) be as described in (2.9). That is \( Y_0 = E(Y)X \) and \( X = Z(Y_0) = C_T(E(Y)) \). Then by (2.9) \( Y = Y_0T \) with only diagonal automorphisms induced on each component of \( E(Y) \). Write \( E(Y) = D_1 \cdots D_k \), a central product of components.

We claim that \( A \leq N(D_i) \) for \( i = 1, \ldots, k \). For suppose \( a \in A \) and \( D_i = D_j \) for \( j \neq i \). If \( x \in T_0 \cap D_i \) then \( x^{-1}x^a \in A \leq C(A) \). It follows that \( D_j = D_j \) and \( x^a = (x^{-1})^a \) (mod \( Z(D_iD_j) \)), which forces \( (T_0 \cap D_i)/Z(D_i)/Z(D_i) \) to be an elementary abelian 2-group. So \( D_i \) is neither a Suzuki or Ree group. Suppose \( D_i \) is defined over \( F \) and the overlying algebraic group has Lie rank \( s \). Then by (2.1) \( T_0 \cap D_i \) has rank at most \( s \) (as an abelian group). On the other hand, (2.4)(iii) shows that

\[
| T_0 \cap D_i | \geq \frac{1}{d_i} (q^b - 1)^s \geq \frac{q^b - 1}{q^b + 1} (q^b - 1)^{s-1} > \left( 1 + \frac{2}{q^b + 1} \right)^{s-1} > \left( \frac{5}{4} \right)^{s-1} \text{ (as } q > 7). \]

This forces \( s = 1 \), so \( d_i \leq 2 \), and we obtain a contradiction. This proves the claim.

For \( i = 1, \ldots, k \), let \( K_i = D_1 \cdots \hat{D}_i \cdots D_k X \) and \( C_i = YA/K_i \). Then \( E(C_i) = D_iK_i/K_i \) and \( Z_i = (T_0 \cap D_i)K_i/K_i \) is a maximal torus of \( E(C_i) \). By minimality \( Z_i \) is \( A \)-invariant. So if \( T_1 = (T_0 \cap D_1) \cdots (T_0 \cap D_k) \) we have \( [A, T_1] \leq \cap K_i = X \). Therefore, \( A \) normalizes \( T_1X \) and hence \( A \) normalizes \( C_\gamma(T_1X) = T \) (see (2.8)(i)). This contradiction shows that \( C_T(A) = 1 \).

Let \( A_1 = [A, T_0] \), so \( A_1 \leq G_0 \). Suppose \( A_1 < A \). By induction \( A_1 \leq N(T_0) \), so \( [A_1, T_0] \leq A_1 \cap T_0 \leq C_T(A) = 1 \). By (2.8) \( A_1 \leq T_0 \), whence \( A_1 \leq C_T(A) = 1 \). Hence \( A = A_1 \). The same argument shows \( N_4(T_0) = 1 \). Therefore, \( T_0 \) acts faithfully and irreducibly on \( A \). In particular, \( AT_0 \) is a Frobenius group and \( A \leq G_0 \).

Consider the action of \( AT_0 \) on the Lie algebra, \( M \), of \( \overline{G} \). Viewing \( M \) as a \( K[AT_0] \)-module and using Clifford's theorem we see that \( M|_{T_0} \) contains the regular representation of \( T_0 \). So if \( \overline{G} \) has Lie rank \( n \) we have the inequality \( \dim(M) \geq |T_0| \geq d^{-1}(q - 1)^n \) (by (2.4)). Use the fact that \( q - 1 > 6 \) and the known values of \( d \) to obtain a contradiction. This proves (6.2).

(6.3) Assume \( q > 7 \) and let \( T_1 \) be a maximal torus of \( G \). Suppose \( T_1 \cap G_0 \leq N_G(T_0) \). Then \( T_1 = T \). In particular, \( T_0 \) is weakly closed in \( N_G(T_0) \).
Proof. We may assume \( Z(G_0) = 1 \). Suppose \( T_0 \neq T_2 = T_1 \cap G_0 \leq N_{G_0}(T_0) \). By (6.2) \( T_0 \leq N(T_2) \). Hence \( [T_0, T_2] \leq T_0 \cap T_2 \leq Z(T_0T_2) \). If \( T_0 \cap T_2 = 1 \), then \( T_2 \leq C_{G_0}(T_0) = T_0 \) by (2.8). Similarly, \( T_0 \leq T_2 \). So we assume \( T_0 \cap T_2 \neq 1 \).

Let \( C = C_G(T_0 \cap T_2)^0 \). Then \( \overline{T}, \overline{T}_1 \leq C \) and \( C = \overline{D}Z \), where \( \overline{T}_1 = C_G(T_1)^0 \) (a maximal torus), \( \overline{D} = E(C) \), and \( Z = Z(C) \leq \overline{T} \cap \overline{T}_1 \). Note that \( \overline{T} \neq \overline{T}_1 \) forces \( D \neq 1 \). Let \( D = O_{\pi}(\overline{D}_0) \), \( T_3 = T_0 \cap D \), and \( T_4 = T_2 \cap D \). By (2.5) \( T_3 \) and \( T_4 \) are maximal tori of \( D \). Since \( T_4 \leq N(T_3) \), we conclude, inductively, that \( T_4 = T_3 \). But then \( T_4 = T_3 \leq T_0 \cap T_2 \leq Z(C) \), which contradicts (2.8).

A useful consequence of (6.3) is the following

(6.4) Assume \( q > 7 \) and let \( T_0 \leq \overline{P} = \overline{P}_\sigma \), where \( \overline{P} \) is a parabolic subgroup of \( \overline{G} \). Then \( \overline{T} \leq \overline{P} \).

Proof. Suppose \( T_0 \leq \overline{P} = \overline{P}_\sigma \). By (5.16) of [25] there is a \( \sigma \)-invariant maximal torus \( \overline{T}_1 \) of \( \overline{P} \) such that \( T_0 \leq N(\overline{T}_1) \). Then \( T_0 \leq N((\overline{T}_1)_\sigma) \) and (6.3) implies \( T_0 \leq (\overline{T}_1)_\sigma \). But then \( \overline{T}_1 \leq C_G(T_0)^0 = \overline{T} \). Therefore, \( \overline{T} = \overline{T}_1 \leq \overline{P} \).

(6.5) Suppose \( q > 7 \) and \( T_0 \leq N_{G_0}(S) \), where \( S \) is a solvable \( p' \)-subgroup of \( G \). Then \( S \leq N(G_0)(T_0) \).

Proof. Let \( S \) be a minimal counterexample and \( S/S_0 \) a chief factor of \( ST_0 \). Then \( S_0 \leq N(T_0) \). If \( [T_0, S] \leq S_0 \), then \( S_0T_0 \leq ST_0 \). But \( S_0T_0 \leq N(T_0) \), so (6.3) implies that \( S \leq N(T_0) \). Thus, we may assume that \( [T_0, S/S_0] = S/S_0 \), and by minimality, \( [T_0, S] = S \). In particular \( S \leq G_0 \). If \( S_0 = 1 \) then \( S \) is abelian and we are done by (6.2). Suppose then, that \( S_0 \neq 1 \) and let \( S_1 \) be a minimal normal subgroup of \( ST_0 \) with \( S_1 \leq S_0 \). By (6.2) \( S_1 \leq N(T_0) \), so \( [S_1, T_0, T_0] = 1 \). Say \( |S_1| = r^a \), with \( r \) a prime. Then \( O_{r'}(T_0) \leq C(S_1) \).

We claim that \( Z(ST_0) \neq 1 \). Suppose otherwise. If \( O_r(T_0) \leq C(S/S_0) \), then \( [O_r(T_0), S] \) covers \( S/S_0 \) and by minimality, \( [O_r(T_0), S] = S \). Then \( S \leq \langle O_r(T_0)^{ST_0} \rangle \leq C(S_1) \) and \( T_0 \) acts irreducibly on \( S_1 \). Then \( [S_1, T_0, T_0] = 1 \) implies \( Z(ST_0) \neq 1 \). Therefore, \( O_r(T_0) \leq C(S/S_0) \). This means that \( O_r(T_0)S_0 \leq O_r(T_0)S_0 \), and since \( O_r(T_0) \leq O_r(S_0T_0) \), we conclude that either \( O_r(T_0) = 1 \) or \( O_r(T_0) \neq O_r(S_0T_0) \neq 1 \). In the latter case, let \( Y \) be minimal normal in \( ST_0 \) with \( Y \) an \( r' \)-group contained in \( O_r(T_0)S_0 \). Then \( [Y, T_0, T_0] \leq [S_0, T_0, T_0] \leq [S_0 \cap T_0, T_0] = 1 \). Thus, \( C(Y) \geq \langle O_r(T_0)^{ST_0} \rangle \). But \( T_0/C_{T_0}(S/S_0) \) is an \( r \)-group, and this forces \( C(Y) \geq S \). As above, this yields \( Z(ST_0) \neq 1 \). Therefore, we assume that \( O_r(T_0) = 1 \) and \( T_0 \) induces a cyclic \( r \)-group on \( S/S_0 \).
Since \([S_1, T_0, T_0] = 1\) we apply Theorem B of Hall-Higman (see p.
359 of [13]) and conclude that \(T_0/C_{T_0}(S_1)\) has exponent 2 or 3. In
particular, \(T_0/C_{T_0}(S/S_0) \cong Z_2\) or \(Z_3\). Let \(T_1 = C_{T_0}(S/S_0)\). Then \(T_1 \leq
O_p(T_1S_0)\) and \(T_1S_0 \leq ST_0\). Since \(S_1\) is minimal normal in \(ST_0\) we have
\(S_1 \leq Z(O_p(T_1S_0))\) and \([T_1, S_1] = 1\). If \(r = 2\) let \(g\) be a 2-element in
\(ST_0 - S_0T_0\). If \(r = 3\), then \(S/S_0\) is an elementary abelian 2-group (this
follows from the proof of Theorem B of Hall-Higman) and we let \(g \in S - S_0\). In either case
\(\langle S_0, T_0, g \rangle = ST_0\). Therefore, \(T_1 \cap T_i^g \leq ST_0\).
If \(T_1 \cap T_i^g \neq 1\), then we may take \(S_i \leq T_1 \cap T_i^g\) and obtain \(T_0 \leq C(S_i)\).
This would imply \(ST_0 \leq \langle T_0^{ST_0} \rangle \leq C(S_1)\), a contradiction. Therefore, \(T_1 \cap
T_i^g = 1\) and \([T_1, T_i^g] \leq T_1 \cap T_i^g = 1\).

Let \(b = |T_i^g|\). Then \(b \geq r^{-1} |T_0| \geq (dr)^{-1}(q - 1)^n\), where \(n\) is the Lie
rank of \(\bar{G}\) (here we use (2.4)(iii) and note that the numerical restrictions
rule out Suzuki and Ree groups). On the other hand, \(T_i^g \leq N(T_0)\), while
\(T_0 \cap T_i^g = 1\). So by (2.7) we may regard \(T_i^g\) as an abelian \(r\)-subgroup of
\(\bar{W}\), the Weyl group of \(\bar{G}\). We leave it to the reader to check that the
assumption \(q > 7\) leads to a contradiction. (In this check the following
inequality is useful. For \(A\) an abelian \(r\)-subgroup of \(S_{m+1}\) we have
\(|A| \leq r^{(m+1)/r}\). To see this let \(o_1, \ldots, o_i\) be the orbits of \(A\) with \(|o_i| = r^{k_i}\).
Then \(m + 1 = \sum r^{k_i}\) and \(|A| \leq \prod r^{k_i}\) (as \(A\) is abelian). Since \(r^{k_i} \geq r^{k_i}\) we
have \(m + 1 \geq \sum r^{k_i} = r(\sum k_i)\), and the inequality follows.) This proves the
claim, hence \(Z(ST_0) \neq 1\).

Choose \(1 \neq x \in Z(ST_0)\) and consider \(ST_0\) as a subgroup of \(C_G(x)\). By
(2.8) \(x \in C_{C_0}(T_0) = T_0\), and so \(T \leq (C_G(x)^0)_x = Y\). Also, \(Y = Y_0T\), where
\(Y_0 = E(Y)\). Since \([S, T_0] = S\) we have \(S \leq Y_0 = D_1 \cdots D_k\), where
\(D_1, \ldots, D_k\) are the components of \(Y_0\). Fix \(i \in \{1, \ldots, k\}\) and let bars
denote images in \(Y_0T_0\) modulo \(D_1 \cdots \hat{D}_i \cdots D_k\). Then \(\bar{S}\) is normalized by
\(\bar{T_0}\), hence by \(\bar{T_0} \cap \bar{Y_0}\). By induction, \(\bar{S} \leq N(\bar{T_0} \cap \bar{Y_0})\). Therefore,
\(\bar{S} \leq N(C_{Y_0T_0}(\bar{T_0} \cap \bar{Y_0})) = N(\bar{T_0})\) and \([S, T_0] \leq D_1 \cdots \hat{D}_i \cdots D_k\). Repeating this
for each \(i\) we conclude \(S = [S, T_0] \leq Z(Y_0)\), and finally \([S, T_0] = 1\). This
is a contradiction proving (6.5).

The next result completes the proof of (6.1)(i).

(6.6) Suppose \(q > 7\) and let \(T_0 \leq S \leq G\), with \(S\) solvable. Then
\(S = O_p(S)N_S(T_0)\).

Proof. Let \(S\) be a minimal counterexample. Suppose \(L \lhd S\) with
\(T_0 \leq L\) and let \(X\) be a Hall \(p'\)-subgroup of \(L\) with \(T_0 \leq X\). Then \(S =\)
\(LN_S(X) = LN_S(T_0)\) by (6.3) and (6.5). By minimality, \(L = O_p(L)N_L(T_0),\)
so $S = O_p(L)N_S(T_0) = O_p(S)N_S(T_0)$. We conclude that $S = \langle T_0^S \rangle$. In particular, $S \leq G_0$.

Let $NO_p(S)/O_p(S)$ be minimal normal in $S/O_p(S)$, where $N$ is a $p'$-group. Then $S = O_p(S)N_S(N)$ and we may assume $T \leq N_S(N)$ (take $N$ in a Hall $p'$-group containing $T_0$). By minimality, we conclude that $S = N_S(N)$. Let $C = C_3(N)$. Suppose, $C = S$ and let $1 \neq x \in C$. Then $x \in C_{G_0}(T_0)$ implies $x \in T_0$, so $T \leq (C_{G_0}(x)^0)_\sigma = Y$. We have $Y = Y_0T$, where $Y_0 = D_1 \cdots D_k Z$ as in (2.9). Since $S = \langle T_0^S \rangle$, we conclude $S \leq Y_0T_0$. Fix $1 \leq i \leq k$ and let bars denote images in $Y$ modulo $D_1 \cdots \hat{D_i} \cdots D_k Z$. Set $T_1 = T_0 \cap D_1 \cdots D_k Z$ and $S_1 = \langle T_1^S \rangle$. By minimality and (6.3), $S_1 = O_p(\overline{S_1})N_{\overline{S_1}}(\overline{T_1}) = O_p(\overline{S_1})(\overline{T_1})$. It follows that $S_1' \leq J$, where $JZ/Z = O_p(S_1Z/Z)$. Therefore, $S_1 = O_p(S_1)T_1$ and $S = S_1N_S(T_1) \leq O_p(S_1)N_S(C_3(T_1))$. Since $N_S(C_3(T_1)) = T_0N_{S\cap C_3}(T_1) = T_0$, we obtain $S = O_p(S_1)T_0 = O_p(S)T_0$, which we are assuming false.

In view of the above, it will suffice to show that $C = S$. So assume $C < S$. Since $T_0 \leq N_{G_0}(N)$, we have $[N, T_0, T_0] = 1$, by (6.2). Therefore, $T_0C \neq S$. Since $S = \langle T_0^S \rangle$ we may choose $C \leq K < L < S$ with $K, L \not\leq S$ and such that $LT_0 = S$, $L/K$ is a chief factor of $S$ and $T_0 \cap L \leq K$. (For example, set $L/C = (S/C)'$ and $K/C$ any maximal normal subgroup of $L/C$). Minimality of $S$ implies $K_{T_0} = O_p(K_{T_0})N_{K_{T_0}}(T_0) = O_p(K)N_{K_{T_0}}(T_0)$.

Let $X$ be a $p'$-Hall subgroup of $K$ with $X^{T_0} = X$ and set $T_1 = T_0 \cap K \leq X$. Let $Y = X \cap O_p(K)$. Then $S = O_p(K)N_S(Y)$, so minimality of $S$ forces $Y \leq S$. Since $K_{T_0} = O_p(K)N_{K_{T_0}}(T_0)$ we conclude that $T_1 \leq K$. At this point we are in a position to use the argument in the proof of (6.5). We have seen that $O_p(Z(S)) = O_p(Z(LT_0)) = 1$. Replace the groups $S, S_0, S_1$ of (6.5) by $L, K, N$, respectively. Arguing as in (6.5) we first obtain (via Hall-Higman, Theorem B) that $T_0C/C \cong Z_2$ or $Z_3$, and then argue that $T_0$ is an $r$-group for some prime $r$. Finally we obtain a numerical contradiction. This completes the proof of (6.6).

To obtain (6.1)(ii) we must consider $T_0$-invariant unipotent subgroups of $G_0$. A key result is the following.

(6.7) Let $q > 5$ and assume that $\langle \overline{X}_i, \overline{X}_j \rangle$ is unipotent. Then $[\overline{X}_i, \overline{X}_j]_\sigma = [X_i, X_j]$. 

Proof. Let $\overline{L} = [\overline{X}_i, \overline{X}_j]$ and set $L = [X_i, X_j]$. So $L \leq \overline{L}_\sigma$. As $\overline{L}$ is $\overline{T}$-invariant, $\overline{L}$ is a product of $\overline{T}$-root subgroups of $\overline{G}$, and since $\overline{L}$ is $\sigma$-invariant these root subgroups fall into $\langle \sigma \rangle$-orbits. The first observation implies $\overline{L} = \overline{L}^0$. If $\overline{L} = 1$, then the result is trivial, so we assume dim($\overline{L}$) > 0. Set $\overline{X} = \langle \overline{X}_i, \overline{X}_j \rangle$. 

Suppose there is a normal subgroup, $\tilde{A}$ of $\bar{X}$, such that $\tilde{A} = \tilde{A}^\sigma$, $\tilde{A}$ is a product of root subgroups of $\bar{T}$, and $1 \neq \tilde{L}_{\sigma}/\tilde{A} = Z(\tilde{X}/\tilde{A}) = \bar{X}_w\tilde{A}/\tilde{A}$ for some $w \in \{1, \ldots, t\}$. We claim that $L_{\sigma}/\bar{A} = \bar{X}/\bar{A}$. Let $\tilde{\Phi}$ denote images in $\bar{X}/\bar{A}$. Then $\tilde{\Phi}$ is $\bar{T}$-isomorphic to $\tilde{V}_w = \bar{X}_wR_u(P_w)/R_u(P_w)$. Hence $\tilde{L}_{\sigma} = \tilde{X}_w$ (Lang’s theorem) is $T$-isomorphic to $V_w$. By (5.1) $T$ acts irreducibly on $V_w$, so it will suffice to show that $\tilde{L} \neq 1$. For this it will suffice to check that there exist elements $a \in X$, and $b \in X_j$ such that $[\tilde{a}, \tilde{b}] \neq 1$.

Since $\tilde{L} = \tilde{X}_w$, it is not the case that $[\tilde{U}_\delta, \tilde{U}_\beta] = 1$ for each $\delta \in \Delta_i$ and $\beta \in \Delta_j$. Therefore, choose $\delta \in \Delta_i$ and $\beta \in \Delta_j$ with $1 \neq [\tilde{U}_\delta, \tilde{U}_\beta] \leq \tilde{U}_w$. Interchanging $\delta$ and $\beta$, if necessary, we may assume that there is an integer $d = 1, 2$, or 3 such that $\delta + d\beta \in \Sigma_w$ ($d = 1$ if $\delta$, $\beta$ are both long roots) and $[\tilde{U}_\delta, \tilde{U}_\beta] \geq \tilde{U}_{\delta + d\beta}$. Let $|\Delta_i| = l$ and $|\Delta_j| = m$. Then $\Delta_i = \{\delta_1, \ldots, \delta_l\}$ and $\Delta_j = \{\beta_1, \ldots, \beta_m\}$, where $\delta_1 = \delta$, $\beta_1 = \beta$, $\delta_i = \delta_{i-1}^\tau$ for $2 \leq i < l$, and $\beta_i = \beta_{i-1}^\tau$ for $2 \leq i \leq m$ (here $\tau$ is the permutation of $\Sigma$ associated with $\sigma$).

From Lang’s theorem we have $(\bar{X}_i/\bar{X}_i')_{\sigma} = X_iX'_i/X_i'$ and $(\bar{X}_j/\bar{X}_j')_{\sigma} = X_jX'_j/X_j'$. Moreover, the 3-subgroup lemma shows that $1 = [\bar{X}_i', \bar{X}_j'] = [\bar{X}_i', \bar{X}_j']$. Let $a \in X_i$ and $b \in X_j$. There are elements $x$, $y \in K$ and $c_2, \ldots, c_l$, $d_2, \ldots, d_m \in K^*$ such that $a = \tilde{U}_{\delta_i}(x)\tilde{U}_{\beta_1}(c_2x^q) \cdots \tilde{U}_{\delta_l}(c_ly^{\delta_{i-1}^\tau}) (\mod \tilde{X}_i)$ and $b = \tilde{U}_{\beta_1}(y)\tilde{U}_{\beta_2}(d_2y^q) \cdots \tilde{U}_{\beta_m}(d(my^{\beta_{i-1}^\tau}) (\mod \tilde{X}_j')$ (slightly different for the Suzuki and Ree groups). There are $q^l$ choices for $x$ and $q^m$ choices for $y$.

For each $\delta_u \in \Delta_i$ there exists at most one $\beta_v \in \Delta_j$ such that $[\tilde{U}_{\delta_u}, \tilde{U}_{\beta_v}] \tilde{U}_{\delta_u + d\beta_v}$. The projection of $[\tilde{a}, \tilde{b}]$ to $\tilde{U}_{\delta_u + d\beta_v}$ is $\tilde{U}_{\delta_u + d\beta_v}(h)$, where $h = \Sigma zc_\delta u^d_y^{\delta_{i-1}^\tau} \tilde{U}_{\delta_u + d\beta_v}(h)$, and the sum ranges over the pairs $(u, v)$ for which $[\tilde{U}_{\delta_u}, \tilde{U}_{\beta_v}] \geq \tilde{U}_{\delta_u + d\beta_v}$, and $z$ is an integer with $(z, p) = 1$ ($z = \pm 1$ if $\delta$, $\beta$ and $\delta + \beta$ are all the same length). Fix $y \neq 0$ and for each pair $(u, v)$ let $e_y = zc_\delta u^d_y^{\delta_{i-1}^\tau}$. Then $h = f(x)$, where $f(t) = \Sigma u e_u^y^{q^{u-1}}$, a non-zero polynomial of degree at most $q^{u-1}$. There are $q^l$ choices for $x$, so we may choose $x$ with $h = f(x) \neq 0$. So for suitable choice of $x$ and $y$ we see that $[\tilde{a}, \tilde{b}]$ has nontrivial projection to $\tilde{X}_j'$. We have now proved the claim.

We now claim that if $\tilde{A} = \tilde{A}^\sigma$ is a product of root subgroups of $\bar{X}$ with $\tilde{A} \leq \bar{X}$, then $L\bar{A}/\bar{A} = \bar{L}_{\sigma}\bar{A}/\bar{A}$. This is proved by induction on $\dim(\bar{X}/\bar{A})$. If this dimension is 0 the claim is obvious. So assume the claim holds for all $\tilde{A}_1$ with $\tilde{A}_1$ satisfying the conditions that $\tilde{A}$ satisfies and $\dim(\bar{X}/\bar{A}_1) < \dim(\bar{X}/\bar{A})$.

Suppose that for $i = 1, 2$ there exist $\tilde{A} < \tilde{A}_i = \tilde{A}_i^\sigma \leq \bar{X}$ such that $\tilde{A}_i/\bar{A} \leq Z(\bar{X}/\bar{A})$ and $\tilde{A}_i = \bar{A} \cdot \bar{X}_i$, for $i \in \{1, \ldots, t\}$. Also, suppose $\tilde{A}_1 \neq \tilde{A}_2$. By the induction hypotheses $L\bar{A}_i/\bar{A}_i = \bar{L}_{\sigma}\bar{A}_i/\bar{A}_i$, for $i = 1, 2$. Consider $\bar{A}_1\bar{A}_2/\bar{A}$. Then $\bar{A}_1\bar{A}_2/\bar{A} \cong \bar{V}_1 \times \bar{V}_2$ (a $T(\sigma)$-isomorphism). By (5.5)
and (5.1) $T$ has inequivalent irreducible representations on $V_{t_1}$ and $V_{t_2}$. Moreover, for $i = 1, 2$ $V_i$ is $T$-isomorphic to $(\tilde{A}_i/A_\sigma) = X_i/\tilde{A}/\tilde{A}$ (for the equality use Lang's theorem and the fact that $X_i \cap A$ is a product of $\tilde{T}$-root subgroups, hence connected). Therefore,

$$L\tilde{A}/\tilde{A} \cap \tilde{A}_1 \tilde{A}_2 = L\tilde{A}/\tilde{A} \cap (\tilde{A}_1)_\sigma (\tilde{A}_2)_\sigma \tilde{A}/\tilde{A}$$

$$= (L\tilde{A}/\tilde{A} \cap X_{t_1} \tilde{A}/\tilde{A})(L\tilde{A}/\tilde{A} \cap X_{t_2} \tilde{A}/\tilde{A}).$$

Now,

$$L\tilde{A}_1/\tilde{A}_1 = \tilde{L}_\sigma \tilde{A}_1/\tilde{A}_1,$$

$$L \leq \tilde{L}_\sigma, \quad |\tilde{L}_\sigma \tilde{A}/\tilde{A}| = |\tilde{L}_\sigma \tilde{A}_1/\tilde{A}_1| \cdot |\tilde{L}_\sigma \tilde{A}/\tilde{A} \cap \tilde{A}_1/\tilde{A}|,$$

and $\tilde{L}_\sigma \tilde{A}/\tilde{A} \cap \tilde{A}_1/\tilde{A} = 1$ or $X_{t_1} \tilde{A}/\tilde{A}$. If the claim is false, we must have the latter case, but $L\tilde{A}/\tilde{A} \cap X_{t_1} \tilde{A}/\tilde{A} = 1$. Passing modulo $A_2$ we have a contradiction. We now suppose that no such groups $A_1, A_2$ exist.

Let $\tilde{Z}/\tilde{A} = Z(\tilde{X}/\tilde{A})$. Then $\tilde{Z}_\sigma = \tilde{Z}$ and $\tilde{Z}$ is a product of root subgroups of $\tilde{T}$. By the above, $\tilde{Z} = \tilde{A} \cdot \tilde{X}_w$ for some $w \in \{1, \ldots, t\}$. By the first claim we may assume $\tilde{L} \cdot \tilde{A}/\tilde{A} \neq \tilde{Z}/\tilde{A}$. As $\dim(\tilde{X}/\tilde{Z}) < \dim(\tilde{X}/\tilde{A})$, we have $L\tilde{Z}/\tilde{Z} = \tilde{L}_\sigma \tilde{Z}/\tilde{Z}$. Also, the usual arguments show that $(\tilde{Z}/\tilde{A})_\sigma = X_w \tilde{A}/\tilde{A}$. So either $L\tilde{A}_1/\tilde{A} \cap X_w \tilde{A}/\tilde{A} = 1$, and we are done by order considerations, or $X_w \tilde{A}/\tilde{A} \leq \tilde{L}_\sigma \tilde{A}/\tilde{A}$. We assume the latter holds. Then $L \geq \tilde{X}_w$ and so $L\tilde{A} > \tilde{Z}$. Let $\tilde{A}_1 \leq \tilde{X}T(\sigma) \leq \tilde{L}$ and $\tilde{A}_1/\tilde{Z}$ is an $\tilde{X}T(\sigma)$-chief factor. Since $\tilde{A}_1$ is a product of $\tilde{T}$-root subgroups, we must have $A_1 = \tilde{X}_k \tilde{Z}$ for some $k$. Since $\tilde{X}_k \tilde{A}/\tilde{A} \neq Z(\tilde{X}/\tilde{A})$ either $[\tilde{X}_i, \tilde{X}_k]$ or $[\tilde{X}_j, \tilde{X}_k]$ is not contained in $\tilde{A}$. With no loss of generality we suppose $[\tilde{X}_i, \tilde{X}_k] \neq \tilde{A}$. Hence $[\tilde{X}_i, \tilde{X}_k] \tilde{A}/\tilde{A} = \tilde{X}_w \tilde{A}/\tilde{A}$.

Now $\dim([\tilde{X}_i, \tilde{X}_k]) < \dim(L)$, so by induction $[\tilde{X}_i, \tilde{X}_k]_\sigma = [X_i, X_k]$. Moreover, $\tilde{X}_k \leq L$ implies that $X_k \tilde{Z}/\tilde{Z} \leq \tilde{L}_\sigma \tilde{Z}/\tilde{Z} = L\tilde{Z}/\tilde{Z}$. Therefore,

$$L\tilde{A}/\tilde{A} \geq [X_i, L] \tilde{A}/\tilde{A} \geq [X_i, X_k] \tilde{A}/\tilde{A} = [\tilde{X}_i, \tilde{X}_k]_\sigma \tilde{A}/\tilde{A} = X_w \tilde{A}/\tilde{A}.$$

At this point the equality $L\tilde{Z}/\tilde{Z} = \tilde{L}_\sigma \tilde{Z}/\tilde{Z}$ and order considerations, imply that $L\tilde{A}/\tilde{A} = \tilde{L}_\sigma \tilde{A}/\tilde{A}$, proving the claim. The result follows by setting $\tilde{A} = 1$.

(6.8) Suppose $q > 5$ and $1 \leq i \leq t$. Then

(i) $X'_i = X_i \cap R_u(\bar{P}_i)'$.
(ii) $\tilde{X}'_i = \tilde{X}_i \cap R_u(\bar{P}_i)'$.
(iii) $V_i \cong X_i/X'_i$ as $F_q[T]$-modules.
(iv) $\bar{V}_i \cong \tilde{X}_i/\tilde{X}'_i$ as $K[T]$-modules.
Proof. By (6.7), $[X_i, X_j] = \overline{[X_i, X_j]}$, so it will suffice to show that 
$[\overline{X_i}, \overline{X_j}] = \overline{X_i \cap R_u(P_j)}'. Let \overline{Y_i} = [\overline{X_i}, \overline{X_j}]. Then \overline{Y_i}$ is \(\overline{T}\)-invariant, hence a product of \(\overline{T}\)-root subgroups. The group \(\overline{X_i}/\overline{Y_i}\) is then \(K[\overline{T}]\)-isomorphic to the direct product of those root subgroups of \(\overline{X_i}\) not contained in \(\overline{Y_i}\). These root subgroups fall into orbits under \(\langle \sigma \rangle\). By definition, \(\overline{X_i}\) is generated by \(\sigma\) proving the result.

The next result will complete the proof of (6.1).

(6.9) Let \(A = A^{T_0}\) be a \(p\)-subgroup of \(G_0\) and assume \(q > 7\). Then

(i) \(A\) is a product of \(T\)-root subgroups.

(ii) Let \(\{C_1, \ldots, C_k\}\) be the composition factors in a fixed \(AT\)-composition series for \(A\). For each \(i\) there exists a unique \(n_i \in \{1, \ldots, t\}\) such that \(C_i \cong V_{n_i}\) as \(F_p[T]\)-modules.

(iii) Let \(n_1, \ldots, n_k\) be \(\sigma\) as in (ii). Then \(A = X_{n_1} \cdots X_{n_k}\) and if \(\overline{A} = \overline{X}_{n_1} \cdots \overline{X}_{n_k}\), then \(A\) is a subgroup of \(\overline{G}\) with \(\overline{A} = A\).

(iv) Let \(n_1, \ldots, n_k\) be as in (ii). Then \(\{n_1, \ldots, n_k\} = \{j \mid X_j \leq A\}\).

Proof. Let \(1 \neq A\) be a \(p\)-subgroup of \(G_0\) with \(A = A^{T_0}\). By (3.9) of [4] there is a canonical parabolic subgroup \(\overline{P}\) of \(\overline{G}\) such that \(A \leq \overline{Y} = R_u(P)\) and \(N_G(A) \leq \overline{P}\). Then \(T_0 \leq \overline{P}\), so by (6.4) \(\overline{T} \leq \overline{P}\). Also, \(\overline{P} = \overline{P}^0\).

Let \(1 = \overline{Y}_0 < \overline{Y}_1 < \cdots < \overline{Y}_k = \overline{Y}\) be a \(\langle \sigma \rangle\)-composition series for \(\overline{Y}\). Then each \(\overline{Y}_i\) is a product of \(\overline{T}\)-root subgroups of \(\overline{G}\), and for \(i = 1, \ldots, k, M_i = \overline{Y}_i/\overline{Y}_{i-1}\) is \(\overline{T}\)-isomorphic to the external direct product of the root groups in some \(\langle \sigma \rangle\)-orbit of roots, say \(\sum_{\sigma}\). Hence, \(\overline{Y}_i = \overline{X}_{n_i}/\overline{Y}_{i-1}\). Recall that for \(i = 1, \ldots, k, \overline{V}_{n_i} = \overline{X}_{n_i} R_u(\overline{P}_{n_i})/R_u(\overline{P}_{n_i})'\). Then \(M_i \cong \overline{V}_{n_i}\).

Also, Lang’s theorem implies that \((M_i)_{\sigma} = X_{\sigma}\overline{Y}_{i-1}/\overline{Y}_{i-1} \cong \overline{V}_{n_i}\). By order consideration we have \(Y = \overline{Y}_{\sigma} = X_{n_1} \cdots X_{n_k}\). This shows that \(X_{n_1}, \ldots, X_{n_k}\) are \(T\)-root subgroups satisfying the following conditions: (i) \(A \leq X_{n_1} \cdots X_{n_k} = Y\); (ii) \(X_{n_1} \cdots X_{n_i} \leq Y\) and \(\overline{X}_{n_1} \cdots \overline{X}_{n_i} \leq \overline{Y}\) for \(i = 1, \ldots, k\); (iii) \(\overline{X}_{n_1} \cdots \overline{X}_{n_i}/\overline{X}_{n_1} \cdots \overline{X}_{n_{i-1}} \cong \overline{V}_{n_i}\) and \(\overline{X}_{n_1} \cdots \overline{X}_{n_i}/\overline{X}_{n_1} \cdots \overline{X}_{n_{i-1}} \cong \overline{V}_{n_i}\) for \(i = 1, \ldots, k\). Among all sets of \(T\)-root subgroups \(\{X_{l_1}, \ldots, X_{l_m}\}\) that satisfy (i), (ii), and (iii), choose one such that \(|X_{l_1} \cdots X_{l_m}|\) is minimal. We claim that \(A = X_{l_1} \cdots X_{l_m}\).

Let \(L = X_{l_1} \cdots X_{l_m}\) and \(L_i = X_{l_i} \cdots X_{l_i}\) for \(1 \leq i \leq m\). Similarly, set \(\overline{L} = \overline{X}_{l_1} \cdots \overline{X}_{l_m}\) and \(\overline{L}_i = \overline{X}_{l_i} \cdots \overline{X}_{l_i}\). Suppose \(A < L\). Then for some \(iA \cap L_i \leq L_{i-1}\) (i.e. \(A\) avoids the \(LT\) composition factor \(L_i/L_{i-1}\)). Choose \(i\) maximal for this. By minimality of \(|L|\), \(i < m\). Also, \(AL_i = L\). Now, 
\[L/L_{i-1} = AL_{i-1}/L_{i-1} \times L_i/L_{i-1}\] and this will be a contradiction to minimality if we can show that \(AL_{i-1}\) is a product of root subgroups
satisfying the necessary conditions. To see this consider $\tilde{L}/\tilde{L}_{i-1}$. Suppose $j, k \geq i$ and consider $[\tilde{X}_j, \tilde{X}_k] \cap \tilde{L} = \tilde{I}$. The group $\tilde{I}$ either covers or avoids the $\tilde{L} \cdot \langle \sigma \rangle$ composition factor $\tilde{L}_i/\tilde{L}_{i-1}$ and $\tilde{I}$ is a product of $\tilde{T}$-root subgroups. So if $\tilde{I}$ covers $\tilde{L}_i/\tilde{L}_{i-1}$, then $\tilde{I} \subseteq [\tilde{X}_j, \tilde{X}_k]$. Consequently, (6.7) implies that $X_i \leq [X_j, X_k]$. But $AL_i = L$ and $L_i/L_{i-1} \leq Z(L/L_{i-1})$. So this forces $L_i/L_{i-1} = X_i L_{i-1}/L_{i-1} \leq A'L_{i-1}/L_{i-1}$, whereas we have assumed that $A$ avoids $L_i/L_{i-1}$. Therefore, $\tilde{I}$ avoids $\tilde{L}_i/\tilde{L}_{i-1}$.

Letting $l_j = l_k = l_{i+1}$ we see that $\tilde{L}_{i+1}/\tilde{L}_{i-1}$ is abelian. Since $\tilde{L}_{i+1}$ is a product of root subgroups we have $\tilde{L}_{i+1}/\tilde{L}_{i-1} = (\tilde{X}_{l_{i+1}} \tilde{L}_{i-1}/\tilde{L}_{i-1}) \times (\tilde{X}_{l_i} \tilde{L}_{i-1}/\tilde{L}_{i-1})$ (consider the action of $\tilde{T}(\sigma)$). Letting $l_j = l_{i+1}$ and $l_k$ vary, we see that $\tilde{L}_{i-1} \tilde{X}_{l_{i+1}} \subseteq \tilde{L}$. Consequently, the $m$-tuple $(X_{l_1}, \ldots, X_{l_m})$ satisfies conditions (i), (ii), and (iii). Notice, also, that $L_{i+1}/L_{i-1} = (X_{l_{i+1}} L_{i-1}/L_{i-1}) \times (X_{l_i} L_{i-1}/L_{i-1})$, and $T_0$ acts irreducibly on each factor with inequivalent representations. Since $A$ is $T_0$-invariant we conclude that $(A \cap L_{i+1}) L_{i-1}/L_{i-1} = X_{l_{i+1}} L_{i-1}/L_{i-1}$. Therefore, a rearrangement of $X_{l_i}, \ldots, X_{l_m}$ also satisfies conditions (i), (ii), and (iii) with an avoided factor nearer the end of an $LT_0$ composition series of $L$. Repeating this a sufficient number of times we obtain a contradiction to the minimality of $|L|$, because at the last step we have $A$ contained in a proper subgroup of $L$ which has the correct form. This proves the claim and the result follows.

We complete this section with one additional result that is useful in computations.

(6.10) Let $X_{n_1}, \ldots, X_{n_k}$ be $T$-root subgroups. Suppose that either $q > 5$ and $\langle X_{n_1}, \ldots, X_{n_k} \rangle$ is unipotent or $q > 7$ and $\langle X_{n_1}, \ldots, X_{n_k} \rangle$ is nilpotent. Then

(a) $\langle X_{n_1}, \ldots, X_{n_k} \rangle_{\sigma} = \langle X_{n_1}, \ldots, X_{n_k} \rangle$.

(b) $[X_{n_1}, \ldots, X_{n_k}]_{\sigma} = [X_{n_1}, \ldots, X_{n_k}]$.

Proof. Suppose $q > 7$ and $\langle X_{n_1}, \ldots, X_{n_k} \rangle = D$ is nilpotent. By (3.9) of [4] there is a canonical parabolic subgroup $P$ of $G$ with $D \leq R_u(P)$ and $N_G(D) \leq P$. Hence $T_0 \leq P$ and, by (6.4), $\tilde{T} \leq \tilde{P}$. The argument of (6.9) shows that $\tilde{X}_n \leq R_u(\tilde{P})$ for $1 \leq i \leq k$. Hence $\langle \tilde{X}_{n_1}, \ldots, \tilde{X}_{n_k} \rangle$ is unipotent. So in either case we have $\tilde{X} = \langle \tilde{X}_{n_1}, \ldots, \tilde{X}_{n_k} \rangle$ a unipotent group.

We may assume $k > 1$. Suppose (b) holds. We prove (a) by induction on the number of $\langle \sigma \rangle$-orbits of root subgroups in $\tilde{X} = \langle \tilde{X}_{n_1}, \ldots, \tilde{X}_{n_k} \rangle$. We have $\tilde{X}' = \langle \tilde{X}_{i_1}, \ldots, \tilde{X}_{i_t} \rangle \subseteq \{n_1, \ldots, n_k\}$ and since $\tilde{X}'$ is invariant under both $\tilde{T}$ and $\langle \sigma \rangle$, $\tilde{X}' = \langle \tilde{X}_{j_1}, \ldots, \tilde{X}_{j_t} \rangle$ for some $\{j_1, \ldots, j_t\} \subseteq \{1, \ldots, t\}$. Inductively, $(\tilde{X}')_{\sigma} = \langle X_{j_1}, \ldots, X_{j_t} \rangle$. Also, $\tilde{X}/\tilde{X}'$ is the product of
the groups $M_i = \bar{X}_n / \bar{X}'$. For $i = 1, \ldots, k$, $M_i$ is either trivial or $M_i \cong \bar{v}$, so (5.1)(i) implies that $\bar{X}/\bar{X}'$ is the direct product of the nontrivial $M_i$. Consequently, Lang's theorem implies $\bar{X}_\alpha = \langle X_{n_1}, \ldots, X_{n_k} \rangle \bar{X}' = \langle X_{n_i}, \ldots, X_{n_k} \rangle \langle X_{i_1}, \ldots, X_{i_j} \rangle$. We may choose $j_1, \ldots, j_s$ such that for each $j_i$, $X_{i_i} \leq [X_{i_1}, \ldots, X_{i_j}]$ for some $\{i_1, \ldots, i_s\} \subseteq \{n_1, \ldots, n_k\}$. So by (b), $X_{i_i} \leq [X_{i_1}, \ldots, X_{i_j}] \leq \langle X_{n_1}, \ldots, X_{n_k} \rangle$ and (a) follows. Therefore, it will suffice to prove (b).

To prove (b) argue by induction on $k$. For $k = 1$ the result is trivial and for $k = 2$ apply (6.7). So suppose $k \geq 3$ and that the result holds for $k - 1$. Set $Y = [X_{n_1}, \ldots, X_{n_{k-1}}]$, $\bar{Y} = [\bar{X}_{n_1}, \ldots, \bar{X}_{n_{k-1}}]$, $D = [Y, X_{n_k}]$, and $\bar{D} = [\bar{Y}, \bar{X}_{n_k}]$. Let $\Gamma = \{i \mid X_i \leq D\}$. Then (6.9) implies that $\bar{D} = \prod_{i \in \Gamma} X_i$. Also, (6.9)(iii) shows that $\bar{D}_1 = \prod_{i \in \Gamma} \bar{X}_i$ is a group with $(\bar{D}_1)_\sigma = D$.

Suppose $h \in \{1, \ldots, t\}$ and $X_h \leq \bar{Y}$. By (6.7), $[\bar{X}_h, \bar{X}_{n_k}] = [X_h, X_{n_k}] \leq [Y, X_{n_k}] = D$. Then $[\bar{X}_h, \bar{X}_{n_k}] \leq \bar{D}_1$. If $\bar{Y}, \bar{X}_{n_k} \leq N_G(D_1)$, then letting $h$ vary we have $\bar{D} = [\bar{Y}, \bar{X}_{n_k}] \leq \bar{D}_1$, whence $\bar{D}_\sigma = (\bar{D}_1)_\sigma = D \leq \bar{D}_\sigma$, proving the result. So let $\bar{X}_j \leq Y$ or $\bar{X}_j = \bar{X}_{n_k}$. It will suffice to show that $\bar{X}_j \leq N_G(\bar{D}_1)$. Since $Y, X_{n_k} \leq N(D)$, we have $[X_j, X_i] \leq D$ for each $i \in \Gamma$. Thus, $[X_j, X_i]$ is a product of certain of the groups $X_\alpha$, for $\alpha \in \Gamma_1 \subseteq \Gamma$. Then (5.5) and (6.8)(iii) imply $[\bar{X}_j, \bar{X}_i] = \prod_{\alpha \in \Gamma_1} \bar{X}_\alpha \leq \bar{D}_1$, as desired. This completes the proof.

7. Nonsolvable $T_0$-invariant subgroups. In this section we maintain the previous notation. In addition, let $Y$ be a $T_0$-invariant subgroup of $G$ such that $Y = Y_1 \cdots Y_n$, a central product of groups of Lie type in characteristic $p$. For $1 \leq i \leq n$ write $Y_i = O^p(Y_i) = Y_i(p^{e_i})$. The goal of this section and the next is to relate $Y$ to the Lie structure of $G$ and to the root system of $G$. Throughout this section we assume $p \geq 5$ and $q > 7$.

The main results of this section are as follows:

(7.1) $T_0$ contains a maximal torus of $Y$.

(7.2) Suppose $T_0 \leq T_1 \leq T$ and $T_1 \leq N(Y)$. For $1 \leq i \leq n$, let $J_i$ be a Cartan subgroup of $Y_i$. Then $J = \prod_{i=1}^n C_{Y_0}(J_i)$ is a maximal torus of $G_0 T_1$.

These results will be used in later sections to characterize such groups $Y$. The difficulty is that, at the outset, the groups $Y_i$ are not known to have any connection with the existing Lie structure of $G$. In particular, $p^{e_i}$ is not known to be a power of $q$.

We will prove (7.1) and (7.2) together, in a series of steps. Suppose that one of (7.1) or (7.2) is false and choose a counterexample (for some
choice of $T_0$) with $|Y| \cdot |G_0|$ minimal. Then $Z(G_0) = 1$. First assume that (7.1) fails for $Y$ and set $S = N_{T_0}(Y_1)$.

(7.3) (i) $T_0$ is transitive on $\{Y_1, \ldots, Y_n\}$.
(ii) $S \cong C(Y)$.

**Proof.** (i) is trivial from the minimality of $Y$, since otherwise we could replace $Y$ by the products of the $T_0$-orbits on $\{Y_1, \ldots, Y_n\}$. For (ii), suppose $S \leq C(Y)$ and let $x \in Y_1 - Z(Y_1)$ be a $p'$-element. Then $A = \langle x^{T_0} \rangle$ is abelian and $T_0$-invariant. Therefore, (6.2) implies that $[A, T_0] \leq A \cap T_0 \leq A \cap S \leq C(Y)$. Letting $x$ vary we have $[Y, T_0] \leq C(Y)$, hence $[Y, T_0] \leq Z(Y)$. But this forces $[Y, T_0] = 1$, contradicting (2.8).

(7.4) $Y = Y_1$.

**Proof.** Suppose $n > 1$ and let $I_i < Y_i$ be an $S$-invariant abelian $p'$-group. Then $I = \langle I_1^{T_0} \rangle$ is a $T_0$-invariant abelian $p'$-group and as in the last result we apply (6.2) to obtain $[I, T_0] \leq T_0 \cap I \leq C_Y(T_0)$. It follows that $I_1Z(Y_1)/Z(Y_1)$ is an elementary abelian 2-group (also $T_0/S \cong Z_2$).

By (2.14) we may take $I_1$ to be a maximal torus of $Y_1$. Suppose the overlying algebraic group of $Y_1$ has Lie rank $l$ and set $q_0 = p^{e_1}$. By (2.1)(iii), $I_1$ has rank at most $l$ (as an abelian group). So $|I_1/Z(Y_1)| \leq 2^l$. On the other hand $|I_1/Z(Y_1)| = e^{-1}(q_0)$, where $e \leq l + 1$. By (2.4)(iii) we have $e^{-1}(q_0) \geq e^{-1}(q_0 - 1)^l \geq e^{-1}4^l$ (since $q_0 \geq p \geq 5$). Therefore, $2^l \geq e^{-1}4^l \geq (l + 1)^{-4}$, forcing $l = 1$. The only possibility is $Y_1/Z(Y_1) \cong PSL(2, 5)$. However, here one can argue that $T_1 \cap Y_1 \leq Z(Y_1)$ and that $I_1$ can be chosen as a subgroup of order 3. This is a contradiction.

By (2.14) we may choose a maximal torus $I$ of $Y$ with $I^{T_0} = I$. Let $I_0 = I \cap T_0$.

(7.5) (i) $C_I(T_0) = I_0 \supseteq [T_0, I]$.
(ii) $Z(Y) \cap T_0 = 1$.
(iii) $[T_0, I]$ is cyclic.

**Proof.** We first use (6.2) to obtain $[T_0, I] \leq T_0 \cap I = I_0$. Also, $I_0 \leq C_I(T_0) \leq T_0$ by (2.8). Thus (i) holds.

Suppose $1 \neq z \in Z(Y) \cap T_0$ and let $E(C_G(z)) = X_1 \cdots X_s$, a commuting product of groups of Lie type over extension fields of $F_q$ (see (2.9)). By (2.9)(v), $T_0 \cap X_i$ is a maximal torus of $X_i$ for $i = 1, \ldots, s$. Since $Y = O^{p'}(Y)$ we have $Y \leq X_1 \cdots X_s$, so (2.8) implies that $Y \not\supset C(T_0 \cap X_i)$.
for some \( i \). Therefore, \( Y = [Y, T_0 \cap X_i] \leq X_i \), so minimality of \( |Y| \cdot |G_0| \) shows that \( T_0 \cap X_i \) contains a maximal torus of \( Y \). This is a contradiction, proving (ii).

Suppose (iii) false and choose \( Z_r \times Z_r \cong E \leq [T_0, I] \), where \( r \) is prime. Let \( 1 \neq e \in E \) satisfy \( E(C_Y(e)) \neq 1 \). (It is not difficult to check the existence of such an \( e \). Consider \( E \) contained in a maximal torus \( T \) of a suitable algebraic group. Then \( E \) acts on each \( T \)-root subgroup, inducing a cyclic product of components. Minimality implies that (7.1) and (7.2) hold for the group \( D_1 \cdots D_n \). By (7.1) \( T_0 \cap D_i \) contains a maximal torus of \( D_i \) for \( 1 \leq i \leq m \). On the other hand, (2.5)(v) shows that \( I \cap D_i \) contains a maximal torus of \( D_i \), and \( T_0 \cap D_i \), \( I \cap D_i \) normalize each other.

Fix \( 1 \leq i \leq m \) and let \( H_i \) be a Cartan subgroup of \( D_i \). By (7.2) \( A_i = C_{T_i T_0}(H_i) \) is a maximal torus of \( G_0 \). We claim that each \( H_i \)-root subgroup of \( D_i \) is also an \( A_i \)-root subgroup of \( G_0 \). We remark that the argument used here will be quoted in the proofs of (7.8) and (7.9). By (3.6) \( A_i \) permutes the \( H_i \)-root subgroups of \( D_i \) and centralizes \( H_i \). So (5.6) and the assumption \( p \geq 5 \) implies that either \( A_i \) normalizes each \( H_i \)-root subgroup of \( D_i \) or there exist \( H_i \)-root subgroups \( R_1, R_2 \) such that \( A_i \) normalizes \( \langle R_1, R_2 \rangle \cong \text{SL}(2, 5) \) or \( \text{PSL}(2, 5) \). In the first case the claim follows since \( A_i \leq C(H_i \cap \langle R_1, R_2 \rangle) \). So suppose the latter case holds and let \( \bar{A}_i = C_{A_i}(\langle R_1, R_2 \rangle) \). Then \( |A_i : \bar{A}_i| \leq 4 \) and \( C_{\bar{G}}(\bar{A}_i)^0 \) is not a maximal torus. Write \( C_{\bar{G}}(\bar{A}_i)^0 = \bar{XZ} \), where \( \bar{X} \) is semisimple and \( Z = Z(\bar{XZ})^0 \). Let \( \bar{A}_i \) be the \( \sigma \)-invariant maximal torus containing \( A_i \). Then \( \bar{Z} \leq \bar{A}_i \) and \( \bar{A}_i \cap \bar{X} \) is a maximal torus of \( \bar{X} \). Now use (2.4) applied to \( \bar{X} \) and the fact that \( q \geq 25 \) (since \( p = 5 \)) to conclude \( |A_i \cap \bar{X} : A_i \cap Z(X)| > 4 \). This contradicts \( \bar{X} \leq C(\bar{A}_i) \) and proves the claim.

Since each \( A_i \)-root subgroup has Frattini quotient on \( \mathbb{F}_q \)-module, the above claim shows that \( D_i \) is defined over a field of size at least \( q \). Thus (6.3) and (2.8) both apply to \( D_i \). From (6.3) we conclude \( T_0 \cap D_i = I \cap D_i \) for each \( i \). From (2.8) we see that if \( T_{00} = (T_0 \cap D_1) \cdots (T_0 \cap D_m) \) and \( C = C(e) \cap C(D_1 \cdots D_m) \), then \( S = C_{C(e)}(T_{00}C/C) \) is an abelian subgroup of \( C_Y(e)/C \). Since both \( T_0C/C \) and \( IC/C \) are contained in \( S \), we conclude \( [T_0, I] \leq C \).

Then \( E \leq C \) and, in particular, \( E \) centralizes a proper \( p \)-subgroup of \( Y \). By (2.3) of [23] this implies that \( Y \) is generated by the subgroups \( D_1 \cdots D_m \) as \( E \) ranges over \( E^* \). Hence, \( [T_0, I] \leq C(Y) \cap I \cap T_0 \leq Z(Y) \cap T_0 = 1 \), by (ii). Then (2.8) gives \( I \leq T_0 \), which we are assuming false. This proves (iii).
(7.6) Let $[T_0, I] = \langle x \rangle$.

(i) $x \neq 1$.

(ii) $I \leq C_G(x)^0$.

Proof. If $x = 1$, then $I \leq C_{G_0}(T_0) = T_0$, which we are assuming false. So (i) holds. Let $\tilde{G} = C_G(x)^0$. To prove (ii) we make use of the universal covering group, $\tilde{G}$, of $G$. Let $\pi: \tilde{G} \rightarrow \tilde{G}$ be the natural surjection and regard $\sigma$ as acting on $\tilde{G}$ and commuting with $\pi$. Then $\tilde{G}_\sigma = G_1$ maps, via $\pi$, onto $G_0$. Now $(Y)\pi^{-1}$ is the central product of part of $Z(\tilde{G})$ with a covering group, $Y_1$, of $Y$. Since $Y_1$ is also a group of Lie type, $(I)\pi^{-1}$ is abelian. Choosing $\tilde{x}$ to be a preimage of $x$ we have $(I)\pi^{-1} \leq C_{\tilde{G}}(\tilde{x}) = C_{\tilde{G}}(\tilde{x})^0$ (see (4.4) of [25]). Therefore, $I \leq (C_{\tilde{G}}(\tilde{x}))\pi = C_G(x)^0$, proving (ii).

At this point we obtain a contradiction. Let $\tilde{C} = C_G(x)^0$ and $C = \tilde{C} \cap G_0$. By (2.9) $C = E(C)T_0$ and by (2.5)(v) $(T_0 \cap E(C))Z(C)/Z(C)$ is a maximal torus of $E(C)Z(C)/Z(C)$. Moreover, $[I, T_0] \leq I_0 \leq Z(C)$, so $IZ(C)/Z(C)$ centralizes $(T_0 \cap E(C))Z(C)/Z(C)$. It follows from (2.8) that $I \leq T_0Z(C) = T_0$, a contradiction.

At this point we know that (7.1) holds for $Y$ (and for all smaller groups). Consequently, (7.2) must fail for $Y$. Recall, that $T_0 \leq T_1 \leq T$.

(7.7) Let $Z = Z(YT_1)$

(i) $Z \leq T_1$.

(ii) $Y = Y_1$, so $Y$ is quasisimple.

(iii) $T_0 \cap Y$ contains a maximal torus, $I$, of $Y$.

Proof. Suppose $yt \in Z$, with $y \in Y$ and $t \in T_1$. Then $y \in C_{G_0}(T_0) = T_0$ (by (2.8)), proving (i). (iii) is immediate from (7.1).

Suppose $n > 1$. By minimality of $|Y| \cdot |G_0|$, $P = C_{Y_nT_1}(J_n)$ is a maximal torus of $G_0T_1$. Also, $P$ normalizes $Y_1 \cdots Y_{n-1}$, so another application of minimality together with (2.3) shows that

$$I = \bigcap_{i=1}^{n-1} C_{Y_1 \cdots Y_{n-1}P}(J_i)$$

is a maximal torus of $G_0P = G_0T_1$. Now $C_{YT_1}(J_n) = Y_1 \cdots Y_{n-1}P$, so $\bigcap_{i=1}^{n} C_{YT_1}(J_i) = I$. Therefore $n = 1$, proving (ii).

(7.8) $Z(YT_1) = C_{YT_1}(Y) = 1$.

Proof. Let $Z = Z(YT_1)$ and $C = C_{YT_1}(Y)$. Clearly, $Z \leq C$. Also, $CT_1$ is a solvable $p'$-group, so (6.1) implies $C \leq N(T_1)$. On the other hand,
C \leq YT_1$; we conclude that $[C, T_1] \leq C \cap T_1 \leq Z(YT_1) = Z$. If $[C, T_1] = 1$, then $C \leq C_{\sigma_0 T_1}(T_1) = T_1$ and $Z = C$. Suppose $C \neq 1$. Then we conclude $C \cap T_1 \neq 1$.

Set $Z_1 = C \cap T_1$ and $\overline{D} = E(C_\overline{G}(Z_1)^0)$. By (7.7) $T_0 \cap Y$ contains a maximal torus of $Y$, so $Y$ is generated by conjugates of $T_0 \cap Y$. It follows that $Y \leq \overline{D}$. Let $D = O^p(\overline{D}_0)$. Then $Y \leq D$ and (2.5) implies that $T_1$ contains a maximal torus of each component of $D$. Let $D_1$ be a component of $D$ and $T_2 = T_1 \cap D_1$. Then $T_2 \leq Z(D_1)$, so $T_2 \leq Z_1$ and $Y = [Y, T_2] \leq D_1$. Letting $D_1$ vary, we conclude $D$ is quasisimple.

Since $\overline{TD}$ is connected we write $\overline{TD} = \overline{DZ}$, where $\overline{Z}$ is a torus, $\overline{Z} \leq \overline{T}$, and $[\overline{Z}, \overline{D}] = 1$. By induction, $J_2 = C_{YT_2}(J_1)$ is a maximal torus of $D$. Let $\overline{A}$ be a maximal torus of $\overline{D}T$ with $J_2 \leq \overline{A}$. Then $\overline{Z} \leq \overline{A}$ and $\overline{A}^\sigma = \overline{A}$. Set $J_3 = G_0 T_1 \cap \overline{A}$. By definition, $\overline{A}_{\sigma}$ is a maximal torus of $\overline{G}_\sigma$, so $J_3$ is a maximal torus of $G_0 T_1$.

At this point we apply the argument of (7.5) to show that each nilpotent $J_2$-root subgroup of $D$ is also a $J_3$-root subgroup of $G_0$. Similarly, if we use the groups $Y, J_1, J_2,$ and $D$ we conclude that each $J_1$-root of $Y$ is a $J_2$-root subgroup of $D$, hence a $J_3$-root subgroup of $G$. As in the proof of (7.5) we have $Y$ defined over a field of order at least $q$. Suppose $yt \in C$ with $y \in Y$ and $t \in T_1$. Then $y \in C_Y(T_0 \cap Y)$, so by (2.8) (which now applies to $Y$) we have $y \in T_0 \cap Y$. This shows that $C \leq T_1$, and so $Z = C = Z_1$.

At this point we invoke Theorem (8.1), the proof of which is independent of (7.1) and (7.2). Let $X_{i_1}, \ldots, X_{i_k}$ be the $J_1$-root subgroups contained in a fixed $J_1$-invariant Sylow $p$-subgroup, $U$, of $Y$. Set $\overline{Y} = \langle \overline{X}_{i_1}, \ldots, \overline{X}_{i_k}, \overline{X}_{i_1}^*, \ldots, \overline{X}_{i_k}^* \rangle$. Then $Y = O^p(\overline{Y}_\sigma)$ (by (8.1)(iii) applied to $\overline{D}$. If $\overline{D}$ has $l$ simple factors apply (8.1) to a diagonal of $\overline{D}$ normalized by $\sigma^l$, then take projections.) Also, $\overline{Y} \leq \overline{DA} = \overline{DZ}$.

As $T_1 \leq N(Y)$, $YT_1 = Y(YT_1 \cap N(J_1)) \leq Y(N(\overline{D}) \cap N(Y) \cap N(J_2))$. But $N(\overline{D}) \cap N(Y) \cap N(J_2)$ permutes the $J_2$-root subgroups of $Y$, so normalizes $\overline{Y}$. Therefore, $T_1 \leq YT_1 \leq N(Y) = N(\overline{Y})$.

Set $\overline{V} = C_{\overline{D}}(\overline{Y})^0$, a $T_1$-invariant subgroup of $\overline{D}$. By (2.14), $T_1$ normalizes a $\sigma$-invariant maximal torus $\overline{L}$ of $\overline{V}$. We have $\overline{A} \overline{Y} = \overline{Z} \overline{Y}$, where $\overline{Z}_1 = Z(\overline{A} \overline{Y})^0$, and $\overline{Z}_1 = \overline{Z}(\overline{Z}_1 \cap \overline{D})$. So $\overline{Z}_1 \cap \overline{D} \leq \overline{V}$ and we see that $\overline{ZVY}$ contains a maximal torus of $\overline{G}$. Therefore, $\overline{E} = C_{\overline{Z} \overline{V} \overline{Y}}(T_1 \cap Y)$ is a $T_1$-invariant maximal torus of $\overline{G}$. Now, $T_0 \leq N(\overline{E}_{\sigma} \cap G_0)$ and (6.3) implies that $T_0 = \overline{E}_{\sigma} \cap G_0$, whence $\overline{E} \leq C_{\overline{G}}(T_0) = \overline{T}$. We conclude that $\overline{E} = \overline{T}$ and $\overline{T} \leq \overline{Y} \overline{V} \overline{Z} \leq N(\overline{Y})$.

By (2.5) $(\overline{YT})_{\sigma} = \overline{Y}_{\sigma} \overline{T}_\sigma = YT$. Let $\overline{C}$ be a maximal torus of $\overline{YT}$ with $J_1 \leq \overline{C}$. Then $\overline{YT} = \overline{YC}$ and $YT = (\overline{YT})_{\sigma} = (\overline{YC})_{\sigma} = Y \overline{C}_{\sigma}$. So $YT_1 = YC_1,$
where \( C_1 = \tilde{C}_o \cap G_0 T_1 \). But \( C_1 \) is a maximal torus of \( G_0 T_1 = G_0 C_1 \) (see (2.3)) and \( C_{YT_1}(J_1) = C_{YC}(J_1) = C_1(C_Y(J_1)) = C_1 J_1 = C_1 \). We are assuming this to be false, so this contradiction proves (7.8).

(7.9) (i) \( I \) is minisotropic.

(ii) There does not exist a subgroup \( D < Y \) such that \( D^{T_1} = D \) and \( D \) a group of Lie type in characteristic \( p \).

(iii) For \( 1 \neq t \in T_1 \), \( C_Y(t) \) does not contain a component of Lie type in characteristic \( p \).

Proof. For (i), suppose \( J \) is contained in a proper parabolic subgroup, \( K \), of \( Y \). The argument of (7.5) shows that each nilpotent \( I \)-root subgroup of \( Y \) is also a \( T_1 \)-root subgroup of \( G \). So \( O_p(K) \) is a product of \( T_1 \)-root subgroups and \( K^{T_1} = K \) (as \( K = N_Y(O_p(K)) \)). If \( K^0 \) is the opposite parabolic then \( T_1 \leq N(K^0) \), so \( T_1 \) normalizes \( K \cap K^0 = L \), a Levi factor of \( K \), containing \( I \). We may assume \( J_1 \leq L \). Let \( L_1 = L' \), so that \( L = L_1 J_1 = L_1 I \). If \( L_1 = 1 \), then \( J_1 = I \leq T_1 \). Since \( J_1 \)-root subgroups of \( Y \) are also \( T_1 \)-root subgroups we have \( Y \) defined over a field with at least \( q \) elements. Then (7.8) and (2.8) imply \( C_{YT_1}(J_1) = T_1 \), a maximal torus of \( G_0 T_1 \).

Suppose then that \( L_1 \neq 1 \), and let \( J_2 = L_1 \cap J_1 \), a Cartan subgroup of \( L_1 \). Minimality implies that \( R = C_{L_1 T_1}(J_2) \) is a maximal torus of \( G_0 T_1 \). As \( J_1 \leq C_{L_1 T_1}(J_2) = C_{L_1 I}(J_2) \leq R \), we also have \( R \leq C_{YT_1}(J_1) = J \).

Replacing \( T_1 \) by \( R \) in the above we have \( J_1 \)-root subgroups of \( Y \) being \( R \)-root subgroups of \( G_0 \). Again we conclude that the defining field for \( Y \) has at least \( q \) elements. Then (2.3), (2.8), and (7.8) yield \( YT_1 = YJ \) and \( J \) Cartan in \( YJ \). So \( J \) is abelian, and another application of (2.8) shows that \( R = J \), a contradiction. Thus (i) holds.

Suppose \( D^{T_1} = D < Y \) and \( D \) a group of Lie type in characteristic \( p \). Let \( A_1 \) be a Cartan subgroup of \( D \) and \( A = C_{DT_1}(A_1) \), a maximal torus of \( G_0 T_1 \), by minimality. But now consider \( YA \). From (i) we conclude \( J_2 = C_{YA}(J_1) \) is a maximal torus of \( G_0 A \). Since \( YA \leq YT_1 \) we also have \( J_2 \leq J \). As in the proof of (7.6) the \( J_1 \)-root groups of \( Y \) are also \( J_2 \)-root subgroups of \( G_0 \), so \( Y \) is defined over a field of at least \( q \) elements. So (2.8) applies to \( YT_1 \) and shows that \( J \) is a maximal torus of \( YT_1 \); in particular an abelian group. But then \( J = C(J_2) \) and (2.8) forces \( J = J_2 \), a contradiction to our supposition. This proves (ii) and (iii) follows.

(7.10) Write \( Y = Y(q_0) \) and \( | T_1 | = \frac{1}{d} \prod \Phi_i(q_0) \).

(i) \( d = 1 \) or \( d \) is prime.

(ii) If \( Y \cong PSL(2, q_0) \), then \( | T_1 | \) is odd.

(iii) \( T_1 \) is cyclic, \( I = T_1 \cap Y \), and \( T_1 \) is a minisotropic torus of \( YT_1 \).
Proof. Suppose \( t \in T_1 \) is an involution and write \( Y = O^\tau(\overline{Y}_\tau) \), where \( \tau \) is an endomorphism of the algebraic group \( \overline{Y} \). If \( t \) extends to an involutory automorphism of \( \overline{Y} \) commuting with \( \tau \), then \( \overline{Y}_\tau^0 \) is reductive and (7.9)(iii) implies that \( \overline{Y}_\tau^0 \) is a torus. Let \( \overline{U} \) be the unipotent radical of a \( t \)-invariant Borel subgroup of \( \overline{Y} \). Then \( t \) inverts \( \overline{U} \), \( \overline{U} \) is abelian, and \( Y \approx \text{PSL}(2, q_0) \). So if \( |T_1| \) is even and \( Y \approx \text{PSL}(2, q_0) \), then some involution \( t \in T_1 \) induces a field or graph-field automorphism of \( Y \), against (7.9)(i). This establishes (ii). If \( Y \approx \text{PSL}(2, q_0) \) then it easily follows that (i) and (iii) also hold. Suppose then that \( Y \approx \text{PSL}(2, q_0) \). Thus \( |T_1| \) is odd. (2.8) or its proof in case \( q_0 = 5 \) shows that elements of odd order in \( Y^\tau_\lambda \) centralizing \( \lambda \) lie in a maximal torus \( V \) of \( Y^\tau_\lambda \). Hence \( \lambda \) is odd. (iii) follows from (7.9)(i) and the argument used to prove (7.5)(iii).

For (i) we note that for \( Y \approx \text{PSO}^{2k} (2, q_0) \), \( \text{PSL}(k, q_0) \), or \( \text{PSU}(k, q_0) \), we automatically have \( d = 1, 2, \) or \( 3 \). For the other cases the result follows from (7.9)(ii) and (7.9)(iii). Namely, (7.9)(ii) shows that \( T_1 \) must act irreducibly on the underlying vector space so \( Y \neq O^+(2k, q_0) \). Hence \( |T_1| = \frac{1}{2}(q_0^k + 1) \), \( \frac{d}{2}(q_0^k - 1) \), or \( \frac{1}{2}(q_0^k + 1) \), respectively, with \( k \) odd in the unitary case. In the latter two cases (7.9)(iii) forces \( k \) to be prime and since \( d | k \) we are done here. In the remaining case \( d | 4 \) and \( d | q_0^k + 1 \). For \( k \) even, \( q_0^k + 1 \equiv 0 \pmod{4} \), so \( d = 1 \) or \( 2 \). If \( k \) is odd, \( T_1 \) contains an element \( 1 \neq t \) with \( |t| = (q_0 + 1) \) and we contradict (7.9)(iii). So (i) holds.

(7.11) Let \( T_2 \) be a maximal torus of \( YT_1 \) and write \( |T_2| = \frac{1}{d} \prod \Phi_e(q_0) = \frac{1}{d} \prod \Phi_c(q) \). Assume that \( \phi_{j_1} \neq \phi_{j_2} \) for \( j_1 \neq j_2 \) and that \( T_2 \leq \hat{T}_2 \) is a maximal torus of \( G_0T_1 \). Then \( T_2 = \hat{T}_2 \).

Proof. Suppose \( |T_2| = \frac{1}{d} \prod \Phi_e(q) = \frac{1}{d} \prod \Phi_c(q) \). By (2.10) it will suffice to show that \( d = e \) and \( \Sigma \psi(d) = \Sigma \psi(s) \). Write \( |T_1| = \frac{1}{d} \prod \Phi_e(q_0) = \frac{1}{d} \prod \Phi_c(q) \), viewing \( T_1 \) as a maximal torus of \( YT_1 \) and \( G_0T_1 \), respectively. Set \( q_0 = p^a \) and \( q = p^b \).

For \( m \) and \( c \) positive integers \( \Phi_m(p^c) = \prod \Phi_{mc_0}(p) \), the product ranging over those divisors \( c_0 \) of \( c \) such that \( (c/c_0, m) = 1 \). Using this and the two expressions for \( |T_1| \) we have \( |T_1| = \frac{1}{d} \prod \Phi_{e_0}(p) = \frac{1}{d} \prod \Phi_{c_0}(p) \). Moreover, \( \Sigma \psi(d) \psi(e_0) = a \cdot \text{rank}(\overline{G}) = \Sigma \psi(s) \). Consequently, it will suffice to show that \( d = e \) and \( \{e, c_0\} = \{e_0, c_0\} \).

Consider a term \( \Phi_{e_0}(p) \). By the primitive divisor theorem (see Zsigmondy [28]) and our assumption \( p \geq 5 \), either \( e, c_0 = 2 \) and \( p \) is a Mersenne prime or there is a prime divisor \( r \) of \( \Phi_{e,c_0}(p) \) with \( r \mid p^x - 1 \) for \( x < e_0c_0 \). Since \( d | q_0 + 1 \), \( (d, r) = 1 \) if \( e_r > 2 \). For such an \( r \), there is a
pair \((c_j, a_0)\) with \(r \mid \Phi_{c_ja_0}(p)\). This forces (see the proof of (2.10)) \(e, c_0 \mid c_j a_0\) and either equality holds or \(r\) divides \(c_j a_0 / e_i c_0\). Of course, we can reverse all this, starting with a term \(\Phi_{c_0a_0}(p)\).

By (7.10)(i) \(d\) is one or prime. Suppose \(G_0 \cong \text{PSL}(n, q)\) or \(\text{PSU}(n, q)\) with \(e > 3\). Then \(e \leq 3\) and \(e \mid p \pm 1\). Using this and the remarks of the previous paragraph, cancel off terms in the two expressions for \(T_1\) where the subscripts \(e, c_0\) and \(c_j a_0\) coincide. Starting from the largest \(e, c_0\) and \(c_j a_0\) we see that all terms cancel except those where \(e, c_0\) or \(c_j a_0\) is 1 or 2 or possibly a single term \(\Phi_{2c_0}(p)\), where \(d\) is a primitive divisor of the term (note that \(T_1\) minisotropic in \(YT\), forces each \(e_i \geq 1\)). So we are left with an expression \(\frac{1}{\phi}(p + 1)^x = \frac{1}{\phi}(p - 1)^y(p + 1)^z\) or \(\frac{1}{\phi}(p - 1)^y(p + 1)^z = \frac{1}{\phi}(p - 1)^y(p + 1)^z\). Using the fact that \(T_1\) is obtainable from no proper subsystem of the root system of the overlying algebraic group of \(Y\) (see (7.9)(ii)) we use the orders given in Carter \([6]\) and extensions to cover the twisted groups, to conclude \(x \leq 1\). In the first case use the facts that \(\frac{e_i}{d} \leq \frac{3}{d}, p \geq 5\), and (7.10)(ii) to conclude \(e = d\) and \(\{e, c_0\} = \{c_j a_0\}\). In the second case note that \(d \geq 3\) (otherwise, obtain a contradiction using a primitive division of \(\Phi_{2c_0}(p)\)). This forces \(Y \cong \text{PSL}(k, q_0)\) or \(\text{PSU}(k, q_0)\) and as in the proof of (7.10)(iii), \(d = k\). But then \(|T_1| = \frac{1}{\phi}(q_{2d}(q_0)\) or \(\frac{1}{\phi}(q_{2d}(q_0)\) and no \(e_i = 2\). This is a contradiction. Therefore we may now assume that \(G_0 \cong \text{PSL}(n, q)\) or \(\text{PSU}(n, q)\) and \(e \geq 3\).

If \(G_0 \cong \text{PSL}(n, q)\), then \(|T_1| = \frac{1}{\phi}(q^q - 1)\), with \(\Sigma n_i = n\). For the unitary group, replace \(q\) by \(-q\), taking absolute values, if necessary. We obtain \(|T_1| = \frac{1}{\phi}(q^q - 1)\), where the first product is over the odd \(n_i\)'s and the second over the even \(n_i\)'s. Moreover, \(e\) is a divisor of \((n, q - 1)\) or \((n, q + 1)\), respectively. If \(Y \cong \text{PSL}(2, q_0)\), then by (7.10)(ii), \(|T_1|\) is odd, hence there are at most two terms in the product. In the unitary case, if there are two terms, then both powers of \(q\) must be odd.

Let \(y \in N_y(T_1)\) with \(|yT_1| = r\), a prime. By (7.9)(iii) and (2.9) \(C_{T_1}(y)\) is an \(r\)-group with order dividing that of the center of the universal covering group of \(Y\). We also have \(N_{G_0T}(T_1) \cong \prod Z_{n_i}\), the factors acting on (by raising to powers of \(q\) or \(-q\)) the appropriate factor of \(T_1\), centralizing the rest. By (7.10)(iii), \(T_1\) is cyclic. Therefore, \((n_i, n_j) = 1\) for \(n_i \neq n_j\). So \(r\) divides \(n_i\) for a unique \(i\), centralizing a subgroup of the appropriate factor having order \(\frac{1}{f}(q^{n_i/r} - 1)/(q \pm 1)\) with \(f\) a divisor of \(e\). It follows that \(n_i = r\). For each \(n_j \neq n_i\), \(y\) centralizes a subgroup of \(T_1\) of order \(\frac{1}{f}(q^{n_j} - 1)/(q \pm 1)\), where \(f\) divides \(e\). Suppose there exists an \(n_j \neq n_i\) with \(n_j > 2\). Then we can choose a primitive divisor, \(s\), of \(q^{n_j} - 1\), and find an element in \(C_{T_1}(y)\) of order \(s\). By the above, \(s \geq r\) and \(s\) is a divisor of the
universal covering group of $Y$. But $s > 3$, so this and (7.9)(iii) yield $Y \cong \text{PSL}(r, q_0)$ or $\text{PSU}(r, q_0)$. Accordingly, $|T_1| = \frac{1}{d}(q_0' \pm 1)/(q_0 \pm 1)$. Using the facts that $n_j|s-1$ and $s = r$ we have $n_j < n_i = r$. Using the earlier primitive divisor argument in the two factorizations of $|T_1|$ (compare largest $e,c_0$ and $e,a_0$) we conclude that $ar = br$, hence $q = q_0$. But $r|q_0 \pm 1$, so $r$ cannot be primitive for $q^{n_j} \pm 1$ if $n_j > 2$. This is a contradiction, proving that no such $n_j$ exists.

If $Y \cong \text{PSL}(2, q_0)$, then $y$ inverts $T_1$, $r = 2$, and $n \leq 3$, contradicting $e > 3$. So $Y \not\cong \text{PSL}(2, q_0)$, $|T_1|$ is odd, and by earlier remarks, there are at most two $n_i$. As $n > 3$, $r$ is an odd prime. At this point the only possibilities are $G_0 \cong \text{PSL}(r, q)$, $\text{PSU}(r, q)$, $\text{PSL}(r+1, q)$, or $\text{PSU}(r+1, q)$. We chose $r$ to be an arbitrary prime divisor of $|N_Y(T_1)/T_1|$ and found that $N_{G_0}(T_1)/T_1 \cong Z_r$. Checking Carter [6] we see that this forces $Y \cong \text{PSL}(r, q_0)$ or $\text{PSU}(r, q_0)$, thus $|T_1| = \frac{1}{d}\Phi_d(q_0)$ or $\frac{1}{d}\Phi_d(q_0)$. As above, a primitive divisor argument yields $q = q_0$ and $(G_0, Y) = (\text{PSL}(r+1, q), \text{PSL}(r, q))$ or $(\text{PSU}(r+1, q), \text{PSU}(r, q))$.

This leads to $|T_1| = \frac{1}{d}(q' \pm 1)/(q \pm 1) = \frac{1}{d}(q' \pm 1)$, where we always take the plus sign in the unitary case and the minus sign otherwise. Therefore, $e = d(q \pm 1)$ and this forces $d = 1$ and $e = q \pm 1 = (r + 1, q \pm 1)$. In particular, $Y$ has trivial multiplier, so the preimage, $D$, of $J_1$ in the corresponding linear group is abelian. Order considerations show that $D$ is a diagonalizable subgroup of the appropriate linear group, from which it follows that $J = J_1$ is contained in a maximal torus of $G_0 = G_0T_1$ (a Cartan subgroup if $G_0 \cong \text{PSL}(r+1, q)$). Comparing orders we conclude that $J$ is a maximal torus of $G_0$, contradicting the original assumption. This proves (7.11).

(7.12) Write $Y = Y(q_0)$ with $q_0 = p^a$, and let $T_2$ be a cyclic subgroup of $YT_1$ with $|T_2| = \frac{1}{d}\Pi\Phi_d(p)$. Suppose that $\Sigma\varphi(d_i) = a \cdot \text{rank}(\tilde{Y})$ (e.g. $T_2$ a maximal torus of $YT_1$) and that $d_i \neq d_j$ for $i \neq j$. Then

(i) $T_2$ is a maximal torus of $YT_1$ and of $G_0T_1$.

(ii) $YT_1 = YT_2$.

(iii) $T_2$ is a minisotropic torus of $YT_1$.

(iv) $T_2^#$ consists of regular elements of $YT_1$ (in the sense of (7.9)(iii)).

Proof. Since $T_2$ is cyclic, $T_2$ is contained in a maximal torus $T_2^*$ of $YT_1$. By hypothesis and (2.10) we have $T_2 = T_2^*$. Now embed $T_2$ in a maximal torus $T_2'$ of $G_0T_1$. Then (7.11) shows $T_2 = T_2'$. This proves (i) and (ii) follows from (2.3). Also, $G_0T_1 = G_0T_2$. We can now replace $T_1$ by $T_2$, and obtain (iii) and (iv) from (7.9).
The remainder of the proof consists of obtaining a contradiction by constructing a certain maximal torus $T_2$ of $YT_1$ that contradicts (7.12).

First suppose $Y \cong \text{PSL}(n, q_0)$. Then $YT_1 \leq \text{PGL}(n, q_0)$ and $\text{PGL}(n, q_0)$ contains an isomorphic copy of $\text{GL}(n - 1, q_0)$ stabilizing a 1-space of the usual module. So $YT_1$ contains a cyclic maximal torus, $T_2$, of order $\frac{1}{d}(q_0^{n-1} - 1)$, with $T_2$ contradicting (7.12)(iii). So $Y \cong \text{PSL}(n, q_0)$.

We remark that (7.10)(ii) shows that $|T_1|$ is odd. In particular, $|YT_1 : Y|$ is odd, so if $Y$ is an orthogonal or symplectic group, then $T_1 \leq Y$.

If $Y$ is a classical group of dimension $2n$ in which the natural module has a singular $n$-space, then the above remarks show that $YT_1 \leq \text{PSp}(2n, q_0)$, $\text{PSO}^+(2n, q_0)$, or $\text{PGU}(2n, q_0)$. We may then choose $T_2$ to be a maximal torus of order $\frac{1}{d}(q_0^n - 1)$ ($\frac{1}{d}(q_0^{2n} - 1)$ in the unitary case) with $T_2$ stabilizing a singular $n$-space. Again we have a contradiction. If $Y \cong \text{PSO}^-(2n, q_0)$, then $T_2 \leq Y$ and we consider cases. If $n$ is odd, then $Y = YT_1$ contains a cyclic subgroup, $T_2$, of order $\frac{1}{d}(q_0^n + 1)$ and $T_2$ contains a subgroup of order divisible by $(q_0 + 1)/(4, q_0 + 1)$ none of whose nonidentity elements is regular. This contradicts (7.12)(iv). If $n$ is even, $o^-(2n, q_0)$ contains $o^+(2n - 2, q_0) \times Z_{q_0+1}$. Here $T_2$ can be taken as a cyclic group of order $\frac{1}{d}(q_0^{n-1} - 1)(q_0 + 1)$ and contradicting (7.12)(iv).

The remaining classical groups are $Y = YT_1 \cong \text{PSO}(2n + 1, q_0)'$ and $\text{PSU}(2n + 1, q_0)$. Here, use the containments $\text{GL}(n, q_0) \leq \text{PSO}(2n + 1, q_0)$ and $\text{GL}(n, q_0^2) \leq \text{PGU}(2n + 1, q_0)$ to get a maximal torus $T_2$ of order $\frac{1}{d}(q_0^n - 1)$ or $\frac{1}{d}(q_0^{2n} - 1)$, respectively. Again we contradict (7.12)(iv). At this stage we take $Y$ to be an exceptional group.

If $Y \cong G_2(q_0)$, then $Y \cong \text{SU}(3, q_0)$ and $\text{SL}(3, q_0)$. Since $p \geq 5$ one of these has center of order 3, and we choose a cyclic group $T_2$ of order $q_0^2 - q_0 + 1$ or $q_0^2 + q_0 + 1$, accordingly. This violates (7.12)(iv). If $Y = E_7(q_0)$, then $|T_1|$ odd gives $T_1 \leq Y$. By Table (3.3) of [23] $Y \geq ^3D_4(q_0)$ and we take $T_2$ as the direct product of cyclic groups of order $q_0^4 - q_0^2 + 1$ and $\frac{1}{2}(q_0^3 - 1)$. Again this contradicts (7.12)(iv). If $Y = E_8(q_0)$ then Table (3.3) of [23] shows that $Y \geq \text{PSL}(9, q_0)$ or $\text{PSU}(9, q_0)$, according to $3 \mid q_0 + 1$ or $3 \mid q_0 - 1$. Here take $T_2$ to be cyclic of order $(q_0^9 - 1)/(q_0 - 1)$ or $(q_0^9 + 1)/(q_0 + 1)$, respectively, and contradict (7.12)(iv).

Suppose $Y = F_4(q_0)$. By Carter [6], Table (3.3) of [23] and (7.9)(ii) it follows that $T_1$ is the Coxeter torus of $YT_1 = Y$. Now $F_4(q_0)$ contains $^3D_4(q_0)$. To see this use the argument of [23] in the verification of Table (3.3) (note that the subgroup of $F_4(K)$ spanned by all long root subgroups in a fixed system has type $D_4(K)$ and the triality graph automorphism is
induced by a Weyl group element). Now $^3D_4(q_0)$ contains a cyclic maximal torus, $T_2$, of order $q_0^4 - q_0^2 + 1$. By (7.12), $T_2$ is a maximal torus of $Y$, so we may assume $T_2 = T_1$. But this contradicts (7.9)(ii).

Suppose $Y \cong E_6(q_0)$. We claim that $T_1 Y \geq ^3D_4(q_0) \times T_3$, where $T_3$ is cyclic of order $\frac{1}{2}(q_0^2 + q_0 + 1)$. Given this, we take $T_2 = T_4 \times T_3$, where $T_4$ is a cyclic torus of $^3D_4(q_0)$ of order $q_0^4 - q_0^2 + 1$. For existence of $^3D_4(q_0) \times T_3$, argue as in (3.3) of [23]. Namely, we first argue that there is an element of the Weyl group of $Y$ mapping the diagram

$$
\begin{array}{c}
\alpha_1 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \\
\downarrow \alpha_2
\end{array}
\quad \text{to} \quad
\begin{array}{c}
r \alpha_2 \alpha_4 \alpha_3 \alpha_1 \\
\downarrow \alpha_5
\end{array}
$$

where $r$ is the negative of the root of highest height. Since the Weyl group is transitive on fundamental systems, we can either do this or map the first system to the reverse of the second. In the latter case multiply by the graph automorphism of $Y$ to get a map as desired. However, the resulting map induces an element of order 3 on $Z\Sigma$ so cannot involve a graph automorphism. Now, complete the construction as in [23].

Next, suppose $Y \cong ^2E_6(q_0)$. Here, we note that if $\hat{W}$ is the Weyl group of $E_6(K) = \hat{Y}$, then for $\tau$ the graph automorphism, $\hat{W}\langle \tau \rangle \cong \hat{W} \times Z_2$, where the nonidentity central element sends all roots to their negatives. It follows from the previous case that $^2E_6(q_0)$ contains $^3D_4(q_0) \times T_3$ where $T_3$ is cyclic of order $\frac{1}{2}(q_0^2 - q_0 + 1)$. So we again get a maximal torus $T_2$ of $YT_1$ that contradicts (7.12)(iv).

The final case to consider is $Y \cong ^3D_4(q_0)$. Then Table (3.3) of [23] shows that $Y$ contains $X \cong \text{PSL}(3, q_0)$ or $\text{PSU}(3, q_0)$ according to whether $3 | q_0 + 1$ or $3 | q_0 - 1$. Accordingly, $C_T(X)$ is cyclic of order $q_0^2 + q_0 + 1$ or $q_0^2 - q_0 + 1$. Therefore, we let $T_2$ be cyclic of order $(q_0^2 - 1)(q_0^2 \pm q_0 + 1)$ resp. and contradict (7.12)(iv). We have now considered all cases and the proof of (7.1) and (7.2) is complete.

8. $T_0$-invariant groups of Lie type. In this section we continue the analysis of §7. Let $Y$ be a $T_0$-invariant subgroup of $G_0$ such that $Y$ is a commuting product of groups of Lie type in characteristic $p$. Assume that $p \geq 5$ and $q > 7$. In (8.1) we assume $T_0 \cap Y$ is a Cartan subgroup of $Y$ and show that $Y$ is related to the root system of $G$. In later sections we will apply (8.1) and the results of §7 to determine $Y$ in the general case. Write $Y = Y_1 \cdots Y_k$ a commuting product of groups of Lie type in characteristic $p$. 
THEOREM (8.1). Suppose $T_0 \cap Y_i$ is a Cartan subgroup of $Y_i$ for $i = 1, \ldots, k$, and let $U_i \subseteq \text{Syl}_p(Y_i)$ with $U_i$ invariant under $T_0 \cap Y_i$. For each $1 \leq i \leq k$, there exist $T_0$-root subgroups $X_{j_1}^i, \ldots, X_{j_t}^i$ of $G$ such that the following hold:

(i) $U_i = X_{j_1}^i \cdots X_{j_t}^i$.
(ii) $Y_i = Y_i(q^e)$ for some $e_i \geq 1$.
(iii) $Y_i = \text{O}^p((Y_i)_e)$, for $Y_i = \langle X_{j_1}^i, \ldots, X_{j_t}^i, (X_{j_1}^i)^*, \ldots, (X_{j_t}^i)^* \rangle$.
(iv) $\overline{Y_i}$ is the commuting product of a $\langle \sigma \rangle$-orbit of $e_i$ semisimple subgroups of $\overline{G}$, each generated by $\overline{T}$-root subgroups of $\overline{G}$.
(v) $\langle \overline{Y_1}, \ldots, \overline{Y_k} \rangle = \overline{Y_1} \cdots \overline{Y_k}$, a commuting product.

By way of example, say $Y \cong D_4(q^j)$. Then $\overline{Y}$ will be the commuting product of $j$ copies of $D_4(K)$, the components of $\overline{Y}$ corresponding to a subsystem of $\overline{\Sigma}$ having the structure of $j$ orthogonal copies of $D_4$.

The proof of (8.1) will be carried out in a series of steps. Assume the hypothesis of (8.1). The idea of the proof is this. First we reduce to the case where $Y$ has just one factor. Next we consider the case $Y = \text{SL}(2, p^e)$ or $\text{PSL}(2, p^e)$. This is the hardest case. After that we work through the various rank 2 possibilities for $Y$ as well as the 3-dimensional unitary group. The general case follows by induction and an application of (2.13).

(8.3) (i) Each $(T_0 \cap Y)^*$-root subgroup of $Y$ is a $T$-root subgroup of $G$.
(ii) $U_i = X_{j_1}^i \cdots X_{j_t}^i$ for $T$-root subgroups $X_{j_1}^i, \ldots, X_{j_t}^i$ of $G$.
(iii) $Y_i = Y_i(q^e)$ for some $e_i \geq 1$.
(iv) Let $\overline{Y}_i$ be as in (8.1)(iii). Then $\langle \overline{Y_1}, \ldots, \overline{Y_k} \rangle = \overline{Y_1} \cdots \overline{Y_k}$ is a commuting product.

Proof. Since $T_0 \leq C(T_0 \cap Y_i)$ for $i = 1, \ldots, k$, $T_0$ normalizes each $Y_i$. The argument in the proof of (7.8) shows that each $(T_0 \cap Y_i)$-root subgroup is also a $T_0$-root subgroup of $G_0$. This proves (i) and (ii) follows from this and (6.9). For (iii) note that the defining field of $Y_i$ has order equal to the minimum of the orders of the root subgroups of $Y_i$.

Fix $i$ and a $(T_0 \cap Y_i)$-root subgroup, $D$. Let $E$ be the opposite $(T_0 \cap Y_i)$-root subgroup of $Y_i$. We claim that $E = D^*$, the opposite $T_0$-root subgroup in $G_0$. By (5.5) and (6.8)(ii) it will suffice to show that the representation of $T_0$ on the Frattini quotient of $E$ is inverse to the representation on the Frattini quotient of $D$. To see this set $Z = Z(Y_i T_0^*)$. Then $(T_0 \cap Y_i)Z/Z$ is a Cartan subgroup of $Y_i T_0/Z$ and (2.8) shows that $T_0$ induces diagonal automorphisms on $Y_i$. As $T_0 \leq C(T_0 \cap Y_i)$, (2.3) shows that $T_0/Z$ is a Cartan subgroup of $Y_i T_0/Z$, and the claim follows.
So for each $i$, $Y_i = \langle X_i^j, \ldots, X_i^j, (X_i^j)^*, \ldots, (X_i^j)^* \rangle$, where $\{X_i^j, \ldots, X_i^j\}$ are the $T$-root subgroups contained in $U_i$. To obtain (iv) we need only apply (6.10).

In view of (8.3), we now assume that $Y = Y_1$. Write $X_j = X_j^1$ for each $j$. 

(8.4) Let $j_m \neq j_n \in \{j_1, \ldots, j_t\}$, with $X_m^j$, $X_n^j$ root subgroups of $Y$ corresponding to fundamental roots.

(i) If $V$ is the $(T_0 \cap Y)$-root subgroup of $Y$ opposite to $X_m^j$, then $V = X_m^j$. 

(ii) $[X_m^j, X_n^j] = 1$.

Proof. (i) was established at the end of the proof of (8.3). It follows from (i) that $X_m^j$ is a root group of $Y$ corresponding to the negative of a fundamental root. Since the difference of fundamental roots is never a root we conclude that $[X_m^j, X_n^j] = 1$. So (ii) follows from (6.10).

(8.5) Suppose $G_0$ is a classical group and $Y \cong \text{SL}(2, q^j)$ or $\text{PSL}(2, q^j)$. Then (8.1) holds.

Proof. Here $U = X_i$ for some $1 \leq i \leq t$ and by (8.4), $Y = \langle X_i, X_i^* \rangle$. Let $D = \langle \overline{X}_i, \overline{X}_i^* \rangle$. By (3.5), $D$ is a reductive group and $\overline{D} = \overline{D}_1 \cdots \overline{D}_m$ a commuting product of a $\langle \sigma \rangle$-orbit of reductive quasisimple groups, each generated by $\overline{T}$-root subgroups of $\overline{G}$. We must show that $m = j$ and that $\overline{D}_1 \cong \text{SL}(2, K)$ or $\text{PSL}(2, K)$. Suppose $m > 1$. Then $O^p(\overline{D}_2)$ is isomorphic to $O^p((\overline{D}_1)_{\sigma^m})$, modulo centers. Also, $\overline{X}_i = \Pi_i(\overline{X}_i \cap \overline{D}_i)$. Replacing $G$ by $D_1, \sigma$ by $\sigma^m$, $\overline{X}_i$ by $\overline{X}_i \cap \overline{D}_i$, $\overline{T}$ by $(\overline{T} \cap D_1)_{\sigma^m}$, and $Y$ by the projection of $Y$ to $\overline{D}_1$, we may assume that $\overline{G} = \langle \overline{X}_i, \overline{X}_i^* \rangle$. Then $Y < G_0$.

By (4.1) and (4.2) $X_i \leq O_p(P)$ for $P$ a parabolic subgroup of $G_0$ corresponding to the stabilizer of a singular $l$-space of the usual module, $M$, of the appropriate classical group. In view of (4.1) and (4.2), we may assume that $G_0 \cong \text{PSp}(2s, q)$, $\text{PSU}(2s, q)$ or $\text{PSO}^+(2s, q)'$. In all cases $s > 1$.

It will be more convenient to deal with the appropriate linear group $G_1 = \text{Sp}(2s, q)$, $\text{SU}(2s, q)$, $\text{SO}^+(2s, q)'$, respectively. Accordingly, we set $\overline{G}_1 = \text{Sp}(2s, K)$, $\text{SL}(2s, K)$, or $\text{SO}(2s, K)$. Then $\overline{G}_1$ is a covering group of $\overline{G}$ and universal except for the orthogonal group. We replace $G$ by $G_1$ and $\overline{G}$ by $\overline{G}_1$, in order to consider module actions. We retain the other notation, viewing $X_i$ and $T_0$ as subgroups of $G_1$. $\overline{X}_i$ and $\overline{T}$ as subgroups of $\overline{G}_1$. Let $\overline{M} = K \otimes M$, the natural module for $\overline{G}_1$, where in the symplectic and orthogonal cases the form is extended naturally.
From (4.2)(vi) and (3.4)(vi), we see that under the action of $T_0$, $M$ decomposes into the direct sum of the two inequivalent, irreducible, $T_0$-submodules, $M_1$ and $M_2$, each of dimension $s$. Moreover, the stabilizer in $G_1$ of $M_i$ induces on $M_i$ either $\text{GL}(s, q)$, a subgroup of index $q + 1$ in $\text{GL}(s, q^2)$, or a subgroup of index 2 in $\text{GL}(s, q)$, according to $G_1 \cong \text{Sp}(2s, q)$, $\text{SU}(2s, q)$, or $\text{SO}^+(2s, q)^\prime$.

We claim that $Y = \langle X_i, X_i^* \rangle$ acts irreducibly on $M$. First note that $T_0 < N(Y)$ and by (5.1) $C_{T_0}(X_i) \leq C_T(\overline{X}_i) = C_T(\overline{G}_1) = Z(\overline{G}_1)$. Therefore, $(T_0 \cap Y)Z(G_1)$ has index at most 2 in $T_0$. Using primitive divisors we see that $T_0 \cap Y$ acts irreducibly on $M_1$ and on $M_2$. Also, the assumptions $p \geq 5$ and $q > 5$ imply that $T_0 \cap Y$ contains an element, $t$, inducing scalar action for different scalars on $M_1$ and $M_2$. So if $Y$ acts reducibly on $M$, then $Y$ stabilizes either $M_1$ or $M_2$. But this is inconsistent with $t \in T_0 \cap Y \leq Y$ (as $t \notin Z(Y)$), and the claim holds.

View $M$ as a $K[T_0]$-module. Since $M_1$ and $M_2$ are inequivalent, and irreducible as $(T_0 \cap Y)$-spaces, $M$ is the direct sum of 1-dimensional $K[T_0]$-modules affording distinct linear representations of $T_0$. As $\overline{T} \leq C(T_0)$, each $K[T_0 \cap Y]$-submodule is also a $K[\overline{T}]$-submodule of $\overline{M}$.

Since $[M, X_i]$ and $[M, X_i^*]$ are $(T_0 \cap Y)$-invariant, it follows that $M = [M, X_i] \oplus [M, X_i^*]$. Write $\overline{M}|_Y = V_1 \oplus \cdots \oplus V_r$, with each $V_i$ an absolutely irreducible $K[Y]$-module. Then, for $1 \leq k \leq r$, $V_k = [V_k, X_i] \oplus [V_k, X_i^*]$. It follows (see (13.1) of [26]) that $V_k$ is isomorphic to the extension (to $K$) of an algebraic conjugate of the usual module for $\text{SL}(2, q^j)$. By the previous paragraph, each $V_k$ is $\overline{T}$-invariant. Therefore, $V_k$ is invariant under $\langle Y, \overline{T} \rangle$. But $\langle X_i, \overline{T} \rangle = \overline{X}_i \overline{T}$ and $\langle X_i^*, \overline{T} \rangle = \overline{X}_i^* \overline{T}$. Hence, $\langle Y, \overline{T} \rangle \geq \langle \overline{X}_i, \overline{X}_i^* \rangle = \overline{G}_1$, and this shows that $r = 1$ and $\overline{G}_1 \cong \text{SL}(2, K)$. But this contradicts $s > 1$, proving (8.5).

(8.6) Suppose $Y \cong \text{SL}(2, q^j)$ or $\text{PSL}(2, q^j)$. Then (8.1) holds.

Proof. In view of (8.5) we may assume that $G_0$ is an exceptional group. As in the proof of (8.5) we reduce to the case $\overline{G} = \langle \overline{X}_i, \overline{X}_i^* \rangle$, where $Y = \langle X_i, X_i^* \rangle$. Also $C_{T_0}(X_i) = C_{T_0}(X_i^*) = C_{T_0}(\langle X_i, X_i^* \rangle) \leq Z(\overline{G})$, so replacing $\overline{G}$ by $G/Z(G)$ we may assume that $C_{T_0}(X_i) = 1$. Thus $Y \cong \text{PSL}(2, q^j)$ and $T_0$ is cyclic of order $q^j - 1$ or $\frac{1}{2}(q^j - 1)$. An argument with primitive divisors shows that $\overline{G}$ has Lie rank $j$ (observe that the assumption $p \geq 5$ excludes the cases $G_0 \cong \text{Sz}(q)$, $^2\text{G}_2(q)$, or $^2\text{F}_4(q)$).

Let $P$ be the unique parabolic subgroup of $G$ satisfying $T \leq P$ and $X_i \leq O_p(P)$ (see (3.5)(v)), and let $P = \overline{P}_\sigma$ for $\overline{P} = \overline{P}_\sigma$, a parabolic subgroup of $\overline{G}$. By (6.4) $\overline{T} \leq \overline{P}$ and we may assume $\overline{B} \leq \overline{P}$. We will consider
possibilities for $P$, locate $T$ in $P$ and $\bar{T}$ in $\bar{P}$, and indicate the element of the Weyl group of $\bar{P}$ that does the twisting. That is we present $\sigma = \tau q$ and determine the orbits of $\tau$ on root subgroups in $R_u(\bar{P})$. We can then determine $X_i$, $\bar{X}_i$, and $\langle \bar{X}_i, \bar{X}_i^* \rangle$. Order considerations show $G_0 \neq \frac{3}{2}D_4(q)$.

First suppose $G_0 = G_2(q)$. Then $\bar{P} = \langle \bar{B}, s_1 \rangle$ or $\langle \bar{B}, s_2 \rangle$, and we may take $\tau = s_1$, or $s_2$, accordingly. Since $|\Delta_i| = 2$, $X_i = \langle U^\alpha, U^\beta \rangle$, where $\{\alpha, \beta\}$ is one of $\{\alpha_2, \alpha_1 + \alpha_2\}$, $\{\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$, $\{\alpha_1, \alpha_1 + 3\alpha_2\}$, or $\{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. But $\bar{X}_i$ is abelian, and the commutator relations show this to be false in the first, third, and fourth cases. In the second case $\langle \bar{X}_i, \bar{X}_i^* \rangle = \langle \bar{U}^\pm_{\pm\alpha}, \bar{U}^\pm_{\pm\beta} \rangle \cong \text{SL}(3, q)$ (the group generated by all long root subgroups), contradicting $\langle \bar{X}_i, \bar{X}_i^* \rangle = \bar{G}$. So $G = G_2(q)$ is not possible.

For the rest of the proof of (8.6) and for the proof of (8.8) it will be convenient to introduce the following table, which indicates possible choices for $G$, $P$, $\tau$, and $\Sigma_i$. In each case the containment $T_0 \leq P$ limits the choices for $\bar{P}$ (usually just one possibility) and we choose an appropriate representative for $\tau = w$ or $\tau = w\delta$ with $w \in W(\bar{P})$ and $\delta$ a graph automorphism (only relevant in the case $G = \frac{2}{2}E_6(q)$). The choices for $\tau$ are based on the facts: $T_0$ is cyclic, minisotropic in $P$ of order divisible by $\frac{1}{2}(q^2 - 1)$, $\tau$ has an orbit of length $j$ on $\Sigma_i$ (recall that $|\Sigma_i| = j$) and $\langle \bar{X}_i, \bar{X}_i^* \rangle = \bar{G}$. The latter fact implies that if $\Sigma$ has roots of different lengths, then $\Sigma_i$ is an orbit of short roots. Otherwise, $\langle \bar{X}_i, \bar{X}_i^* \rangle$ would be contained in the proper subgroup of $\bar{G}$ generated by all root subgroups, $\bar{U}_a$, with $a$ long. Similarly, if $\bar{P} = \langle \bar{B}, s_i \mid i \neq i_0 \rangle$ ( $\bar{P}$ has this form, even if $G = \frac{2}{2}E_6(q)$), then each $\alpha \in \Sigma_i$ must have $\alpha_{i_0}$-coefficient equal to 1. These conditions eliminate many possibilities for $\Sigma_i$.

**Table (8.7)**

<table>
<thead>
<tr>
<th>$G$</th>
<th>$i_0$</th>
<th>$\tau$</th>
<th>orbit representative of $\Sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4(q)$</td>
<td>1</td>
<td>$(s_2s_3)(s_2^3s_4)$</td>
<td>(none possible)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$(s_3s_2)(s_2^3s_4)$</td>
<td>(none possible)</td>
</tr>
<tr>
<td>$^2E_6(q)$</td>
<td>4</td>
<td>$s_2s_1s_5s_6s_7$</td>
<td>(none possible)</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>1</td>
<td>$s_1s_4s_6s_7s_8s_9$</td>
<td>000000 001111 111111 1 1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$s_1s_4s_6s_7s_8s_9$</td>
<td>000000 111111 1121000 1 1</td>
</tr>
<tr>
<td>$E_7(q)$</td>
<td>2</td>
<td>$s_1s_4s_6s_7s_8s_9$</td>
<td>001111 1232111 1</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>2</td>
<td>$s_1s_4s_6s_7s_8s_9$</td>
<td>001111 1232111 1</td>
</tr>
</tbody>
</table>
To complete the proof of (8.6) one simply checks (with a bit of calculation) that in none of the cases is $X_i$ an abelian group.

(8.8) If $Y \cong SU(3, q^j)$ or $PSU(3, q^j)$, then (8.1) holds.

Proof. Write $U = X_i$ and $Z(U) = X_k$, for $i, k \in \{1, \ldots, t\}$. Then $|\Sigma_i| = 2j$ and $|\Sigma_k| = j$. Set $P = \langle \bar{X}_i, \bar{X}_k^* \rangle$ and $P = P_a \cap G_0$. First assume that $G_0$ is a classical group. Then the structure of $P$ is given in (4.2). Using the notation of (4.2) we first note that $X_i$ nonabelian implies $r \neq 2s$. The result then follows from (4.2)(iv) and (4.2)(v). We may now assume $G_0$ to be an exceptional group.

Arguing as in the proof of (8.6) we may assume $\bar{G} = \langle \bar{X}_i, \bar{X}_k^* \rangle$ and passing to quotient groups, if necessary, we may assume $Z(\bar{G}) = Z(G_0) = 1$. Also, $Z(YT_0) \leq Z(\bar{G}) \cap T_0 = 1$, so $YT_0 \cong SU(3, q^j)$ or $PGU(3, q^j)$, and $T_0$ is cyclic of order $q^{2j} - 1$ or $\frac{1}{2}(q^{2j} - 1)$. As in (8.6) we conclude that $\bar{G}$ has Lie rank $2j$. This immediately rules out the case $G_0 \cong E_7(q)$ and $3D_4(q)$ is ruled out by order considerations (namely, $|T_0|$ divides $|P|$).

Suppose $G \cong G_2(q)$. Then $Y \cong PSU(3, q)$. The remarks preceding Table (8.7) show that $\Sigma_i = \{ \beta_1, \beta_2 \}$ for short roots $\beta_1, \beta_2 \in \bar{\Sigma}$ and (6.7) shows that $\bar{P} = N_{\bar{G}}(\bar{U}_\gamma)$, with $\gamma$ a long root. We may take $\tau = sq$, where $s \in N_{G_0}(T_0)$ and $s$ is in the derived group of the Levi factor of $\bar{P}$. Then $s \in C(\langle \bar{U}_\gamma, \bar{U}_\gamma^* \rangle)$, which gives $s \in C_{G_0}(\langle X_k, X_k^* \rangle)$. Hence $C_{G_0}(s) \cong SL(2, q)$. On the other hand, $s$ normalizes $\langle X_i, X_i^* \rangle = Y$, since $s$ normalizes $T_0, X_i$, and $X_i^*$. So $s$ induces a graph automorphism of $Y \cong PSU(3, q)$, forcing $C_Y(s)' \cong PSL(2, q)$. This is a contradiction. Therefore $G_0 \not\cong G_2(q)$.

The remaining cases are $G_0 \cong F_4(q), 2E_6(q), E_6(q)$, and $E_8(q)$, where we refer to Table (8.7). The first three are ruled out immediately. Suppose $G_0 \cong E_6(q)$. Here $|\Sigma_k| = j = 4$, so $[\bar{X}_i, \bar{X}_j]$ is the product of 4 $T$-root subgroups of $\bar{G}$. However, for each of the possible orbits listed in (8.7) a direct check with the commutator relations shows that $[\bar{X}_i, \bar{X}_j]$ is the product of more than 4 $T$-root subgroups. This proves (8.8).

We have now proved (8.1) when $Y$ has Lie rank 1 (noting that $p \geq 5$ excludes Suzuki groups and Ree groups). Next, we establish (8.1) for groups of Lie rank 2. We will use the following notation. For $O$ a representation of the abelian group $A$ and $n \in Z$, $O^n$ is the representation given by $O^n(a) = O(a^n)$.

(8.9) Suppose $Y \cong SL(3, q^j)$ or $PSL(3, q^j)$. Then (8.1) holds.
Proof. Write $U = X_i X_k X_l$ with $X_i = [X_i, X_k]$, and regard these subgroups as irreducible $\mathbf{F}_p[T_0]$-modules. Set $q = p^a$. There are linear $\mathbf{F}_{p^a}$-representations $\varphi_i, \varphi_k, \varphi_i$ of $T_0$ such that $\mathbf{F}_{p^a} \otimes \mathbf{F}_q X_i = \varphi_i^s \oplus \cdots \oplus \varphi_i^s$. Similarly for $X_k, X_l$. We have $\langle X_i, X_i^* \rangle \cong \langle X_k, X_k^* \rangle \cong \langle X_i, X_i^* \rangle \cong \text{SL}(2, q^j)$, so by (8.6) each of $\langle \overline{X}_i, \overline{X}_i^* \rangle$, $\langle \overline{X}_k, \overline{X}_k^* \rangle$, and $\langle \overline{X}_l, \overline{X}_l^* \rangle$ is the commuting product of a $\sigma$-orbit of $j$ copies of $\text{SL}(2, K)$, each generated by a $T$-root subgroup of $G$ and its opposite. Write $X_i = U_{\beta_i} \times \cdots \times U_{\beta_i}$, $X_k = U_{\gamma_i} \times \cdots \times U_{\gamma_i}$, and $X_l = U_{\delta_i} \times \cdots \times U_{\delta_i}$. By (6.7) we have $\langle \overline{X}_i, \overline{X}_i \rangle = \overline{X}_i X_l$.

Let $Z = Z(\mathcal{T}_T)$ and let $z = yt \in Z$, with $y \in Y$ and $t \in T_0$. Then $y \in C_Y(Y \cap T_0) = Y \cap T_0$ by (2.8). So $Z \leq T_0$. Passing to $T_0/Z$ and applying (2.7), we conclude that $N_T(Y \cap T_0) \leq N_T(0) \leq N_T(G)$. Since $N_T(Y \cap T_0)$ is transitive on the $(T_0 \cap Y)$-root subgroups of $Y$, we see that $\overline{\Sigma}_i, \overline{\Sigma}_k, \overline{\Sigma}_l$ and $\overline{\Sigma}_i$ are conjugate under $N_T(T)$. In particular, the roots in $\overline{\Sigma}_i \cup \overline{\Sigma}_k \cup \overline{\Sigma}_l$ are all of the same length. Consequently, we may choose notation so that $\langle U_{\beta_i}, U_{\gamma_i} \rangle = \overline{U}_{\beta_i}$.

Let $\varphi, \psi, \theta$ be the $K$-representations of $T$ afforded by $U_{\beta_i}, U_{\gamma_i}, U_{\delta_i}$, respectively restricted to $T_0$. The commutator relations show that $\varphi \psi = \theta$. It follows from (5.1) that we may assume $\varphi = \varphi_i^K, \psi = \varphi_k^K, \theta = \varphi_l^K$. We claim that $[\overline{U}_{\beta_i}, \overline{U}_{\gamma_i}] = 1$, for $r > 1$. Otherwise, $[\overline{U}_{\beta_i}, \overline{U}_{\gamma_i}] = \overline{U}_{\delta_i}$ for $\delta_i = \beta_i + \gamma_r$. Then $\theta q^{-r-1} = \varphi \psi q^{r-1}$ and since $\varphi \psi = \theta$, we obtain $\varphi_i = \varphi_i^v$, for $u = q^{r-1} - 1$ and $v = q^{r-1} - q^{r-1}$. This implies $(\varphi_i^u)^v = (\varphi_i^v)^v$. Let $T_1 = T_0 \cap \langle X_i, X_i^* \rangle$, a Cartan subgroup of $\langle X_i, X_i^* \rangle \cong \text{SL}(2, q^j)$. If $\varphi_i$ and $\varphi_k$ denote $\varphi_i | T_1$ and $\varphi_k | T_1$, respectively, then computation within $Y$ yields $\varphi_i = (\varphi_k^{-2})^{p^c}$ for some $0 \leq c < aj$. Therefore, $(\varphi_i^u)^v = (\varphi_k^{-2} p^c)^v$ and so $\varphi_k v^2 + 2 p^c = 1$. But this contradicts $| \varphi_k | = q^j - 1$, proving the claim.

Transforming the commutator relation of the previous paragraph by powers of $\sigma$ and using (8.4)(ii) we obtain the following commutator relations:

(i) $[\overline{U}_{\beta_r}, \overline{U}_{\gamma_r}] = \overline{U}_{\delta_r}$, for $1 \leq r \leq j$.

(ii) $[\overline{U}_{\beta_s}, \overline{U}_{\gamma_s}] = 1$, for $r \neq s$ in $\{1, \ldots, j\}$.

For $1 \leq r \leq j$ let $\overline{D}_r = \langle \overline{U}_{\beta_r}, \overline{U}_{\gamma_r} \rangle$. Then $\overline{D}_r \cong \text{SL}(3, K)$ or $\text{PSL}(3, K)$, and the above relations give $[\overline{D}_r, \overline{D}_s] = 1$ for $r \neq s$. Since $\langle \overline{X}_i, \overline{X}_l^* \rangle = \overline{D}_1 \cdots \overline{D}_j$, the proof of (8.9) is complete.

(8.10) Suppose $Y$ is a non-trivial image of $\text{Sp}(4, q^j)$ or $G_2(q^j)$. Then (8.1) holds.

Proof. The arguments are similar to those in (8.9), although slightly more complicated. We consider only the (more difficult) case of $G_2(q^j)$. 

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Let $\alpha$, $\beta$ be fundamental long and short roots of the root system of $Y$, so that the complete set of positive roots is $\{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$. Say $X_i$ and $X_k$ are the $(T_0 \cap Y)$-root subgroups of $Y$ corresponding to $\alpha$ and $\beta$, respectively. Let $X_r$, $X_s$, $X_t$, and $X_w$ correspond to the compound roots $\alpha + \beta$, $\alpha + 2\beta$, $\alpha + 3\beta$, $2\alpha + 3\beta$, respectively. Then $[X_i, X_k] = X_i X_k X_i X_k$, so (6.7) implies that $[\bar{X}_i, \bar{X}_k] = \bar{X}_i \bar{X}_k \bar{X}_i \bar{X}_k$.

Write $\Delta_i = \{\bar{U}_{\beta_1}, \ldots, \bar{U}_{\beta_j}\}$, where $\bar{U}_{\beta_2} = \bar{U}_{\beta_3}, \ldots, \bar{U}_{\beta_j} = \bar{U}_{\beta_{j-1}}$. Similarly, write $\Delta_k = \{\bar{U}_{\gamma_1}, \ldots, \bar{U}_{\gamma_j}\}$. Set $q = p^a$ and regard each of $X_i$, $X_k$, $X_r$, $X_s$, $X_t$, and $X_w$ as $aj$-dimensional $F_p[T]$-modules. As in (8.9) we choose linear $F_q[T_0]$ representations $o_i$, $o_k$, $o_r$, $o_s$, $o_t$, $o_w$ so that $F_q \otimes F_p X_i = o_i^p \oplus \cdots \oplus o_i^{p^a}$ for $l \in \{i, k, r, s, t, w\}$. We may assume that $o_i^K$, $o_k^K$ are the $K$-representations that $T_0$ induces on $\bar{U}_{\beta_l}$, $\bar{U}_{\gamma_l}$, respectively, and we may assume $[\bar{U}_{\beta_l}, \bar{U}_{\gamma_l}] \neq 1$.

From the commutator relations for $G_2(q^j)$ it follows that there exist $\alpha, \beta, \gamma, \delta, \epsilon, \eta, \mu, \nu \in \{1, \ldots, aj\}$ and that $o_r = o_i^{p^a} o_k^p$, $o_s = o_i^{p^r} o_k^{p^s}$, $o_t = o_i^{p^s} o_k^{p^t}$, and $o_w = o_i^{p^w} o_k^{p^w}$.

Suppose $[\bar{U}_{\beta_l}, \bar{U}_{\gamma_l}] \neq 1$ for $1 \leq l \leq j$. Then $o_i^K(o_i^{p^{l-1}})^K = ((o_i^{p^a} o_k^p)^{q^m})^K$, $((o_i^{p^r} o_k^{p^s})^{q^m})^K$, $((o_i^{p^s} o_k^{p^t})^{q^m})^K$, or $(o_i^{p^w} o_k^{p^w})^{q^m}$, for some $0 \leq m \leq j - 1$. There are elements $t, v \in T_0 \cap Y$ such that $o_i(t) = o_i(v) = 1$ (i.e., $t \in C(X_i), v \in C(X_k)$) and $|o_i(v)| = |o_i(v)| = q^j - 1$. If $l = 1$, evaluate at $t$ and $v$ and conclude that the first possibility must hold and $\alpha = \beta$. Now suppose $l > 1$. Evaluating at $t$ we again see that the first possibility must occur and we obtain the congruence $q^{l-1} \equiv p^a q^m \pmod{p^{aj} - 1}$. Evaluating at $v$ we have $1 \equiv p^a q^m \pmod{p^{aj} - 1}$, contradicting the other congruence. Therefore, $[\bar{U}_{\beta_l}, \bar{U}_{\gamma_l}] = 1$ for each $l > 1$, and transforming by powers of $\sigma$ we have $\langle \bar{X}_i, \bar{X}_k \rangle$ a central product of the groups $\langle \bar{U}_{\beta_1}, \bar{U}_{\gamma_1} \rangle, \ldots, \langle \bar{U}_{\beta_j}, \bar{U}_{\gamma_i} \rangle$. By (8.4)(ii), $[\bar{U}_{\beta_g}, \bar{U}_{\gamma_g}] = 1$ for $1 \leq g, h \leq j$, and by (8.6) $[\bar{U}_{\beta_s}, \bar{U}_{\beta_{s-1}}] = [\bar{U}_{\gamma_t}, \bar{U}_{\gamma_{t-1}}]$ if $1 \leq g \neq h \leq j$. So letting $\bar{D}_g = \langle \bar{U}_{\beta_g}, \bar{U}_{\gamma_g} \rangle$ for $1 \leq g \leq j$, we have $\langle \bar{X}_i, \bar{X}_k \rangle$ equal to the central product of the semisimple groups $\bar{D}_1, \ldots, \bar{D}_j$.

The group $\bar{D}_j$ has as its root system a rank 2 subsystem of $\bar{S}$. On the other hand, $\langle \bar{X}_i, \bar{X}_k \rangle = \langle \bar{U}_{\beta_1}, \bar{U}_{\gamma_1} \rangle \times \cdots \times \langle \bar{U}_{\beta_j}, \bar{U}_{\gamma_j} \rangle$ and $\langle X_i, X_k \rangle$ has nilpotence class 5. This forces $\bar{D}_1$ to be of type $\bar{G}_2(K)$ (it also forces $j = 1$, since $G_2$ is not a sub-root system of any other indecomposable system). Since $\langle \bar{D}_1, \ldots, \bar{D}_j \rangle$ is an orbit under $\langle \sigma \rangle$, we have proved (8.10).

(8.11) Suppose $Y$ is a non-trivial image of $SU(4, q^j)$, $3D_4(q^j)$, or $SU(5, q^j)$. Then (8.1) holds.

Proof. We will discuss the most difficult case where $Y$ is an image of $SU(5, q^j)$. Here $U$ is the product of four root subgroups, $X_i$, $X_k$, $X_t$, $X_m$, $X_l$, $X_s$, $X_r$. Then (8.1) holds.
where $X_t$ and $X_k$ are fundamental, $\langle X_t, X_t^* \rangle \cong \text{SL}(2, q^{2j})$, $\langle X_k, X_k^* \rangle \cong \text{SU}(3, q^j)$, $X_t$ is a conjugate of $X_i$, and $X_m$ a conjugate of $X_k$. In addition, each of $X'_k$ and $X'_m$ is a $(T_0 \cap Y)$-root subgroup. Say $X'_i = X_r$ and $X'_m = X_s$. Then each of $X_i, X_k, X_t, X_m, X_r$, and $X_s$ is a $T$-root subgroup of $G$.

View $X_i$ and $X_k/X_k'$ as $2aj$-dimensional $F_q[T_0]$-modules, where $q = p^a$. There are linear $F_q$-representations $\varphi$ and $\psi$ of $T_0$ such that $F_q \otimes (X_i, X_i^*) = \varphi \otimes \cdots \otimes \varphi \otimes \varphi^{X_i}$ and $F_q \otimes (X_k, X_k^*) = \psi \otimes \cdots \otimes \psi \otimes \psi^{X_k}$. Let $\Delta_i = \{U_{i_1}, \ldots, U_{i_{2j}}\}$ and $\Delta_k = \{U_{k_1}, \ldots, U_{k_{2j}}\}$. We may assume that $\varphi^K, \psi^K$ are the $K$-representations of $T_0$ induced on $\overline{U_{i_1}}, \overline{U_{k_1}}$, respectively. We have $[X_i, X_k] = X_i X_m$ (computation in $Y$) and so (6.10) implies $[\overline{X_i}, \overline{X_k}] = \overline{X_i X_m}$. We relabel if necessary so that $[\overline{U_{i_1}}, \overline{U_{k_1}}] \neq 1$ and for $1 \leq l \leq j$, $\overline{U_{i_l}} = U_{i_l}^{\sigma^{l-1}}$ and $\overline{U_{k_l}} = U_{k_l}^{\sigma^{l-1}}$.

Let $\delta, \omega$ be linear $F_q$-representations of $T_0$ such that $F_q \otimes \delta, \omega$ are the sums of the Galois conjugates of $\delta, \omega$ respectively. Computations in $Y$ imply that there exists $\alpha$ such that we may take $\omega = \varphi \psi^\alpha$. From the relation $[X_i, X_i^*] = X_i X_m$ we see that for some $\beta, \gamma$ we must have $\delta \varphi^{P^\beta} = (\omega^{1+q^j})^P = (\varphi \psi^{\alpha})^{(1+q^j)}$. Let $\delta, \omega$ be linear $F_q$-representations of $T_0$ such that $F_q \otimes \varphi, \psi$ are the sums of the Galois conjugates of $\delta, \omega$ respectively. Computations in $Y$ imply that there exists $\alpha$ such that we may take $\omega = \varphi \psi^{P^\alpha}$. From the relation $[X_i, X_i^*] = X_i X_m$ we see that for some $\beta, \gamma$ we must have $\delta \varphi^{P^\beta} = (\omega^{1+q^j})^P = (\varphi \psi^{\alpha})^{(1+q^j)}$.

Suppose $[\overline{U_{i_1}}, \overline{U_{k_1}}] \neq 1$. Then by the above $\varphi \psi^{q^j-1}$ is a Galois conjugate of one of $\delta, \omega, \omega^{1+q^j}$. By the previous paragraph each of $\delta, \omega, \omega^{1+q^j}$ can be expressed in terms of $\varphi$ and $\psi$. Make this substitution and consider the resulting relation between powers of $\varphi$ and $\psi$. There exist elements $t_1, t_2$ of $T_0$ such that

$$\varphi(t_1) = \psi(t_2) = 1 \quad \text{and} \quad |\varphi(t_2)| = |\psi(t_1)| = (q^2 - 1)/(5, q + 1).$$

Substituting $t_1, t_2$ into the above relations we see that such a relation can hold only if the obvious equalities hold between powers of $\varphi$ and powers of $\psi$. First substitute $l = 1$ and obtain $\varphi \psi = \omega$ and $\alpha = 0$. Now let $l > 1$ and obtain a contradiction. Consequently $[\overline{U_{i_l}}, \overline{U_{k_l}}] = 1$ for $l > 1$, and transforming by powers of $\sigma$, we conclude $[\overline{U_{i_u}}, \overline{U_{k_v}}] = 1$ for any $u \neq v$. From (6.10) and the fact $[X_i, X_k] \neq 1$, we conclude that $[\overline{X_i}, \overline{X_k}] \neq 1$. Therefore, $[\overline{U_{i_u}}, \overline{U_{k_v}}] \neq 1$, for $1 \leq u, v \leq 2j$.

Consider the group $D_1 = \langle \overline{U_{\pm \beta_1}}, \overline{U_{\pm \gamma_1}}, \overline{U_{\pm \gamma_{j+1}}}, \overline{U_{\pm \beta_{j+1}}} \rangle$. The argument of (8.9) shows that $\langle U_{\pm \gamma_1}, U_{\pm \gamma_{j+1}} \rangle \cong \text{SL}(3, K)$, with $\{\gamma_1, \gamma_{j+1}\}$ a fundamental system. By (8.6), $[\overline{U_{\pm \beta_1}}, \overline{U_{\pm \beta_{j+1}}}] = 1$, and by (8.4)(ii) $[\overline{U_{\pm \beta_{u}}}, \overline{U_{\gamma_v}}] = [\overline{U_{-\beta_{u}}}, \overline{U_{\gamma_v}}] = 1$ for $1 \leq u, v \leq 2j$.

We claim that $\beta_1$ and $\gamma_1$ are roots of the same length. Otherwise, the commutator relations applied to $[\overline{X_i}, \overline{X_k}] = \overline{X_i X_m}$ shows that either $(\varphi \psi^2)^K$ or $(\varphi^2 \psi)^K$ is a component of one of the representations $F_{q^j} \otimes (X_m/X'_m)$.
The previous computations show that this is impossible, proving the claim. There are three classes of \((T_0 \cap Y)\)-root subgroups of \(Y\) (under \(N_\gamma(T_0 \cap Y)\)), with representatives, \(X_i\), \(X_k\), \(X_r\). Since 
\[
\langle \tilde{U}_{\pm \gamma_1}, \tilde{U}_{\pm \gamma_1} \rangle \cong \text{SL}(3, K) \quad \text{we conclude that the roots in } \tilde{\Sigma} \cup \tilde{\Sigma}_k \cup \tilde{\Sigma}_l \cup \tilde{\Sigma}_s \text{ are all long roots. Hence } \langle \tilde{U}_{\pm \beta_1}, \tilde{U}_{\pm \gamma_1} \rangle \cong \langle \tilde{U}_{\pm \gamma_1}, \tilde{U}_{\pm \beta_1} \rangle \text{.}
\]
We can now apply (2.13) and conclude that \(D_j\) is an image of \(A_4(K)\) with fundamental set \(\{\beta_{v_1}, \gamma_{+1}, \beta_{y+i}\}\).

Let \(D_i = \tilde{D}_i^{\sigma-1}\), for \(1 \leq i \leq j\). Using the aforementioned commutator information together with (8.6) and (8.8) we have 
\[
\langle \tilde{D}_1, \ldots, \tilde{D}_j \rangle = \tilde{D}_1 \cdots \tilde{D}_j, \text{ a central product. Thus, (8.1) holds, completing the proof of (8.9).}
\]

(8.12) If \(Y\) has Lie rank at least 3, then (8.1) holds.

Proof. Let \(\beta_1, \ldots, \beta_n\) be a fundamental system for the root system of \(Y\), with \(U_{\beta_1}, \ldots, U_{\beta_n}\) the corresponding \((T_0 \cap Y)\)-root subgroups, corresponding to the labeling of the Dynkin diagram of \(Y\) (see §1). For \(i = 1, \ldots, n\), let \(U_{\beta_i} = X_{l_i}\). Then \([X_{l_i}, X_{l_n}] = 1\), while \([X_{l_{n-1}}, X_{l_n}] \neq 1\).

Fix \(1 \leq i \leq n\) and write \(\Delta_i = \{\tilde{U}_{\beta_{i-1}}, \ldots, \tilde{U}_{\beta_{i-1}}\}\), where \(k_1 = j \) or \(2j\). Arrange notation so that \(\tilde{U}_{\beta_{k-1}} = \tilde{U}_{\beta_{k+1}}\), for \(1 \leq k < k_1\). Set 
\[
Z = \langle \tilde{X}_{l_{i-1}}, \tilde{X}_{l_{i-1}}^*, \tilde{X}_{l_{i}}^*, \ldots, \tilde{X}_{l_{n-1}}^* \rangle.
\]
Inductively, we know that 
\[
Z = Z_1 \cdots Z_j, \text{ a commuting product of a } \langle \sigma \rangle\text{-orbit of Chevalley groups and } O^\sigma(\tilde{Z}_0) = \langle U_{\pm \beta_1}, \ldots, U_{\pm \beta_{n-1}} \rangle.
\]
First suppose that \(Y\) is an untwisted group. Then \(k_1 = j\) for \(1 \leq i \leq n\) and we may reorder, if necessary, so that 
\[
Z_1 = \langle \tilde{U}_{\pm \beta_1}^0, \ldots, \tilde{U}_{\pm \beta_{n-1}}^0 \rangle.
\]
By (8.9) there exists a unique \(k\) such that \(\tilde{U}_{\pm \beta_{k-1}}^0\) does not commute with \(U_{\beta_{n-1}}\), and we may reorder \(\Delta_n\), if necessary, so that \(k = 1\). Set 
\[
\tilde{D}_1 = \langle \tilde{Z}_1, \tilde{U}_{\pm \beta_{n-1}}^0 \rangle.
\]
For \(i < n - 1\), \([U_{\beta_i}, U_{\beta_n}] = 1\), so by (6.10) \([\tilde{X}_{l_i}, \tilde{X}_{l_n}] = 1\), hence 
\[
[\tilde{U}_{\beta_i}, \tilde{U}_{\beta_n}] = 1.
\]
This together with (8.4)(ii) yields \([\tilde{U}_{\pm \beta_{n-1}}, \tilde{U}_{\pm \beta_{n-1}}] = 1\) for each \(i < n - 1\). By (8.9), 
\[
\langle \tilde{U}_{\pm \beta_{n-1}}, \tilde{U}_{\pm \beta_{n-1}} \rangle \cong \text{SL}(3, K),
\]
so we can apply (2.13) and conclude that \(\tilde{D}_1\) has the same Dynkin diagram as does \(Y\). Moreover, (8.6), (8.4), and induction show that \(\tilde{D}_1 \cdots \tilde{D}_j\) is a commuting product. The result follows.

Now suppose \(Y\) is a twisted group. The argument is essentially the same as above, although slightly more complicated. We have 
\[
Z = Z_1 \cdots Z_j.
\]
and 
\[
Z = O^\sigma(\tilde{Z}_0) = \langle U_{\pm \beta_1}, \ldots, U_{\pm \beta_{n-1}} \rangle.
\]
To illustrate the charges we consider the case \(Y \cong 2E_6(q')\), leaving the remaining cases to the reader. Here, \(n = 4\), \(Z \cong O^- (8, q')\) and so \(\tilde{Z}_1 \cong \cdots \cong \tilde{Z}_j \cong D_4(K)\). Each \((T_0 \cap Y)\)-root group of \(Y\) is abelian, so \(\tilde{X}_{l_3}, \ldots, \tilde{X}_{l_4}\) are each the direct
product of the root subgroups in $\Delta_1, \ldots, \Delta_4$, respectively. Moreover, $|\Delta_1| = |\Delta_2| = j$, while $|\Delta_3| = |\Delta_4| = 2j$.

Labeling roots as before, we may assume $\overline{Z}_1 = \langle \overline{U}_{\pm \beta_3}, \overline{U}_{\pm \beta_4}, \overline{U}_{\beta_4}, \overline{U}_{\pm \beta_3}, \overline{U}_{\pm \beta_4}, \overline{U}_{\beta_4} \rangle$. Here, $\sigma^j$ interchanges the groups $\langle \overline{U}_{\pm \beta_3}, \overline{U}_{\pm \beta_4} \rangle$ and stabilizes the other two groups, and induces a graph automorphism on the Dynkin diagram of $\overline{Z}_1$. Now, $X_{\overline{Z}_1} = \overline{U}_{\beta_4} \times \cdots \times \overline{U}_{\beta_4}$, and by induction we have the structure of $\langle \overline{X}_{\overline{Z}_1}, \overline{X}_{\overline{Z}_1}, \overline{X}_{\overline{Z}_1}, \overline{X}_{\overline{Z}_1}, \overline{X}_{\overline{Z}_1}, \overline{X}_{\overline{Z}_1} \rangle$. We may choose our notation so that $[\overline{U}_{\beta_4}, \overline{U}_{\beta_3}] \neq 1$. Set $D_1 = \langle \overline{U}_{\pm \beta_3}, \overline{U}_{\pm \beta_4}, \overline{U}_{\beta_4} \rangle$ and argue, using (2.13) and commutator information, that $D_1 \cong E_6(K)$. The argument is then completed by setting $D_i = D_1^{\sigma^{i-1}}$ for $1 \leq i \leq j - 1$ and observing that $\langle D_1, \ldots, D_j \rangle = D_1 \cdots D_j$, a central product. This completes the proof of (8.10).

Theorem (8.1) follows from (8.6)-(8.12).

9. A technical result. In this section we apply the results of §7 and §8 and establish a technical result that will be useful in §10. Continue the assumptions $p \geq 5$ and $q > 7$.

We introduce the following notation which will be used throughout the rest of the paper. If $Y$ is a $T_0$-invariant subgroup of $G$, set $Y(T_0) = \langle X_i \mid X_i \leq Y \rangle$ and $\overline{Y}(T_0) = \langle \overline{X}_i \mid X_i \leq Y \rangle$.

Throughout the section we let $Y \leq G$ with $Y^{T_0} = Y$ and $Y/O_p(Y)$ a central product of groups of Lie type in characteristic $p$. Let $T_1$ be a $p'$-Hall subgroup of a Sylow $p$-normalizer of $Y$. Then $T_1O_p(Y)/O_p(Y)$ is a Cartan subgroup of $Y/O_p(Y)$. This forces $C_{Y^{T_0}}(T_1)$ to be solvable and we choose a $p'$-Hall subgroup, $T_2$, of $C_{Y^{T_0}}(T_1)$. Then $T_2 \supseteq T_1$. We call $T_2$ a Cartan subgroup of $YT_0$.

(9.1) (i) $T_2$ is a maximal torus of $G_0$;
(ii) $Y = Y(T_2) = O_p(\overline{Y}(T_2))_0$ and $YT_0 = YT_2$;
(iii) $\overline{Y}(T_2) = \overline{Y}(T_0)$ and $\overline{Y}(T_2)\overline{T}_2 = \overline{Y}(T_2)\overline{T}$, where $\overline{T}_2 = C_{\overline{G}}(T_2)_0$ (the maximal torus of $\overline{G}$ containing $T_2$);
(iv) $T_2$ can be chosen so that there is a $T_2$-invariant subgroup $J \leq Y$ and $Y$ is the semidirect product $Y = O_p(Y)J$ and $JT_2 = JT_0$;
(v) $T_2$ is a $p'$-Hall subgroup of a Sylow $p$-normalizer of $YT_0$.

We remark that a missing item in (9.1) is the assertion $Y = Y(T_0)$. At this stage we do not even have $G_0 = G_0(T_0)$; that is we have yet to establish the fact that $G_0$ is generated by its $T_0$-root subgroups.
The rest of this section concerns the proof of (9.1). We first show that (i) holds, noting that this is just (7.2) in case $O_p(Y) = 1$. So suppose $O_p(Y) \neq 1$ and let $\overline{P}$ be the canonical parabolic subgroup of $\overline{G}$ with $O_p(Y) \leq R_u(\overline{P})$ and $N_{\overline{G}}(O_p(Y)) \leq \overline{P}$. Then $\overline{P}$ is $\sigma$-invariant and (6.4) shows $\overline{T} \leq \overline{P}$. Let $\overline{L}$ be the Levi factor of $\overline{P}$, with respect to the maximal torus $\overline{T}$. Set $J = \overline{L} \cap YR_u(\overline{P})$. Then $J \leq O_p'(\overline{L}_o) = L$ and $J$ is a $T_0$-invariant commuting product of groups of Lie type in characteristic $p$.

Now let $\overline{T}_1$ be a Cartan subgroup of $J$ and $\overline{T}_2 = C_{\overline{L}}(\overline{T}_1)$. By (7.2), $\overline{T}_2$ is a maximal torus of $G_0$. But $JR_u(\overline{P})_o = YR_u(\overline{P})_o$ implies that $\overline{T}_2$ and $T_2$ are conjugate by an element in $R_u(\overline{P})_o$. Hence (i) holds.

Let $\overline{T}_2 = C_{\overline{G}}(T_2)^0$, a maximal torus. We claim $Y = Y(T_2) = O_p'((\overline{Y}(T_2))_o)$. Let $V_1$ and $V_2$ be Sylow $p$-subgroups of $Y$ normalized by $T_1$ and such that $Y = \langle V_1, V_2 \rangle$. The argument of (7.5) applied either to $YT_0$ or to $YT_0R_u(\overline{P})/(R_u(\overline{P}))_o$ (according to whether or not $O_p(Y) = 1$) shows that $T_2 \leq N(V_1) \cap N(V_2)$. By (6.9), $V_1$ and $V_2$ are each products of $T_2$-root subgroups of $G_0$ and $V_i = \overline{V}_i(T_2)_o$, for $i = 1, 2$. As $Y = \langle V_1, V_2 \rangle$ we have $Y = Y(T_2)$. If $O_p(Y) = 1$, let $K = G$, and if $O_p(Y) \neq 1$ let $\overline{P}$ be as above and $\overline{K}$ the Levi factor of $\overline{P}$ containing $\overline{T}_2$. So $\overline{K} = \overline{K}_o$.

If $O_p(Y) \neq 1$, embed each $V_i$ in the unipotent radical of a $\langle \sigma \rangle T_2$-invariant parabolic subgroup of $\overline{P}$ and use (5.1) and (5.5) to conclude that $\overline{V}_i(T_2) \leq \overline{P}$, for $i = 1, 2$. Moreover, in this situation, each $\overline{T}_2$-root subgroup of $\overline{P}$ is contained either in $R_u(\overline{P})$ or in $\overline{K}$. It follows that $\overline{V}_i(T_2) = O_p(Y)(T_2)(\overline{V}_i(T_2) \cap \overline{K})$ for $i = 1, 2$. By (6.10) $O_p(Y)(T_2)$ is normalized by $\overline{V}_i(T_2) \cap \overline{K}$ for $i = 1, 2$. Therefore, $\overline{Y}(T_2) = O_p(Y)(T_2)(\overline{Y}(T_2) \cap \overline{K})$, and this also holds if $O_p(Y) = 1$. Since $V_i = O_p(Y)(V_i \cap \overline{K})$, for $i = 1, 2$, $Y \cap \overline{K}$ complements $O_p(Y)$. Hence, (8.1)(iii) shows that $Y(T_2) = O_p'(\overline{Y}(T_2))_o$. This proves the claim.

Set $\mathcal{J} = \overline{Y}(T_2) \cap \overline{K}$, $J = O_p'(\mathcal{J}_o)$, and $X = YT_0$. Then $Y = Y(T_2) = O_p(Y)J$, a semidirect product. The Frattini argument gives $X = YN_X(T_2) = O_p(Y)JN_X(T_2)$. Now, $J = J(T_2)$ is the group generated by all $T_2$-root subgroups of $Y$ whose opposite root group is also in $Y$. Since $N_X(T_2) \leq N(\overline{T}_2)$, we conclude that $N_X(T_2)$ normalizes both $J$ and $\overline{J} = J(T_2)$. Thus, $JN_X(T_2)$ is a group and normalizes $\overline{J}T_2$. As $JN_X(T_2)$ complements $O_p(Y)$, we may replace $T_0$ by a $Y$-conjugate that lies in $JN_X(T_2)$. In particular, $T_0$ normalizes a maximal torus, $\overline{T}_3 = \overline{T}_3^o$, of $\overline{J}T_2$. Then $T_0 \leq N(G_0 \cap (\overline{T}_3)^o)$, so (6.3) implies $T_0 = G_0 \cap (T_3)^o$. But then (2.8) implies $\overline{T}_3 \leq C_{\overline{G}}(T_3)^0 = \overline{T}$.

Hence, $\overline{T} = \overline{T}_3$ and $\overline{J}T_2 = JT_2$. Then (2.5) implies $JT_2 = JT_0$. Now $Y = Y(T_2) = O_p(\overline{Y}(T_2))_o = O_p(Y)J$, therefore $YT_2 = YT_0$, completing (ii).

From the above, $Y = O_p(Y)J$, a semidirect product, and $JT_0 = JT_2$. This proves (iv). From $\overline{Y}(T_2) = O_p(Y)(T_2)\mathcal{J}$ and the fact that $\overline{J}T_2 = JT_2$,
we conclude $\overline{\nu}(T_2)\overline{\nu}_2 = \overline{\nu}(T_2)\overline{\nu}$. We next prove $\overline{\nu}(T_2) = \overline{\nu}(T_0)$. Let $\overline{D} = O_{p}(\overline{Y})(T_2)$, so that $\overline{D}_\sigma = O_{p}(\overline{Y})$, by (6.9). Earlier arguments imply $\overline{D} \leq \overline{\nu}(T_2)$, so $\overline{D} \leq \overline{\nu}(T_2)\overline{\nu}_2 = \overline{\nu}(T_2)\overline{\nu}$. Thus, $\overline{T}\langle \sigma \rangle$ normalizes $\overline{D}$, and since $\overline{D}_\sigma = O_{p}(\overline{Y})$ we again apply (6.9) to conclude $\overline{D} = O_{p}(\overline{Y})(T_0)$. So it will suffice to show $\overline{J} = \overline{J}(T_2) = \overline{J}(T_0)$. We have $\overline{J}\overline{T}_2 = \overline{J}\overline{T}$, so each component of $\overline{J}$ is $\overline{T}$-stable and generated by certain $\overline{T}$-root subgroups of $\overline{G}$. If $\alpha \in \overline{\Sigma}$ and if the $\overline{T}$-root subgroup $\overline{U}_\alpha$ is contained in $\overline{J}$, then $\overline{X}_i \leq \overline{J}$. Hence $X_i \leq O'_{p}(\overline{J}_\sigma) = J$. This shows $\overline{J} \leq \overline{J}(T_0)$. 

For the other containment, suppose $X_i \leq J$. If $X_i$ is a $p$-group, then $\overline{J}$ contains a $\overline{T}\langle \sigma \rangle$-invariant parabolic subgroup $\overline{P}$ with $X_i \leq R_u(\overline{P})$. Then $R_u(\overline{P})$ is the product of certain $\overline{X}_i$, $1 \leq l \leq t$, and (6.9) implies $\overline{X}_i \leq R_u(\overline{P})$. Hence $\overline{X}_i \leq \overline{J}$. 

Now suppose $X_i$ is of Lie type and defined over $\mathbb{F}_{q^\alpha}$. Let $C$ be a $\sigma$-invariant maximal torus of $\overline{X}_i\overline{T}$ contained in a $\sigma$-stable Borel subgroup of $\overline{X}_i\overline{T}$ (see (2.9) of [25]). Set $C = G_0 \cap \overline{C}_\alpha$. Then $C \cap X_i$ is a Cartan subgroup of $X_i$, while $C$ is a maximal torus of $G_0$. Moreover, $X_iT_0 = X_iC$ by (2.5). Therefore, $C \leq X_iT_0 \leq JT_0$ and replacing $T_0$ by $C$ in the above we conclude $\overline{C} \leq \overline{J}\overline{T}_2 = \overline{J}\overline{T}$. Each $C$-root subgroup of $X_i$ is a $p$-group, so the argument of the last paragraph gives $\overline{X}_i \leq \overline{J}$. Therefore, $\overline{J}(T_0) \leq \overline{J}$, and (iii) holds.

It remains to prove (v). We have seen that $T_2 \leq N(V_1)$, so (v) follows from this and (ii). This completes the proof of (9.1).

III. THE MAIN THEOREMS

10. Classification. In this section and the next we complete our analysis of subgroups of $G$ invariant under a maximal torus. We show that any such subgroup arises from a subset of $\overline{\Sigma}$. However, to carry out the proof, we must invoke the classification of finite simple groups. Let $\mathcal{K}$ denote the list of simple groups; the alternating groups, groups of Lie type, and the 26 sporadic groups (see (11.1)). At one point we will need the fact that any $T_0$-invariant simple section of $G$ is isomorphic to a group in $\mathcal{K}$. Very little information about groups in $\mathcal{K}$ is actually required for the proofs of the main results, but the author sees no way to avoid an application of the classification theorem.

The fundamental result is (10.1) below, while (10.2) provides extra information, which follows fairly easily from (10.1) and previous results. To state (10.2) we require the following terminology and notation. For a subset $\Delta \subseteq \overline{\Sigma}$ let $\overline{D} = \langle \overline{U}_\alpha | \alpha \in \Delta \rangle$ and $\hat{\Delta} = \{ \beta \in \overline{\Sigma} | \overline{U}_\beta \leq \overline{D} \}$. We say $\Delta$ is closed if $\Delta = \hat{\Delta}$, in which case we write $\overline{D} = G(\Delta, \overline{T})$. If $\Delta$ is a closed, $\langle \sigma \rangle$-invariant, subset of $\overline{\Sigma}$, we set $G(\Delta, \overline{T}) = O'_{p}(G(\Delta, \overline{T})_{\sigma})$. 

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Throughout this section and the next we make the standing hypothesis that $p > 3$ and $q > 11$.

**Theorem (10.1).** Let $Y$ be a $T_0$-invariant subgroup of $G$.

(i) $Y(T_0) \leq Y$ and $\langle T_0^\gamma Y \rangle = Y(T_0)T_0$.

(ii) $Y(T_0) = O^p(\bar{Y}(T_0))$.

**Theorem (10.2).** Let $T_0 \leq Y \leq G$. Then

(i) $O_p(Y) \leq Y(T_0)$ and $Y(T_0)/O_p(Y) = E(Y/O_p(Y))$.

(ii) $Y(T_0)$ is the semidirect product of $O_p(Y)$ and a $T_0$-invariant subgroup $J = J(T_0)$, and $J$ is a central product of groups of Lie type over extension fields of $F_q$. Also, $O_p(Y)$ is a product of $T$-root subgroups of $G$.

(iii) If $T_1$ is any maximal torus of $G_0$ with $T_1 \leq Y$, then $Y(T_0) = Y(T_1)$, $\bar{Y}(T_0) = \bar{Y}(T_1)$, and $Y(T_0)T_0 = Y(T_1)T_1$.

(iv) There is a maximal torus $T_1$ of $G_0$ such that $T_1 \leq Y$ and $Y = Y(T_1)N_{\gamma}(T_1)$.

(v) There is a unique $\langle \sigma \rangle$-invariant, closed, subset $\Delta$ of $\Sigma$, such that $\bar{Y}(T_0) = G(\Delta, \bar{T})$ and $Y(T_0) = G(\Delta, T)$.

We will first show how to derive (10.2) from (10.1). So suppose $T_0 \leq Y$ and that the hypotheses and conclusions of (10.1) are satisfied. Set $\Delta_0 = \{ \alpha \mid \bar{U}_{\alpha} \leq \bar{Y}(T_0) \}$. Clearly $\Delta$ is $\langle \sigma \rangle$-invariant, closed, and $\bar{Y}(T_0) = G(\Delta, \bar{T})$. Hence, $Y(T_0) = G(\Delta, T)$. This gives (v).

By (2.5) $\bar{Y}(T_0) = R_a(\bar{Y}(T_0))L$, where $L = L^\sigma$ is $\bar{T}$-invariant and $L$ is semi-simple. Then $(R_a(\bar{Y}(T_0)))^\sigma = O_p(Y(T_0))$, $J = O^p(L_\sigma)$ is a central product of groups of Lie type over extension fields of $F_q$, and $Y(T_0) = O_p(Y(T_0))O^p(L_\sigma)$ By (6.1) $O_p(Y) \leq Y(T_0)$, so $O_p(Y(T_0)) = O_p(Y)$ and (ii) holds.

Since $q > 11$, $J = E(J)$ and so (10.1)(i) implies $Y(T_0)/O_p(Y) \leq E(Y/O_p(Y))$. Suppose equality fails to hold and let $X/O_p(Y) = C(Y(T_0)/O_p(Y)) \cap E(Y/O_p(Y))$. Then $[X, T_0] \leq X$, while (10.1)(i) implies $[X, T_0] \leq Y(T_0)T_0$. This forces $[X, T_0] \leq O_p(Y)$, whence $O_p(Y)T_0 \leq I$, where $I = XT_0$. The Frattini argument implies $I = O_p(Y)N_I(T_0)$. Now, $T_0O_p(Y)/O_p(Y) \leq Z(I/O_p(Y))$ and $T_0 \cap O_p(Y) = 1$. Therefore, $N_I(T_0) \leq C_I(T_0) = T_0$ by (2.8). But then $I = O_p(Y)T_0$, a contradiction. This proves (i).

Let $T_1$ be a maximal torus of $G_0$ with $T_1 \leq Y$. Replacing $T_0$ by $T_1$ in the above we have $Y(T_1)/O_p(Y) = E(Y/O_p(Y))$, so $Y(T_1) = Y(T_0)$. Let $T_2$ be a Cartan subgroup of $J$ and $T_2 = C_{T_0}(T_2)$. By (9.1) $T_2$ is a maximal torus of $G_0$ with $L = \bar{J}(T_2) = \bar{J}(T_0)$. Also, the proof of (9.1) showed $\bar{J}(T_2)T_2 = \bar{J}(T_0)\bar{T}$, where $\bar{T}_2 = C_{\bar{G}}(T_2)^0$, a maximal torus of $\bar{G}$. By (9.1)
and (2.5)(iv), \(JT_2 = G_0 \cap (\overline{J}(T_2)\overline{T_2})_\sigma = G_0 \cap (\overline{J}(T_0)\overline{T})_\sigma = JT_0\). Since \(\overline{T_2} \leq \overline{J}(T_0)\overline{T} \leq \overline{Y}(T_0)\overline{T}\), we have \(\overline{T_2} \leq N(\Gamma(T_0))\). From (6.9) and the fact \(R_u(\overline{Y}(T_0))_\sigma = O_\rho(\overline{Y}(T_0))(T_2)\), we conclude \(R_u(\overline{Y}(T_0)) = O_\rho(\overline{Y}(T_0))(T_2)\), and this proves \(\overline{Y}(T_0) = \overline{Y}(T_2)\).

The results of the last paragraph yield \(Y(T_0)T_0 = Y(T_2)T_2\). Since \(T_3\) is a Cartan subgroup of \(J\) we have \(Y = Y(T_2)N_\gamma(T_3)\). Now \(N_\gamma(T_3)\) normalizes \(C(T_3)\) and \(\overline{T_2} = Y(T_2)\overline{T_2}\) and \(\overline{T_2}\) is the unique maximal torus of \(\overline{G}\) containing \(\overline{T_2}\). Therefore, \(\overline{Y}(T_2)\overline{T_2} = \overline{Y}(T_2)\overline{T_2}\) so (2.5) implies \(Y(T_2)T_2 = Y(T_2)T_1\). By the above, \(Y(T_2)T_2 = Y(T_0)T_0\) and \(Y(T_0) = Y(T_2) = Y(T_1)\). Hence, \(Y(T_1)T_1 = Y(T_0)T_0\). Replacing \(T_1\) by a \(Y(T_2)\)-conjugate we may assume \(JT_2 = JT_1\). Now replace \(T_0\) by \(T_1\) in the previous argument to get \(\overline{Y}(T_2) = \overline{Y}(T_1)\). Hence \(\overline{Y}(T_1) = \overline{Y}(T_0)\) and this establishes (iii), completing the proof of (10.2).

The rest of §10 and all of §11 concerns the proof of (10.1). Toward this end suppose the result false and let \(G_0\) be a counterexample of least order for which (10.1) fails for some pair \((T_0, Y)\). Then \(Z(G_0) = 1\). We may assume \(T_0 \leq Y\) (otherwise replace \(Y\) by \(YT_0\)), and among all such groups \(Y\) choose one with \(|Y|\) minimal. In other words, if \(T_1\) is a maximal torus of \(G_0\) and \(T_1 \leq Y_1\) where \(|Y_1| < |Y|\), then (10.1) holds for the pair \((T_1, Y_1)\).

For \(X\) a finite group, let \(E(X)_p\) denote the product of all components of \(X\) that are of Lie type in characteristic \(p\).

(10.3) Suppose \(X\) is a proper, \(T_0\)-invariant, subgroup of \(G_0\) and \(X/O_\rho(X) = E(X/O_\rho(X))_p\). Then (10.1) holds for the group \(XT_0\).

Proof. Let \(T_2\) be a \(p'\)-Hall subgroup of a Sylow normalizer of \(XT_0\). By (9.1)(i) \(T_2\) is a maximal torus of \(G_0\). By (9.1)(iii) \(\overline{X}(T_0) = \overline{X}(T_2)\) and by (9.1)(ii) \(X = O_\rho'(\overline{X}(T_2))_\sigma = O_\rho'(\overline{X}(T_0))_\sigma\). Since \([X, T_0] = X\) for any \(T\)-root subgroup of \(G\) we have \(\langle T_0^X \rangle = XT_0\). Therefore it remains to show \(X = X(T_0)\).

By (9.1)(iv) we have \(X = O_\rho(X)J\), a semidirect product, where \(J\) is \(T_0\)-invariant, \(JT_0 = JT_2\) and \(J = E(J)_p\). By (6.1) \(O_\rho(X)\) is a product of \(T_0\)-root subgroups, so we may assume \(X = J\). By (7.1), \(T_0\) contains a
maximal torus of each component of $X$, so we may also assume $X$ to be quasisimple. Then (8.1) shows that $X$ is defined over $F_q$, for some $j$ and $\bar{X}(T_2)$ is the commuting product of a $\langle \sigma \rangle$-orbit of $j$ semi-simple groups, say $\bar{X}(T_2) = \bar{D}_1 \cdots \bar{D}_j$. Let $X_0 = O^p((\bar{D}_1)_{\sigma})$, a group of Lie type over $F_q$, having $A = X_0 \cap (T \cap \bar{D}_1)_{\sigma}$ as maximal torus. From the minimality of $|G_0|$, (10.1) holds within $X_0$ and applying the result to the $A$-invariant subgroup $X_0$ of $X_0$ we have $X_0 = X_0(A)$. So $X_0$ is generated by its $A$-root subgroups (corresponding to $\langle \sigma^i \rangle$-orbits of $(T \cap \bar{D}_1)$-root subgroups of $\bar{D}_1$). Using the fact that the map $x_0 \rightarrow x_0 x_0^\sigma \cdots x_0^{\sigma^{r-1}}$ is a surjection from $X_0$ to $X$ and mapping $A$-root subgroups of $X_0$ to $\Gamma_0$-root subgroups of $X_0$ we have $X = X(T_0)$, as required.

(10.4) (i) $Y \cong G_0$.
(ii) If $O_p(Y) = 1$, then $E(Y)_p = 1$.

Proof. Suppose $Y \geq G_0$. Then (10.1) holds if $Y(T_0) = G_0$. So $Y(T_0) < G_0$, and we set $Y_1 = Y(T_0)T_0 < Y$. Minimality implies that (10.1) holds for $Y_1$. However, $\bar{Y}_1(T_0) = \bar{G}$ and $Y_1(T_0) = Y(T_0) \neq O^p(\bar{G}_\sigma)$. This is a contradiction, proving (i).

For (ii), suppose that $O_p(Y) = 1$ and $X = E(Y)_p \neq 1$. By (i), $X < G_0$. If $X_i$ is any $T_0$-root subgroup of $G$, then $[X_i, T_0] = X_i$ (use (7.1), (5.5), and (6.8)), hence $X_i \leq \langle T_0 \rangle$. So if we can show $\langle T_0 \rangle \leq XT_0$, then (7.1) implies $\langle T_0 \rangle = XT_0 = \langle T_0 \rangle$, and (10.3) shows that (10.1) holds for $XT_0$, and hence for $Y$. Since we are assuming this to be false, it will suffice to show $\langle T_0 \rangle \leq XT_0$.

Let $T_1$ be a Cartan subgroup of $X$ and let $D = C_Y(X)$. The Frattini argument shows $Y = XN_Y(T_1)$ and, of course $D \leq C_Y(T_1) \subset N_Y(T_1)$. By (7.2), $T_2 = C_{XT_0}(T_1)$ is a maximal torus of $G_0$. By minimality of $|Y|$, (10.1) holds for the group $N_Y(T_1)$, which contains the torus $T_2$. Also, (10.1) holds for the groups $DT_2$ and $C_Y(T_1)$, each of which contains $T_2$. So applying (10.1) to $DT_2$ we have one of $O_p(D) \neq 1, E(D)_p \neq 1$, or $T_2 \leq DT_2$. As $D \leq Y$, the first two situations are out. Hence $T_2 \leq DT_2$. Considering Aut$(X)$, we see that $C_Y(T_1)/D$ is a solvable $p'$-group. Therefore, applying (10.1) to $C_Y(T_1)$ we conclude $T_2 \leq C_Y(T_1)$. From (6.3) we have $T_2 \leq N_Y(T_1)$, and this proves $XT_2 \leq NX_Y(T_1) = Y$. Now, (9.1) shows $XT_2 = XT_0$, so $\langle T_0 \rangle \leq XT_0$, as required.

(10.5) $O_p(Y) = 1$.

Proof. Suppose $T_0 \leq Y \leq G$ and $O_p(Y) \neq 1$. By (3.9) of [4] there is a canonical parabolic subgroup $P < G_0Y$ with $O_p(Y) \leq O_p(P)$ and
In particular $T_0 \leq P$. By (6.4) $P = G_0 Y \cap \tilde{P}_\sigma$, for $\tilde{P}$ a $\sigma$-invariant parabolic subgroup of $G$ containing $T$. Write $P = VL(T \cap P)$, where $V = O_p(P)$ and $L$ is the derived group of the Levi factor, $L(T \cap P)$, of $P$. If $L = 1$, then $Y = O_p(Y)(T \cap Y)$ and (10.1) follows from (6.9).

Since $Y$ is a counterexample to (10.1) we have $L \not= 1$.

Set $T_{00} = T_0 \cap L$, a maximal torus of $L$. One checks that the $T_{00}$-root subgroups of $L$ are just the $T$-root subgroups of $G$ that are contained in $L$. Let $\tilde{\cdot}$ denote images in $LV/V$ and set $Y_1 = Y \cap VL$. Then $\tilde{Y}_1(T_{00}) \leq \tilde{Y}_1$ and $\langle \tilde{Y}_1(T_{00}) \rangle = \tilde{Y}_1(T_{00})\tilde{T}_{00}$ (by minimality of $|G_0|$). Set $A/V = \tilde{Y}_1(T_{00})$ and $X = Y \cap A$. Then $X \leq Y_1$ and $X/O_p(X) = E(X/O_p(X))$. Also, $X T_{00} \leq Y_1$.

Let $T_2$ be a $p'$-Hall subgroup of a Sylow $p$-normalizer of $XT_0$. By (9.1)(i) $T_2$ is a maximal torus of $G_0$, and (9.1)(ii) implies $XT_0 = XT_2$.

We claim $XT_0 \leq Y$. First note that $O_p(X) = O_p(Y) = O_p(Y_1)$, then argue as in the proof of (10.2) that $X/O_p(Y) = E(Y_1/O_p(Y))$. This shows $X \leq Y$. Let $V_1$ be a Sylow $p$-subgroup of $X$ with $T_2 \leq N_Y(V_1) = D$. Suppose $D < Y$. Minimality of $|Y|$ implies $D(T_2) \leq D$ and $\langle T_2^D \rangle = D(T_2)T_2 \leq D$. As $Y/Y_1$ is an abelian $p'$-group, $D(T_2) \leq Y_1$. Also, $Y = XN_Y(V_1) = XD$ implies $XD(T_2)T_2 \leq Y$. Now $D(T_2) \leq LV$ and $D(T_2)$ is generated by $(T_2 \cap L)$-root subgroups of $\tilde{L}$. Hence $[D(T_2)\tilde{\cdot},(T_2 \cap L)] = D(T_2)\tilde{\cdot}$. On the other hand, $X(T_2 \cap L) = XT_{00} \leq Y_1$. Therefore, $D(T_2) \leq X$ and so $XT_0 = XT_2 = XD(T_2)T_2 \leq Y$, as required. Suppose then that $D = Y$; that is $X = V$.

Let $Y \in Y$. Then $V_1 T_{00}$ and $V_1 T_{00}^\nu$ are normal in $Y_1$, and so $\langle V_1 T_{00}, V_1 T_{00}^\nu \rangle = V_1 F$, where $F$ is a $p'$-group normalizing $T_{00}$. Also, $F$ is generated by $T_{00}$ and $T_{00}^\nu$ for some $\nu \in V_1$. Applying (6.3) to the maximal torus $T_{00}$ of $L$, we conclude $F = T_{00}$. As $Y$ was arbitrary, $V_1 T_{00} \leq Y$. So $Y = V_1 N_Y(T_{00})$ which shows $V_1 C_Y(T_{00}) \leq Y$. But (2.8) implies $C_Y(T_{00}) \leq V_1(T \cap Y)$, and $V_1(T \cap Y) \leq Y$. Since $XT_0 = V_1 T_0 = V_1(T \cap Y) \cap G_0 \leq Y$, the claim is proved.

By the claim $XT_2 = XT_0 \leq Y$, so $Y = XN_Y(T_2)$. By (10.3) $X = X(T_0)$ and by (6.1) $X = X(T_2)$. If $E$ is a $T_2$-root subgroup of $G$ or a $T_0$-root subgroup of $G$, then $[E,T_2] = E$ or $[E,T_0] = E$, respectively. Hence $X = Y(T_2) = Y(T_0)$. These remarks and (10.3) show that (10.1) holds, which we have assumed false. The proof of (10.5) is now complete.

(10.6) Suppose $F^*(Y) = \text{Fit}(Y)$. Then

(i) $F^*(Y)$ is a $p'$-group;

(ii) If $Y_1 \leq Y$ and $Y_1 T_0 < Y$, then $Y_1 \leq N(T_0)$.

(iii) $T_0$ normalizes no non-trivial $p$-subgroup of $Y$.

(iv) $Y = \langle T_0^Y \rangle$. 

Proof. (i) is immediate from (10.5). Suppose \( Y_x \leq Y \) and \( Y T_0 \leq Y \). Minimality of \(|Y|\) implies that (10.1) holds for \( Y T_0 \). Therefore, either \( O_p(Y_1) \neq 1 \), \( E(Y_1)_p \neq 1 \), or \( Y_1 \leq N(T_0) \). By (i) the first possibility is out, and our hypothesis rules out the second possibility. Therefore \( Y_1 \leq N(T_0) \), establishing (ii).

For (iii), suppose \( T_0 \) normalizes the non-identity \( p \)-subgroup, \( D \), of \( Y \). Then \( R = F^*(Y) DT_0 \) is solvable. By (6.1), \( R = O_p(R) N(R) \). Since \( C_Y(F^*(Y)) \leq F^*(Y) = O_p(F^*(Y)) \), we have \( O_p(R) = 1 \) and \( D \leq N(T_0) \). But then \([D, T_0] = 1\), against (2.8). This gives (iii).

Finally, let \( Y_1 = \langle T_0^Y \rangle \) and suppose \( Y \leq Y \). Then \( Y_1 \leq \text{Fit}(Y) \). By (6.1) \( T_0 \leq \text{Fit}(Y) \), and (6.3) implies \( T_0 \leq Y \). However, with \( T_0 \leq Y \), (10.1) is a triviality, whereas we are assuming it false. This is a contradiction. So \( Y_1 = Y \) and (iv) holds.

(10.7) Suppose \( F^*(Y) = \text{Fit}(Y) \). Then \( F^*(Y) = O_p(Y) \) for some prime \( r \neq p \).

Proof. Suppose \( X = F^*(Y) = \text{Fit}(Y) \) and \( X = O_p(X) \times \cdots \times O_{p_l}(X) \), where \( p_1, \ldots, p_l \) are distinct prime divisors of \(|X|\) and \( l > 1 \). By (6.1) \( X \leq N(T_0) \), so \([X, T_0] \leq T_0 \cap X\). If \( T_0 \cap O_{p_i}(X) = 1 \) for some \( i \), then \([T_0, O_{p_i}(X)] \leq T_0 \cap O_{p_i}(X) = 1 \) and \( O_{p_i}(X) \leq C_G(T_0) = T \) (by (2.8)). But (10.6)(iv) shows \( Y \leq G_0 \), whence \( O_{p_i}(X) \leq T \cap G_0 = T_0 \), a contradiction. Therefore, \( T_0 \cap O_{p_i}(X) \neq 1 \) for \( i = 1, \ldots, l \). Suppose \( T_0 \leq X \). Then for \( y \in Y, T_0^y \leq X \leq \hat{N}(T_0) \), so \( T_0^y = T_0 \) by (6.3). Hence \( T_0 \leq Y \) and (10.6)(iv) gives the contradiction \( Y = T_0 \). Therefore, \( 1 < T \cap X < T \). Also, \( X \) is a \( p' \)-group by (10.6)(i).

For \( i = 1, \ldots, l \) the groups \( O_{p_i}(T_0) \) and \( O_{p_i}(X) \) normalize each other. Hence, they commute. Set \( Y_i = \langle O_{p_i}(T_0)^Y \rangle \). Then \( Y_i \leq Y \) and \( Y_i \leq C(O_{p_i}(X)) \). Suppose that for some \( 1 \leq i \leq l \) we have \( Y_i T_0 \neq Y \). By (10.6)(ii) \( Y_i \leq N(T_0) \), hence \( O_{p_i}(T_0) \leq \text{Fit}(Y_i) \leq \text{Fit}(Y) = X \). Let \( C_i = C_Y(O_{p_i}(X)) \). Then \( O_{p_i}(X) \leq C_i \leq Y \) and \( T_0 \leq T_0 Y_i C_i \). By (10.6)(iv), \( Y = Y_i C_i \) and since \( Y_i \leq N(T_0) \) and \( Y = \langle T_0^Y \rangle \) we have \( Y = C_i T_0 = C_i(O_{p_i}(T_0)) \). Since both \( C_i \) and \( O_{p_i}(T_0) \) centralize \( Z = O_{p_i}(T_0) \) we conclude \( Z \leq Z(Y) \).

Since \( O_{p_i}(T_0) \neq 1 \) we choose \( 1 \neq z \in Z \) and consider the group \( C_{G_0}(z) \geq Y \). Let \( D_1, \ldots, D_k \) be the components of \( C_G(z) \). Then \( D_1 \cdots D_k T_0 \leq C_{G_0}(z) \) with quotient group isomorphic to a subgroup of the center of the universal covering group of \( G_0 \) (see (2.9)). Since \( Y = \langle T_0^Y \rangle \), \( Y \leq D_1 \cdots D_k T_0 \) and we have \( Y = T_0 \hat{Y} \), where \( \hat{Y} = Y \cap D_1 \cdots D_k \). If \( 1 \leq j \leq k \), let \( \hat{Y}_j = \{ d_j \in D_j | d_j g \in \hat{Y} \) for some \( g \in D_1 \cdots D_j \cdots D_k \). Then \( \hat{Y}_j \) is a group and essentially the projection of \( \hat{Y} \) to \( D_j \) (note that the projection is not defined since the product may not be direct). Then
\[ \hat{Y} \leq \hat{Y}_1 \cdots \hat{Y}_k \text{ and } T_0 \cap D_j \leq \hat{Y}_j \text{ for } 1 \leq j \leq k. \] Also, \( T_0 \cap D_j \) is a maximal torus of \( D_j \), for \( 1 \leq j \leq k \). Minimality of \( |G_0| \) implies that (10.1) holds for each of the containments \( T_0 \cap D_j \leq \hat{Y}_j \leq D_j \).

We conclude that for \( 1 \leq j \leq k \) one of the following hold: \( O_p(\hat{Y}_j) \neq 1 \), \( E(\hat{Y}_j)_p \neq 1 \), or \( T_0 \cap D_j \leq \hat{Y}_j \). Since \( O_p(Y) = 1 = E(Y)_p \), we necessarily have \( T_0 \cap D_j \leq \hat{Y}_j \) for \( 1 \leq j \leq k \), hence \( I = (T_0 \cap D_1) \cdots (T_0 \cap D_k) \leq \hat{Y} \).

By (2.8) \( T_0 = D_1 \cdots D_k T_0 \cap C(I) \), so \( \hat{Y} \leq N(T_0) \). We then have \( T_0 \leq Y = \langle T_0^\gamma \rangle \), so \( T_0 = Y \), a contradiction. We have proved that \( Y \cap T_0 = Y \) for \( 1 \leq i \leq l \). In particular, \( Y/Y_i \) is abelian for \( 1 \leq i \leq l \). Set \( \hat{Y} = \cap_{i=1}^l Y_i \).

Then \( Y/\hat{Y} \) is abelian, so \( Y = \langle T_0^\gamma \rangle = \hat{Y} T_0 \). However, \( \hat{Y} \leq C_Y(X) \) (as \( Y_i \leq C(O_p(X)) \)). Therefore, \( \hat{Y} \leq X \), \( X T_0 \leq N(T_0) \), and as \( Y = \langle T_0^\gamma \rangle = T_0 \), this is a final contradiction.

(10.8) \( F^*(Y) \neq \text{Fit}(Y) \).

**Proof.** By way of contradiction, suppose \( F^*(Y) = \text{Fit}(Y) \). By (10.7), \( X = \text{Fit}(Y) = O_r(Y) \) for some prime \( r \) and \( r \neq p \) by (10.6). As in the first paragraph of the proof of (10.7) we have \( 1 < T_0 \cap X < T_0 \). Also, \( X \leq N(T_0) \), so \( O_r(T_0) \leq C_Y(X) \leq X \), and we conclude that \( T_0 \) is an \( r \)-group.

Fix \( y \in Y \) with \( T_0^y \neq T_0 \) and set \( \hat{Y} = O_r(Y) \langle T_0, T_0^y \rangle \). If \( \hat{Y} < Y \), then minimality of \( |Y| \) shows that (10.1) holds for \( \hat{Y} \). However, \( C_Y(X) \leq X \) implies that \( E(\hat{Y})_p = O_p(\hat{Y}) = 1 \). Therefore, \( T_0 \leq \hat{Y} \). But (6.3) implies \( T_0 \) is weakly closed in its normalizer, whence \( T_0^y = T_0 \), a contradiction. Therefore, \( Y = \hat{Y} = O_r(Y) \langle T_0, T_0^y \rangle \).

Let \( A = T_0^0 \cap X, B = T_0^0 \cap X, \) and \( V = \langle A, B \rangle \). Both of \( A, B \) are normal in \( X \). If \( A \cap B \neq 1 \), then \( \langle T_0, T_0^y \rangle \leq C(A \cap B) \) so we can choose \( 1 \neq z \in A \cap B \cap Z(Y) \). Then \( Y \leq C_G(z) \) and the argument of (10.7) gives a contradiction. Therefore, \( A \cap B = 1 \). This shows that \( V = \langle A, B \rangle = A \times B \). Also, \( [V, T_0] \leq [X, T_0] \leq X \cap T_0 = A \leq V \), so \( T_0 \leq N_Y(V) \) and similarly \( T_0^y \leq N_Y(V) \). Hence \( V \leq Y \).

If \( t \in T_0 \), then \( [X, t, t] \leq [A, t] = 1 \). Therefore, \( t' \) centralizes each \( Y \)-chief factor contained within \( X \). But the intersection of the centralizers of such chief factors is an \( r \)-group and hence \( X \). Therefore, \( t' \in X \) and \( T_0' \leq X \cap T_0 = A \). If \( (T_0')' \) contains an element, \( j \), of order \( r^2 \), then for \( t \in T_0 \) we have \( [t', j] \in [A, B] = 1 \). As \( VT_0 \) has nilpotence class \( 2 \) we conclude \( 1 = [t', j] = [t, j]' = [t, j'] \). That is, \( j' \in C_T(T_0) = T_0 \) contradicting \( A \cap B = 1 \). Therefore, \( T_0' \), and hence \( T_0 \), has exponent at most \( r^2 \).

Say \( G \) has Lie rank \( n \), so that \( |T_0| = \frac{1}{d}f(q), \) where \( f(t) = \Pi \Phi(d_i) \) and \( \Sigma \varphi(d_i) = n \). By (2.4)(iii) \( |T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{d}(12)^n \). As an abelian group
\(T_0\) has rank at most \(n\) (see (2.3)). Therefore, \(|T_0| \leq (r^2)^n\) and \(d \geq (12/r^2)^n\). As \(d \leq n + 1\), we have \(r > 2\). Also, \(X/A\) is isomorphic to a subgroup of \(\overline{W}\), so \(n \geq 2\). If \(r = 3\), then the inequality forces \(q = 13\) and consideration of primitive divisors leads to a contradiction. Therefore, \(r \geq 5\).

Suppose \(\overline{G}\) is an exceptional group. Then \(r || \overline{W}|\) implies \(\overline{W}\) is of type \(E_6, E_7, \) or \(E_8\). If \(r = 5\), then \(\overline{W}|_r = r, r, r^2\), respectively, while if \(r = 7\), \(|\overline{W}|_r = 1, r, r\), respectively. In any case, \(r \leq 7\). If \(|\overline{W}|_r = r\), then \(X/A \cong Z_r\), so \(X = V \cong Z_r \times Z_r\) and \(Y/X \cong \text{SL}(2, r)\). But then \(|T_0| = r^2\), against \(|T_0| \geq \frac{1}{2} (q - 1)^n\). Therefore, \(r = 5\), \(\overline{W}\) is of type \(E_6, E_7, E_8\), and \(|T_0|\) divides \(r^2\). If \(|A| = r^2\), then \(|B| = r^2\) and \(X = A \times B\) is elementary of order \(r^4\).

So \(Y/X \leq \text{SL}(4, r)\), \(|T_0| \leq r^2 \cdot r^4 = 5^6\), again contradicting \(|T_0| \geq \frac{1}{2} (q - 1)^n \geq (12)^8\). If \(|A| = r\), then \(|X| \leq r^3\) and one argues \(|T_0| \leq r^4\), impossible. So \(\overline{G}\) is not an exceptional group.

Therefore, \(G_0\) is a classical group \((^3D_4(q)\) is out as \(r || \overline{W}|\) and \(r \geq 5\)) and we let \(M\) be the natural module for the appropriate covering group, \(\hat{G}_0\), of \(G_0\). Let \(\hat{T}_0\) be the preimage of \(T_0\) and \(\hat{B}\) a Sylow \(r\)-subgroup of the preimage of \(B\). Write \(M = M_1 \oplus \cdots \oplus M_l\), a decomposition into irreducible \(\hat{T}_0\)-irreducible submodules. If \(\hat{G}_0 \cong \text{SL}(m, q)\) then \(M\) has a non-degenerate bilinear form and we can arrange \(M_1, \ldots, M_l\) so that for some \(k \leq l\), \(M_1, \ldots, M_k\) are non-degenerate, while each of \(M_{k+1}, \ldots, M_l\) is totally singular. Moreover, \(M = M_1 \perp \cdots \perp M_k \perp (M_{k+1} \oplus M_{k+2}) \perp \cdots \perp (M_{l-1} \oplus M_l)\), with \(\hat{T}_0\) inducing contragredient representations (contragredient followed by a field automorphism, in the unitary case) on the pairs \(\{M_{k+1}, M_{k+2}\}, \ldots, \{M_{l-1}, M_l\}\). Set \(N_j = M_{k+1} \oplus M_{k+2}, \ldots, N_j = M_{j-1} \oplus M_j\), where \(j = \frac{1}{2} (l - k)\). We now have \(M = M_1 \perp \cdots \perp M_k \perp N_1 \perp \cdots \perp N_j\), and for \(\hat{G}_0 = \text{SL}(m, q)\), \(M = M_1 \oplus \cdots \oplus M_k\), considering \(j = 0\).

We claim that \(T_0\) is not cyclic. Otherwise, \(A \cong Z_r\) and \(T_0 \cong Z_r^2\). Then \(\text{Aut}(T_0)\) has Sylow \(r\)-subgroups of order \(r\), which implies \(X/A \cong Z_r\). Thus, \(X = V \cong Z_r \times Z_r\) and \(Y/X \cong \text{SL}(2, r)\). But then \(Y\) splits over \(V\), forcing \(T_0 \cong Z_r \times Z_r\), a contradiction. This proves the claim, which implies \(l > 1\) and \(G_0 \cong \text{PSL}(2, q)\).

Next we show that \(k + j = 2\). If \(k + j = 1\), then \(\hat{T}_0\) is cyclic, contradicting the preceding paragraph. So \(k + j \geq 2\). If \(k + j \geq 3\), then letting \(\hat{T}\) be the maximal torus in \(\text{GL}(m, q)\), \(\text{Sp}(m, q)\), \(\text{SO}^\pm(m, q)\), or \(\text{GU}(m, q)\) containing \(\hat{T}_0\), we have the Sylow 2-subgroup of \(\hat{T}\) of rank at least 3. Thus, the Sylow 2-subgroup of \(\hat{T}_0\) has rank at least 2, which forces \(|T_0|\) to be even. However, \(T_0\) is an \(r\)-group and \(r \geq 5\). So \(k + j = 2\), as desired. Accordingly, write \(M = M' \oplus M''\), where each of \(M'\) and \(M''\) has the form \(M_i\) or \(N_i\).
The representations of \( \hat{T}_0 \) on \( M' \) and \( M'' \) have different kernels (otherwise \( \hat{T}_0 \) would be cyclic) and each of \( M' \) and \( M'' \) is either irreducible or the sum of two inequivalent \( \hat{T}_0 \)-submodules. So \( M \) is the sum of at most 4 pairwise inequivalent, irreducible, \( \hat{T}_0 \)-submodules, and since \( B \leq N(T_0) \) and \( r \geq 5 \), \( B \) necessarily stabilizes each of these irreducibles.

Let \( I \leq M \) be an irreducible \( \hat{T}_0 \)-submodule of \( M \) such that \([\hat{T}_0 |_I, \hat{B} |_I] \neq 1\). Say \( \dim(I) = d \). Then \( \hat{T}_0 |_I \) can be regarded as a subgroup of the multiplicative group of the field \( \mathbb{F}_{q^d} \) (\( \mathbb{F}_{q^2} \) in the unitary case) and \( \hat{B} \) induces field automorphisms on \( \hat{T} |_I \). Therefore, \( r \mid d \). On the other hand, \( \Phi_d(q) \) divides the order of \( \hat{T}_0 |_I \), so let \( s \) be a primitive divisor of \( \Phi_d(q) \). As \( d > 2 \), we have \( s \mid | T_0 | \), whence \( s = r \). However, \( d \) is the order of \( q \), modulo \( s \), so \( d \mid s - 1 \). We now have \( r \mid d \) and \( d \mid r - 1 \), which is absurd. This contradiction proves (10.8).

(10.9) If \( T_0 \leq X \) and (10.1) holds for \( X \), then \( E(X) = E(X)_p \).

**Proof.** As (10.1) holds for \( X \), \( \langle T_0^X \rangle = X(T_0)T_0 \). Let \( J \) be the product of those components of \( X \) not contained in \( E(X)_p \). Then \([T_0, J] \leq J \cap X(T_0)T_0 \leq Z(J) \). The 3-subgroup lemma then shows \([T_0, J] = 1 \). But this gives \( J \leq C_{G_0}(T_0) = T_0 \), a contradiction.

(10.10) (i) \( F^*(Y) = E(Y) \);  
(ii) \( Z(E(Y)) = 1 \); and  
(iii) \( E(Y)_p = 1 \).

**Proof.** By (10.5) \( O_p(Y) = 1 \), so (10.4) shows \( E(Y)_p = 1 \). Also, (10.8) implies \( F^*(Y) \neq \text{Fit}(Y) \), so \( X = E(Y) \neq 1 \) and is a product of components, none of which is of Lie type in characteristic \( p \). If \( XT_0 < Y \), then minimality of \( | Y | \) and (10.9) gives a contradiction. Therefore \( Y = XT_0 \).

To prove the result it will suffice to show \( \text{Fit}(Y) = 1 \). By (6.1) we have \( T_0 \leq \text{Fit}(Y)T_0 \) (recall \( O_p(Y) = 1 \)), hence \([T_0, \text{Fit}(Y)] \leq T_0 \cap \text{Fit}(Y) = A \). If \( A = 1 \), then \( \text{Fit}(Y) \leq C_{G_0}(T_0) \cap \text{Fit}(Y) = T_0 \cap \text{Fit}(Y) = A = 1 \), as required. Suppose \( A \neq 1 \). Then \( A \leq \text{Fit}(Y) \leq C(X) \) and so \( A \leq Z(Y) \). Fix \( 1 \neq a \in A \) and consider the embedding \( Y \leq C_{G_0}(a) \).

Since \( T_0 \) centralizes no component of \( Y \), \( X = [T_0, Y] \) and this implies \( X \leq D_1 \cdots D_k \), where \( D_1, \ldots, D_k \) are the components of \( C_{G_0}(a) \) (see (2.9)). By (7.1), \( T_0 \cap D_1 \) is a maximal torus of \( D_1 \), and we may reorder, if necessary, so that \( Y_1 = [T_0 \cap D_1, Y] \neq 1 \). Argue in \( D_1 \) with the subgroup \( Y_1(T_0 \cap D_1) \), using minimality of \( | G_0 | \) and (10.9) to obtain a contradiction.
(10.11) Let \( X = F^*(Y) \). Then \( Y = X T_0 \leq \text{Aut}(X) \) and either \( X \) is simple or \( X \) is the commuting product of two \( T_0 \)-conjugate simple groups.

Proof. Let \( X_1 \) be a component of \( Y \) and \( \{X_1, \ldots, X_k\} \) the orbit of \( X_1 \) under \( T_0 \). By (10.10)(iii), \( E(Y)_p = 1 \), so minimality of \( |Y| \) and (10.9) imply \( Y = X_1 \cdots X_k T_0 \). By (10.10)(ii), each of \( X_1, \ldots, X_k \) is a simple group. Suppose \( k > 1 \).

Let \( T_2 = N_{T_0}(X_i) \), so that \( T_2 \leq N(X_i) \) for \( i = 1, \ldots, k \). Let \( r \) be a prime divisor of \( k \) and let \( t \) be an \( r \)-element of \( T_0 \) with \( |t T_2| = r \). If \( t' = 1 \), consider the group \( C_X(t) T_0 \) and obtain a contradiction (using minimality of \( |Y| \)). Hence, \( t' \neq 1 \). From order consideration we have \( r \mid |X_1| \), so it follows that \( r \) divides \( |A| \), where \( A = C_X(t') \). Write \( A = A_1 \cdots A_k \) with \( A_i = A \cap X_i, i = 1, \ldots, k \).

Apply (10.1) to the group \( A_T_0 \). Let \( C/O_p(A) = A(T_0)/O_p(A) \). Then \( C T_0 \leq A T_0 \) and \( C = C_1 \cdots C_k \), where \( C_i = C \cap X_i \), for \( i = 1, \ldots, k \). If \( C > O_p(A) \), then by (9.1) \( T_0 \) contains a maximal torus of \( C \). However, \( T_0 \cap C_1 \leq T_0 \cap X_i = 1 \) (otherwise \( T_0 \leq N(X_i) \)). Therefore, \( C = O_p(A) \) and \( O_p(A) T_0 \leq A T_0 \). Let \( 1 \neq a_i \in A_1 \) with \( |a_i| = r \). For \( g \in T_0 - T_2 \), \( a_1^{-1} a_f \in X_1 X_f^g \cap O_p(A) T_0 \). So, modulo \( O_p(A) \) this element centralizes \( T_0 \), and it follows that \( X = X_1 X_f^g \). This proves (10.11).

At this stage in the proof of (10.1) we consider the possibilities for \( X = E(Y) \). This is where the classification of finite simple groups becomes relevant. Write \( X = X_1 \) or \( X_1 \times X_2 \).

(10.12) \( X_1 \cong A_m \), for \( m \geq 5 \).

Proof. Suppose \( X_1 \cong A_m \). First we rule out the case \( X = X_1 \times X_2 \). Otherwise, let \( t \in T_0 - N(X_1) \) with \( t \) a 2-element. Then \( X_1^t = X_2 \) and \( j = t^2 \in N(X_1) \). So \( j \) acts as an element of \( S_m \) on each of \( X_1 \) and \( X_2 \) and we set \( A_i = C_X(j) \), for \( i = 1, 2 \). Then \( A_2 = A_1^t \). The structure of \( A_i \) is determined from the cycle decomposition of \( j \). From (10.1) we conclude \( T_0 \unlhd A_1 A_2 T_0 \). So for \( 1 \neq a_1 \in A_1, a_1^{-1} a_f \in [A_1, A_2, T_0] \leq T_0 \leq C(t) \). This forces \( |a_1| = 2 \); hence \( A_1 \) is elementary abelian. From the known structure of \( A_1 \), we conclude \( m \leq 5 \) and \( |T_0| \leq 8 \). But \( X_1 \times X_2 \leq G_0 \) forces \( n \geq 2 \), and we obtain a contradiction from (2.4). Therefore, \( X = X_1 \).

Let \( \Omega = \{1, \ldots, m\} \). Since \( Y \leq \text{Aut}(X) \) and since the order restrictions on \( |T_0| \) force \( m > 6 \), we have \( Y \cong A_m \) or \( S_m \), and \( T_0 \) acts on \( \Omega \). We claim \( T_0 \) is transitive on \( \Omega \). Otherwise, we can write \( \Omega = \Omega_1 \cup \Omega_2 \), a disjoint union of \( T_0 \)-invariant subsets. By minimality of \( |Y| \) and (10.9) we
have $|\Omega_i| \leq 4$ for $i = 1, 2$, hence $|\Omega| \leq 8$. By considering subgroups of $L_2(q)$ we see that $n \geq 2$, and so order restrictions (see (2.4)) lead to a contradiction. Therefore, $T_0$ is transitive on $\Omega$, and as $T_0$ is abelian, $T_0$ is regular. In particular, $|T_0| = m$. If $m$ is not a prime, write $\Omega = \Omega_1 \cup \cdots \cup \Omega_\ell$, a disjoint union, corresponding to a system of imprimitivity for $T_0$ on $\Omega$. Then $T_0$ stabilizes a subgroup $V$ isomorphic to $A_{\Omega_1} \times \cdots \times A_{\Omega_\ell}$. If $|\Omega_1| > \ell$, then the minimality of $|\Omega|$ and (10.9) (applied to $VT_0$) gives a contradiction. Suppose $|\Omega_1| \leq 4$. Then $VT_0$ is solvable, and since $p \geq 5$, we conclude from (6.1) that $l = 2$ and $V$ is an elementary abelian 2-group. That is, $N_{\Gamma_0} = 2^\ell$. If $m$ is not a prime, write $\Omega = \Omega_1 \cup \cdots \cup \Omega_\ell$, a disjoint union, corresponding to a system of imprimitivity for $T_0$ on $\Omega$. Then $T_0$ stabilizes a subgroup $\Gamma$ isomorphic to $A_{\Omega_1} \times \cdots \times A_{\Omega_\ell}$.

If $l = 2$ and $m = 4$, a contradiction. Therefore, $m$ is prime. Also, $T_0 \cong Z_m$, $T_0 \leq X = F^*(Y)$, and $N_{\gamma}(T_0)/T_0$ is cyclic of order $\frac{1}{2}(m - 1)$. We use this information in order to get a numerical contradiction.

Suppose $G_0$ is a classical group. Let $\hat{G}_0$ be the appropriate linear group acting on the natural module $M$, and let $\hat{T}_0$ be the preimage in $\hat{G}_0$ of $T_0$. With notation as in the proof of (10.8) we have $M = M_1 \oplus \cdots \oplus M_k$ if $\hat{G}_0 \cong \text{SL}(n, q)$, while $M = M_1 \perp \cdots \perp M_k \perp N_1 \perp \cdots \perp N_j$, otherwise. Here $\hat{T}_0$ stabilizes each of the subspaces, acts irreducibly on each $M_i$, while each $N_i$ decomposes into two $\hat{T}_0$-invariant totally singular subspaces. If $\hat{G}$ denotes the full linear group (GL$(n, q)$, GU$(n, q)$, Sp$(n, q)$, or SO$^\pm(n, q)$) and $\hat{T}$ the maximal torus of $\hat{G}$ containing $\hat{T}_0$, then $\hat{T}_0 = \hat{T}_1 \times \cdots \times \hat{T}_k$ if $\hat{G}_0 \cong \text{SL}(n, q)$, or $\hat{T}_0 = \hat{T}_1 \times \cdots \times \hat{T}_k \times \hat{T}_i \times \cdots \times \hat{T}_j$, otherwise, where the appropriate subgroups act on the corresponding $M_i$ or $N_j$, and are trivial on all other parts of the decomposition.

Now, $T_0$ has prime order. If $G_0$ is a symplectic or even dimensional orthogonal group, say $\text{dim}(M) = 2n$, then $|\hat{T}|/|T_0|$ divides 4, and since $q \geq 13$ we conclude that $k + j = 1$. It follows that $N_{\gamma}(T_0)/T_0$ has order at most $2n$. If $\hat{G}_0 \cong \text{O}(2n + 1, q)'$, we get the same conclusion, although here one of $M_i$’s has dimension 1 and $\hat{T}$ induces $Z_2$ on this factor.

Suppose $\hat{G}_0 \cong \text{SL}(n, q)$. Then $\hat{T}_1$ is cyclic of order $q^{m_1} - 1$, where $m_1 = \text{dim}(M_1)$. This forces $k \leq 2$ and if $k = 2$, then one of $M_1$ and $M_2$ has dimension 1. So here $N_{\gamma}(T_0)/T_0$ has order $n$ or $2n - 1$. Similarly, if $\hat{G}_0 \cong \text{SU}(n, q)$ we have $N_{\gamma}(T_0)/T_0$ of order at most $2n$. Thus, in all cases, $|N_{\gamma}(T_0)/T_0| \leq 2r$, where $r$ is the Lie rank of $\bar{G}$. This gives the inequality, $2r \geq \frac{1}{2}(m - 1)$, or $4r + 1 \geq m$. Also, $m \geq \frac{1}{4}(q - 1)r' \geq \frac{1}{4}(12r' \text{ (by (2.4))})$ and $d \leq r + 1$. Hence, $(4r + 1)(r + 1) \geq 12$, a contradiction. This shows that $G_0$ is not a classical group.

Let $G_0$ be an exceptional group and $\bar{G}$ of Lie rank $r$. Then $\frac{1}{2}(m - 1)$ divides $|\bar{W}|$, while $m \geq \frac{1}{4}(q - 1)r' \geq \frac{1}{3}12r'$. Considering the possibilities for $|\bar{W}|$ we obtain a contradiction.

(10.13) $X_1$ is not a group of Lie type in any characteristic.
Proof. Suppose that $X$ is of Lie type and defined over $F$, where $q_0 = r^a$ and $r$ is prime. By (10.10)(iii), $r \neq p$. Minimality of $|Y|$ and (10.9) shows that $T_0$ stabilizes no subgroup $J < X$ with $E(J) \neq 1$.

Suppose $r$ divides $|T_0 \cap X|$. We first claim that $T_0$ is an $r$-group. From (3.9) of [4] it follows that there is a canonical parabolic subgroup, $D$ of $X$, with $O_r(T_0 \cap X) \leq O_r(D)$. Then $T_0 \leq N(D)$ and minimality of $|Y|$ implies that (10.1) holds for $DT_0$. Since $O_p(DT_0) \leq O_p(D) = 1$ and since $E(DT_0)_p \leq E(D)_p = 1$, we must have $T_0 \leq DT_0$. In particular, $[O_r(T_0), O_r(D)] \leq O_r(T_0) \cap O_r(D) = 1$. Then $[D, O_r(T_0)] \leq D \cap C(O_r(D)) \cap O_r(T_0) = 1$. In particular, $O_r(T_0)$ centralizes a Borel subgroup of $X$, and checking $Aut(X)$ we see that $O_r(T_0) = 1$. This proves the claim.

Let $U \in Syl_r(X)$ with $T_0 \leq N(U)$. Then $N_Y(U)T_0$ is solvable and, as above, $T_0 \leq N_Y(U)T_0$. Suppose $X = X_1 \times X_2$. Then $r = 2$. Let $t \in T_0 - N(X_1)$ and $a_1 \in N_{X_1}(U \cap X_1)$. Then $a_1^{-1}a_1^t = [a_1, t] \in T_0 = O_r(T_0)$, from which it follows that $q_0 = 2$. Also, $a_1^{-1}a_1^t \in T_0$ and $T_0$ abelian implies $U \cap X_1$ is abelian. But then, $X_1 \simeq SL(2, 2)$, a contradiction. Therefore, $X$ is simple. Since $T_0 \leq N_Y(U)T_0$, $[N_Y(U), T_0, T_0] = 1$, and consideration of $T_0 = O_r(T_0) \leq Aut(X)$ yields $T_0 \leq U$. Let $D_1$ be any proper parabolic subgroup of $X$ with $U \leq D_1$. Then minimality of $|Y|$ yields $T_0 \leq D_1$. If $X \neq PSL(2, q_0)$, then letting $D_1$ vary, we conclude $T_0 \leq X$, a contradiction. Therefore, $X \simeq PSL(2, q_0)$, and $N_Y(U) \leq N(T_0)$ forces $|T_0| = q_0$. Also, $Y = X$ and $N_Y(T_0)/T_0$ is cyclic of order $q_0 - 1/(2, q_0 - 1)$. At this point we have the same situation that existed at the end of the proof of (10.12) (set $m = q_0$ but allow for the fact that $T_0$ may not have prime order) and this led to a numerical contradiction. We conclude that $T_0 \cap X$ is an $r'$-group.

Suppose $1 \neq t \in T_0 \cap X$ and consider $C = C_X(t)$. Let $t \in I$, a maximal torus of $X$. By (2.9), there are commuting groups of Lie type, $D_1, \ldots, D_l$, over extension fields of $F_{q_0}$ such that $D_1 \cdots D_lI$ is normal in $C$ with quotient isomorphic to a subgroup of the center of the universal cover of $Y$. If $E(C) \neq 1$, we contradict the minimality of $|Y|$. Hence, $E(C) = 1$. Consequently, either $D_1 \cdots D_lI = I$ or $q_0 = 2$ or $3$ and $D_i \cong SL(2, q_0)$, $PSL(2, q_0)$, $SU(3, 2)$, or $PSU(3, 2)$, for $i = 1, \ldots, l$.

Suppose one of the latter cases occurs and set $J = [D_1 \cdots D_l, T_0]$. Then $J \leq D_1 \cdots D_l \cap T_0 \leq O_r(D_1 \cdots D_l)$ (as $T_0$ is an $r'$-group). Also, $J \leq D_1 \cdots D_lT_0$. Since $T_0$ is abelian, $J \cap D_i \leq Z(D_i)$ for any $i$ with $D_i \cong SL(2, 3)$ or $SU(3, 2)$. For such an $i$, $[T_0, D_i, D_i] \leq [Z(D_i), D_i] = 1$, and since $D_i$ is generated by $r$-elements, $[T_0, D_i] = 1$. But then $D_i \leq C_{GO}(T_0) = T_0$, whereas $T_0$ is an $r'$-group. We conclude that $D_i \cong PSL(2, q_0)$ or $PSU(3, 2)$ for $i = 1, \ldots, l$. Normality of $J$ in $D_1 \cdots D_lT_0$ and the
previous commutator argument shows that \( J = O_r(D_1 \cdots D_t) \). In particular, there are root subgroups \( A_1 \approx A_2 \approx Z_{q_0} \) of \( Y \) such that \( \text{PSL}(2, q_0) \approx \langle A_1, A_2 \rangle \leq C \) and \( E = O_r(\langle A_1, A_2 \rangle) \leq T_0 \). Let \( t_1 \) be a generator of \( E \). As above, \( E(C_X(t_1)) = 1 \). This implies that \( Y \) has Lie rank at most 2. Since \( X \leq G_0, n \geq 2 \), and \( | T_0 | \geq \frac{1}{2} 12^2 \geq 48 \) (see (2.4)). As \( T_0 \) is an abelian \( r' \)-subgroup of \( C_Y(t) \) we obtain a numerical contradiction.

We now have \( D_1 \cdots D_t I = I \leq C \). Let \( m \) be the Lie rank of the overlying algebraic group of \( X_1 \). Then \( | C | \) is bounded by the order of a maximal torus of the universal cover of \( X \), so (2.4) implies that \( | T_0 \cap X | \leq (q_0 + 1)^m \) (replace \( q_0 \) by \( q_1 = \sqrt{q_0} \) for the Suzuki and Ree groups). Here we use the fact that if \( X = X_1 \times X_2 \), then \( T_0 \cap X_1 \cap T_0 \cap X_2 = 1 \). Regarding \( T_0/T_0 \cap X \) as an abelian subgroup of \( \text{Out}(X) \) we have \( | T_0 | \leq (q_0 + 1)^{m+2} \) (again we replace \( q_0 \) by \( q_1 = \sqrt{q_0} \) in the Suzuki and Ree cases). This inequality also holds in case \( T_0 \cap X = 1 \).

We cannot have \( X_1 \approx \text{Sz}(q), L_3(4) \), or \( U_4(3) \). For the first two this follows since \( T_0 \) is abelian of order at least 48. If \( X_1 \approx U_4(3) \), use (5.16) of [23] together with the existence of an extraspecial 3-group in \( X_1 \) of order 3^5 to conclude that \( n \geq 4 \). Then \( | T_0 | \geq \frac{1}{2} 12^4 \), contradicting the above inequality.

Suppose \( M \) is a faithful module in characteristic \( p \) for a covering group of \( G_0 \). Using the main theorem of [18], the containment \( X < G_0 \), and the above paragraph, we can obtain lower bounds on \( \text{dim}(M) \). Excluding the Suzuki and Ree groups, we then have \( \text{dim}(M) \geq \frac{1}{2}(q_0^m - 1) \) (in most cases this is too low, but for the symplectic groups in odd characteristic, it is exact). For \( X \approx \text{Sz}(q_0), 2G_2(q_0), 2F_4(q_0) \) we have \( \text{dim}(M) \geq (q_0/2)^{1/2}(q_0 - 1), q_0(q_0 - 1), (q_0/2)^{1/2}q_0^4(q_0 - 1) \), respectively. In what follows we use these bounds on \( \text{dim}(M) \) to obtain contradictory inequalities involving \( | T_0 | \). The contradiction is most easily obtained for the three exceptional cases, although they must be considered individually. We, therefore, leave these cases to the reader and present a treatment of the remaining cases.

First, suppose \( G_0 \) to be a classical group and let \( M \) be the usual module for the corresponding linear group. Then \( \text{dim}(M) \leq 2n + 1 \). By the previous paragraph, \( 2n + 1 \geq \frac{1}{2}(q_0^m - 1) \), so \( n > \frac{1}{4}(q_0^m - 1) \). We then have the inequalities \( (q_0 + 1)^{m+2} \geq | T_0 | \geq \frac{1}{2}(q - 1)^n \geq \frac{1}{2}(q - 1)^{n-1} > \frac{1}{2}(12^{(1/4)}q_0^n)^2 \). This yields \( 288(q_0 + 1)^{m+2} > 12^{(1/4)}q_0^n \). When \( m \geq 2 \) and \( q_0 \geq 5 \) this is impossible. Moreover, if \( q_0 = 3 \) or 4, then \( m \leq 2 \), while if \( q_0 = 2 \), then \( m \leq 4 \). Suppose \( m = 1 \). Then the inequality forces \( q_0 \leq 25 \). Considering subgroups of \( \text{PSL}(2, q) \), we see that \( n \geq 2 \). But then \( | T_0 | \geq \frac{1}{2} 12^n \geq 48 \), contradicting \( T_0 \leq \text{Aut}(X) \). Therefore, \( m \geq 2 \) and it follows that \( q_0 = 2, 3, \) or 4. Also, \( m = 2 \) in the latter cases.
We treat these cases separately, using the inequality (*) 
\((q_0 + 1)^{m+2} \geq |T_0| \geq \frac{1}{2}(q - 1)^n \geq \frac{1}{2}12^n\). Suppose \(q_0 = 4\). Then \(2n + 1 \geq \frac{1}{2}(q_0^m - 1) > 7\), so \(n \geq 4\), which contradicts (*). Suppose \(q_0 = 3\). Then \(m = 2\) and \(X\) is simple. For the moment, exclude the case \(X \cong \text{PSp}(4, 3)\). The result in [18] then gives \(2n + 1 \geq \dim(M) \geq 6\). So \(n \geq 3\) and we contradict (*). If \(X \cong \text{PSp}(4, 3)\), then \(X\) contains the split extension of an elementary abelian 3-group of order 3^3 and \(S_3\) and \(O_3(X)\) has class 3. However, \(G_0\) is a classical group with \(n \geq 2\), and one checks that this forces \(n \geq 3\). This is a contradiction, leaving only the case \(q_0 = 2\). Since \(T_0\) is an \(r'\)-group we can improve the earlier bound to get \(|T_0| \leq (q_0 + 1)^{m+1} = 3^{m+1}\). If \(m = 4\), the bound \(2n + 1 \geq \frac{1}{2}(2^m - 1)\) forces \(n \geq 3\), whence \(|T_0| \geq \frac{1}{2}12^3\), contradicting the above. If \(m = 2, 3\), then \(n \geq 2\) and we again have a contradiction unless \(X \cong \text{PSU}(4, 2)\) and \(n = 2\). But \(\text{PSU}(4, 2) \cong \text{PSp}(4, 3)\) and we have already observed that this forces \(n \geq 3\).

At this point \(G_0\) is an exceptional group, and, except for the case \(G_0 \cong G_2(q)\), these cases are easier than the above. Suppose \(G_0 = G_2(q)\), so that \(G_0\) has a 7-dimensional representation in characteristic \(p\). We then have the inequality \(7 = \dim(M) \geq \frac{1}{2}(q_0^m - 1)\), so \(q_0^m \leq 13\). If \(q_0 = 2\), then as above, \(|T_0| \leq (q_0 + 1)^{m+1}\) which forces \(m \geq 4\) (as \(d = 1\)), a contradiction. If \(q_0 = 3\), then (*) forces \(m = 2\) (\(X\) would be solvable if \(m = 1\)), so \(X \cong \text{PSL}(3, 3)\), \(\text{PSp}(4, 3)\), or \(\text{PSU}(3, 3)\). But then we can improve the earlier bound on \(|T_0|\) obtaining \(|T_0| \leq 2(q_0 + 1)^{m+1}\), which contradicts (*). Therefore, \(q_0 \geq 4, m = 1,\) and \(X \cong \text{PSL}(2, q_0)\). This contradicts (*) (as \(q_0 \leq 13\)). So \(G_0 \neq G_2(q)\).

For the other exceptional groups argue as follows. In each case \(X\) acts on a module \(M\) of dimension 27 if \(G_0 \cong F_4(q)\), \(E_6(q)\), or \(2E_6(q)\), dimension 56 if \(G_0 \cong E_7(q)\), and dimension 248 if \(G_0 = E_8(q)\). These bounds give easy contradictions. Details are omitted, but we illustrate with the case \(G_0 = F_4(q)\). Here \(\dim(M) = 27 \geq \frac{1}{2}(q_0^m - 1)\), while \(n = 4\) and \(d = 1\). As above we obtain a contradiction. This completes the proof of (10.13).

11. Classification (continued). In this section we complete the proof of (10.1). In view of the classification theorem and (10.11)–(10.13) we have \(X = F^*(Y)\) a sporadic simple group or the direct product of two sporadic groups interchanged by an element of \(T_0\). Our method is to first show that \(T_0\) is T.I. set in \(Y\), of odd order, and to use this together with properties of the individual groups to obtain a contradiction. An effort has been made to keep the number of special properties to a minimum, avoiding an extensive list of references. For the most part we only need the orders of the sporadic groups (Table (11.1), below) and the structure
of centralizers of involutions (available in Table 1 of [2]). For certain
groups we do appeal to the literature for additional information. A
somewhat shorter proof could be obtained by citing a much larger number
of references, but this did not seem worthwhile.

Let \( n \) be the Lie rank of \( G \). The containment \( X \leq G_0 \) certainly forces
\( n \geq 2 \), whence \( |T_0| \geq \frac{1}{d}(q - 1)^n \geq 48 \) (as \( d = n + 1 \)). In the following
table we list the sporadic simple groups and their orders.

(11.2). If \( t \in T_0 \) is an involution, then \( T_0 \leq O_2(C_Y(t)) \) and \( C_Y(t) \) is
2-constrained.

Proof. The possibilities for \( C_X(t) \) are presented in Table 1 of [2] and
in [17] for \( X = J_4 \). Suppose \( C_X(t) \) is not 2-constrained. We have \( T_0 \leq N(E(C_X(t))) \), so minimality of \( |Y| \) implies (10.1) holds for \( E(C_X(t))T_0 \).
The only possibility is \( X \cong J_3 \) or \( J_3 \times J_3 \), with \( t \) inducing an outer
automorphism on each component of \( X \). Hence, \( C_X(t) \cong PSL(2, 17) \) or
\( PSL(2, 17) \times PSL(2, 17) \), and \( q = 17 \). Also, \( T_0 \cap X \leq C_X(t) \), \( T_0 \cap X \intersects each component trivially if \( X \) is not simple, and \( |T_0 \cap X| \geq 12 \) (24 if
\( X \) is simple). Since \( T_0 \) is a \( p' \)-group, this is impossible. Therefore, \( C_X(t) \) is
2-constrained. From (10.1) and the minimality of \( |Y| \) we conclude \( C_Y(t) \)
\( \leq N(T_0) \), which forces \( T_0 \leq O_2(C_Y(t)) \).

\[ \begin{array}{|c|c|}
\hline
X & |X| \\
\hline
M_{11} & 2^4 \cdot 3^2 \cdot 5 \cdot 11 \\
M_{12} & 2^6 \cdot 3^3 \cdot 5 \cdot 11 \\
M_{22} & 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \\
M_{23} & 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \\
M_{24} & 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \\
J_1 & 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \\
J_2 & 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \\
J_3 & 2^7 \cdot 3^3 \cdot 5 \cdot 17 \cdot 19 \\
J_4 & 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43 \\
HS & 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \\
Mc & 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \\
Suz & 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \\
He & 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17 \\
\hline
\end{array} \]

\[ \begin{array}{|c|c|}
\hline
X & |X| \\
\hline
ON & 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 13 \\
Co_1 & 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 \\
Co_2 & 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 \\
Co_3 & 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23 \\
F_{22} & 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23 \\
F_{23} & 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \\
F_{24} & 2^{21} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \\
Ly & 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67 \\
F_1 & 2^{46} \cdot 3^2 \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \\
F_2 & 2^{41} \cdot 3^13 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \\
F_3 & 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31 \\
F_5 & 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19 \\
Ru & 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29 \\
\hline
\end{array} \]
(11.3) (i) $|T_0|$ is odd.
(ii) $X$ is simple.

Proof. Suppose $|T_0|$ is even and apply (11.2) to obtain $T_0 \leq O_2(C_Y(t))$ and $C_Y(t)$ 2-constrained for each involution $t \in T_0$. Suppose $X = X_1 \times X_2$, with $X_2 = X'_1$, for some $y \in T_0$. The previous remarks show that each involution, $t$, of $T_0$ normalizes $X_1$ and $X_2$, and since $y \in O_2(C_Y(t))$ we necessarily have $C_X(t)$ a 2-group (consider $[a_1, y]$ for $a_1 \in C_X(t)$ of odd order). However, Table 1 of [2] and [17] show this to be impossible. Therefore (ii) holds.

We know that $T_0$ is a 2-group of order at least 48, and this forces $|T_0| \geq 2^6$. By Table 1 of [2] and [17] the only sporadic groups $X$ having an involution $t$ such that $O_2(C_Y(t))$ contains an abelian subgroup of order $2^6$ are $R_u$, $Co_1$, $Co_2$, $F_{22}$, $F_{23}$, $F_2$, $J_4$, or $F_4$. Moreover, in each case $O_2(C_Y(t))$ has exponent at most 4.

Since $|T_0| \geq 2^6$, $T_0$ has rank at least 3. From (2.1)(iii) we conclude $n \geq 3$. Hence, $|T_0| \geq \frac{1}{2}(q - 1)^n \geq \frac{1}{4} \cdot 12^3$ (recall, $d \leq n + 1$), and since $T_0$ is a 2-group, $|T_0| \geq 2^9$. Repeat the argument. Namely, $T_0$ of exponent at most 4 implies $T_0$ of rank at least 5. Therefore, $n \geq 5$ and $|T_0| \geq \frac{1}{6} \cdot 12^5 > 2^{15}$. Eventually, we obtain $|T_0| > |X|$, a contradiction.

(11.4) (i) $X = Y$.
(ii) $N_X(T_0)$ is the unique maximal subgroup of $Y$ containing $T_0$.
(iii) $T_0$ is a T.I. set in $X = Y$.

Proof. $X = Y$ by (11.3) and Table 1 of [2]. Suppose (ii) holds. If $T_0$ is not a T.I. set in $Y$, then there exists $y \in Y - N(T_0)$ with $1 \neq T_0 \cap T_0^y$. Choose $1 \neq t \in T_0 \cap T_0^y$, so that $T_0$, $T_0^y \leq C_Y(t)$. By (ii), $C_Y(t) \leq N(T_0)$, so (6.3) forces the contradiction $T_0 = T_0^y$. Therefore, it will suffice to prove (ii).

Let $T_0 \leq M < X$. We must show $N_X(T_0) \geq M$. Suppose false, so that minimality of $|Y|$ implies that (10.1) holds for $M$. Therefore, either $O_p(M) \neq 1$ or $E(M)_p \neq 1$. Suppose $E(M)_p \neq 1$ and let $D$ be a component of $E(M)_p$. By (7.1) $T_0 \cap D$ is a maximal torus of $D$, hence $T_0 \leq N(D)$. Then (7.2) shows that $T_2 = C_{T_0D}(T_1)$ is a maximal torus of $G_0$, where $T_1$ is a Cartan subgroup of $D$. Let $V$ be a Sylow $p$-subgroup of $D$ normalized by $T_2$. Then $V$ is a product of $T_2$-root subgroups of $G_0$ (use (9.1)), each of which has order a power of $q$. If $D \cong L_2(q^a)$, for some integer $a$, then $T_1$ is necessarily of even order. But then $|T_2|$ is even, and applying (11.3) to...
rather than to \( T_0 \), we have a contradiction. So \( D \cong L_2(q^a) \) and \( \frac{1}{2}(q^a - 1) \) is odd.

Suppose \( |V| = q \) and let \( T_2 = C_{G}(T_2)^0 \), a maximal torus of \( G \). There is a \( T_2 \)-root subgroup \( V \) of \( G \) such that \( V = \overline{V} \). Then \( D \) is generated by \( V \) and the opposite \( T_2 \)-root subgroup of \( G_0 \), and it follows that either \( D \cong \text{SL}(2, q) \) or \( G_0 \cong \text{PSp}(4, q) \) and \( V \) is a root subgroup for a short root. The first case contradicts \( |T_2| \) odd. In the second case, consideration of \( N_{G_0}(V) \) shows that \( |T_2| = \frac{1}{2}(q - 1)^2 \) or \( \frac{1}{2}(q^2 - 1) \), either way we again contradict \( |T_2| \) odd. Therefore, \( a > 1 \), and since \( \frac{1}{2}(q^a - 1) \) is odd, so is \( a \). Thus, \( q^3 \mid |V| \). Also, \( q \equiv -1 \text{ (mod 4)} \) forces \( q \) to be an odd power of \( p \) and \( p \equiv -1 \text{ (mod 4)} \). From Table (11.1) we easily rule out all possibilities for \( X \). Hence, \( E_{p}(M) = 1 \).

We then have \( O_{p}(M) \neq 1 \). Let \( J \) be a minimal \( T_0 \)-root subgroup contained in \( O_{p}(M) \). Then \( |J| = q^a \) for some \( a \geq 1 \). If \( a = 1 \), argue as in the above paragraph and contradict the fact that \( |T_0| \) is odd. Hence \( a \geq 2 \).

Suppose \( a = 2 \). Then \( J = X_i \) for some \( i \in \{1, \ldots, v\} \) and \( \bar{X}_i = U_\alpha \times U_\beta \) for \( \alpha, \beta \) roots of the same length. Let \( \overline{D} = \langle U_{\pm \alpha}, U_{\pm \beta} \rangle \). Then \( D \) is a \( \overline{G} \)-invariant rank two subgroup of \( G \), so \( D = O^\sigma(D_{a}) \) is a perfect central extension of one of the groups \( \text{PSL}(2, q^2) \), \( \text{PSL}(3, q) \), \( \text{PSU}(3, q) \), or \( \text{PSp}(4, q) \). By (7.1), \( T_0 \cap D \) is a maximal torus of \( D \). Also, \( T_0 \leq N(J) \), so \( T_0 \cap D \) is contained in a proper parabolic subgroup of \( D \). However, one checks that this forces \( |T_0 \cap D| \) even, a contradiction. Therefore, \( a \geq 3 \).

From Table (11.1) we conclude that \( X \) is one of the groups \( \text{Ly}, \text{F}_5, \text{F}_2, \) or \( F_1 \). In the first two cases, \( J \) is necessarily of order \( 5^6 \) and a Sylow \( 5 \)-subgroup of \( X \). However, \( J \) is elementary abelian, while \( \text{Ly} \) contains \( \text{G}_2(5) \) (Lyons [19]) and \( F_5 \) contains a HS section (Table 1 of [2]), which contains a \( U_3(5) \) section. This is impossible, forcing \( X \cong F_2 \) or \( F_1 \). At this point we appeal to (11.8) (which is proved independently of (11.4)) to obtain a contradiction. This completes the proof of (11.4).

(11.5) \( G_0 \) has a faithful irreducible projective module \( M \) over a field of characteristic \( p \) and satisfying

(i) \( \dim(M) \leq 2n + 1 \) if \( G_0 \) is a classical group;
(ii) \( \dim(M) \leq 8 \) if \( G_0 \cong G_2(q) \);
(iii) \( \dim(M) \leq 27 \) if \( G_0 \cong F_4(q), E_6(q), \) or \( 2E_6(q) \);
(iv) \( \dim(M) \leq 56 \) if \( G_0 \cong E_7(q) \); and
(v) \( \dim(M) \leq 248 \) if \( G_0 \cong E_8(q) \).
Proof. If $G_0$ is a classical group use the natural module associated with the corresponding linear group. If $G_0 = G_2(q)$, the containment $G_2(q) \leq D_4(q)$ implies the result. Each of the groups in (iii) is contained in $E_6(K)$. To obtain the module $M$, consider the group $E_7(K)$ and let $P$ be the standard parabolic subgroup whose derived group involves $E_6(K)$. Then $M = R_u(P)$ can be viewed as a 27-dimensional module for the Levi factor of $P$. A similar procedure establishes (iv), while in (v) we let $M$ be the Lie algebra of $G = E_8(K)$.

\[(11.6)\] (i) $|T_0| \geq \frac{1}{2} (q - 1)^n \geq \frac{1}{14} \cdot 12^n$.

(ii) If $n \geq k$, then $|T_0| \geq 1/(k + 1) \cdot 12^k$.

Proof. By (2.3) $|T_0| \geq \frac{1}{2} (q - 1)^n$. As $d \leq q + 1$ and $q \geq 13$, we have $\frac{1}{2} (q - 1)^n \geq \frac{1}{2} \frac{(q - 1)/(q + 1))(q - 1)^{n-1}}{12} \geq \frac{1}{2} \frac{12^{n-1}}{12^n} = \frac{1}{14} \cdot 12^n$, which proves (i). To prove (ii) use the inequality $|T_0| \geq \frac{1}{2} (q - 1)^n \geq \frac{1}{2} (n + 1) 12^n$ and repeated use of the inequality $(1/(l + 1))(q - 1)^l \geq \frac{1}{2} (q - 1)^{l-1}$.

\[(11.7)\] $X$ is not a Mathieu group.

Proof. Suppose $X$ is a Mathieu group. Regard $X$ as a subgroup of $M_{24}$, acting on a set $\Omega$ of size 24. Let $\Delta$ be an orbit of $T_0$ of maximal length. Fix $\alpha \in \Delta$ and set $T_1 = (T_0)_\alpha$. By (11.3) $|\Delta|$ is odd and since $T_0$ is abelian, $T_1$ fixes each point in $\Delta$. From $|T_0| \geq 48$ we conclude $T_1 \neq 1$, and since $|\Delta| \geq 3$, $T_1 \leq M_{21} \leq \text{Aut}(\text{PSL}(3,4))$.

If $|\Delta| = 3$, then $|T_1| \geq 16$ forces $7 || T_1 |$, whence $T_0$ has an orbit of size a multiple of 7. This contradicts the choice of $\Delta$. Therefore $|\Delta| \geq 5$ and $T_1$ must be contained in a Cartan subgroup of $\text{Aut}(\text{PSL}(3,4)) \geq M_{21}$. Then $|T_1|$ divides 9 and $|\Delta| \geq 7$. This forces $|T_1| = 3$, whence $|\Delta| \geq 16$, impossible.

\[(11.8)\] $X \not\cong F_2$ or $F_1$.

Proof. If $X \cong F_2$ or $F_1$, then $X$ contains a covering of $^2E_6(2)$ (see Table 1 of [2]). By [18] any faithful projective representation of $^2E_6(2)$ in odd characteristic has degree at least $3 \cdot 2^9$. So (11.5) implies $G$ is a classical group and $n \geq \frac{1}{2} (3 \cdot 2^9 - 1)$. Combining this with (11.6) we have $|T_0| > |X|$, a contradiction.

\[(11.9)\] $X \not\cong Mc$ or $Co_3$. 

\[\]
Proof. Suppose $X \cong \text{Mc}$ or $\text{Co}_3$. By (11.4)(ii) $N_X(T_0)$ is a maximal subgroup of $X$, all such subgroups being determined in Finkelstein [11]. Now $X$ acts faithfully on the Lie algebra, $M$, of $G$. Also, $\hat{A}_8 \leq \text{Mc} \leq \text{Co}_3$ and $A_8 \cong \text{GL}(4,2)$. Decomposing $M$ into eigenspaces for the central involution of the $\hat{A}_8$ and using [18], we have $\dim(M) \geq 2 \cdot 7 = 14$. This implies $n \geq 4$ and so $|T_0| \geq \frac{1}{2}12^4$. The results in [11] yield a contradiction.

(11.10) $X \cong \text{HS}$.

Proof. Suppose $X \cong \text{HS}$. Then $X$ is 2-transitive of degree 176, with 1-point stabilizer $\text{PSU}(3,5)$ and 2-point stabilizer an extension of $\text{SL}(2,5)$ by $Z_6$. Say $X$ acts on $\Omega$ and $\Delta$ is a non-trivial orbit of $T_0$. For $a \in \Delta$, $\tau a$ stabilizes each point in $\Delta$, hence $T_1 \leq \text{SL}(2,5) \times Z_3$ and $|T_1| \leq 15$. In particular, $|T_0| \leq 15 \cdot 176$. If $p \neq 5$, then using the containment $\text{PSU}(3,5) < G_0$ together with [18] and (11.5) we have $n \geq 4$. Hence, $|T_0| \geq \frac{1}{2}(q - 1)^n \geq \frac{1}{2} \cdot 24^2 = 2^63$. Also, $T_0$ has order prime to 5, so $|T_1| \leq 3^2$ and $|\Delta| > 21$. The lower bound on $|T_0|$ implies that each orbit of $T_0$ is non-trivial, of odd length, and of order greater than 21, but dividing $|X|$. Table (11.1) shows this to be impossible.

(11.11) $X \cong \text{Ru}$ or $\text{Suz}$.

Proof. Suppose $X \cong \text{Ru}$ or $\text{Suz}$ then $X$ contains a subgroup $(A \times R)D$, where $Z_2 \times Z_2 \cong R \leq RD \cong A_4$, $D \leq N(A)$, and $A \cong \text{Sz}(8)$ or $\text{PSL}(3,4)$, respectively (this follows from Table 1 of [2]). Let $M$ be a nontrivial, projective, irreducible module for $G_0$ in characteristic $p$, and let $\hat{G}_0$ be the representation group for $M$.

Write $M = C_M(\hat{R}) \oplus [M, \hat{R}]$. We first show that $[\hat{A}, [M, \hat{R}]] \neq 0$. Suppose otherwise. Using (3.3) and (8.10)(i) of [2] we see that there exists $x \in \hat{X}$ with $\hat{R}^x \leq \hat{A}\hat{R}$ and $\hat{R}^x \cap \hat{A} \leq Z(\hat{X}) \geq \hat{R}^x \cap \hat{R}$. Then $\hat{A}\hat{R} = \hat{A}\hat{R}^x$. Since $\hat{A}$ is trivial on $[M, \hat{R}]$ we conclude $[M, \hat{R}^x] \geq [M, \hat{R}]$, hence equality holds. Therefore, $C_M(\hat{R}) = C_M(\hat{R}^x)$, whereas $\hat{R}^x$ induces a subgroup of $\hat{A}$ on $C_M(\hat{R})$ and, surely, $[\hat{A}, C_M(\hat{R})] \neq 0$. Therefore, $[\hat{A}, [M, \hat{R}]] \neq 0$, as asserted.

Suppose $X \cong \text{Ru}$. Then the main theorem of [18] implies $\dim(M) \geq \dim([M, \hat{R}]) \geq 8$ and $\dim(M) \geq 14$ in case $Z(\hat{X})$ has odd order. If $\hat{R}$ is not abelian, then Clifford’s theorem and Schur’s lemma imply that $M$ contains the direct sum of two faithful $K[\hat{A}]$-composition factors, hence $\dim(M) \geq 16$. If $\hat{R}$ is abelian, we can write $[M, \hat{R}] \geq M_1 \oplus M_2 \oplus M_3$, the
sum of three faithful irreducibles for $\hat{A}$, permuted transitively by $\hat{D}$. Here $\dim(M) \geq 24$ and $\dim(M) \geq 42$ if $Z(\hat{X})$ has odd order. Apply (11.5) and note that in (11.5)(iii) the module corresponded to a 3-fold cover of $E_6(K)$. We conclude that $n \geq 7$ and if $n = 7$, then $G_0 \cong E_7(q)$ and $d = 2$. Therefore, $|T_0| \geq \frac{1}{2}12^7$, and this contradicts (11.3) and (11.4)(i).

Now suppose $X \cong \text{Suz}$ and let $M$ be the Lie algebra of $G$. Here $X = \hat{X}$ acts on $M$ and $A$ does not centralize $[M, R]$. Also, $[M, R] = M_1 \oplus M_2 \oplus M_3$, where $M_1$, $M_2$, and $M_3$ are the fixed spaces of the involutions in $R$. The spaces $M_1$, $M_2$, and $M_3$ are left invariant by $A$ and permuted transitively by $D$. It follows from [18] that $\dim(M) \geq 3 \cdot \dim(M_1) \geq 45$. If $G = E_6(K)$, we have a contradiction by replacing $M$ by the module in (10.5)(iii) (a module for the 3-fold cover of $E_6(K)$). Hence $n \geq 5$. Since $|T_0|$ is odd equality holds only if $G_0 = \text{PSp}(10, q)$ or $\text{PSO}^-(10, q')$ with $|T_0| \geq \frac{1}{2}(q - 1)^6$. The only possibilities are $|T_0| = \frac{1}{2}(q^5 + 1)$ with $q = 13$ or 17, or $n = 6$ and $G_0 = \text{PSU}(7, 13)$. In each case we have a numerical contradiction.

(11.12) $X \cong J_1$.

Proof. We use (11.4)(iii) to conclude $|T_0|^2 \leq |X|$. If $n \geq 3$, then $|T_0| > \frac{1}{4}12^3$ and this is impossible. Thus, $n = 2$. A Sylow 2-normalizer of $X$ contains a Frobenius group of order $2^37$. Therefore, any projective irreducible for $X$ in odd characteristic has dimension at least 7. By (11.5) $G \cong G_2(K)$, hence $d = 1$ and $|T_0| \geq (q - 1)^2$. As $|T_0|^2 \leq X$, this forces $q \leq 19$. As $|T_0|$ is odd, $|T_0| = \Phi_3(q)$ or $\Phi_6(q)$, and one checks that $|T_0| \mid |J_1|$.

(11.13) $X \cong J_2$ or $J_3$.

Proof. Suppose $X \cong J_2$ or $J_3$ and let $t$ be a 2-central involution in $X$. By Table 1 of [2] we have $C_X(t)$ an extension of an extraspecial group of order $2^5$ by $A_5$.

We first claim $n \geq 3$. To see this let $M$ be a faithful $K[X]$-module, and write $M = [M, t] \oplus C_M(t)$. As $M$ is faithful each of the factors is non-trivial, and they are both $C_X(t)$ invariant. Since $O_2(C_X(t))$ is extraspecial, we have $\dim([M, t]) \geq 4$. Now consider $C_M(t)$. Involutions in $O_2(C_X(t))$ are non-trivial on $C_M(t)$ and it is easy to see that such involutions are conjugates of $t$ (use Table 1 of [2]). Thus, $O_2(C_X(t))$ acts in a non-trivial manner on $C_M(t)$ and it follows from Clifford's theorem that $\dim(C_M(t)) \geq 5$. Consequently, $\dim(M) \geq 9$. Now $\text{SL}(3, K)$ acts on its
Lie algebra of dimension 8; $G_2(K)$ acts in 8 dimensions; while $\text{PSp}(4, K) \cong O(5, K)'$ acts in 5 dimensions. This proves the claim, and so $|T_0| \geq \frac{1}{4} 12^3$.

From Sylow’s theorem, we have the Sylow 7-subgroups of $J_2$ self-centralizing while the Sylow $r$-subgroups of $J_3$ are self-centralizing for $r = 17, 19$. Since $|T_0|$ is odd, the above inequality shows $|T_0| = 3^3 \cdot 5^2$ or $3^3 \cdot 5$, according to $X \cong J_2$ or $J_3$. However, (11.4) implies that $|X : N_X(T_0)| \equiv 1 \pmod{|T_0|}$ and this is impossible.

(11.14) $X \not\cong J_4$.

Proof. Suppose $X \cong J_4$. We refer the reader to Janko [17] for properties of $J_4$. If $t$ is a 2-central involution then $O_2(C_X(t))$ is extraspecial of order $2^{13}$. So if $M$ is the Lie algebra of $\bar{G}$ we have $\dim(M) \geq 2^6$. Thus $n \geq 6$ and $|T_0| \geq \frac{1}{2} 12^6$.

If $P \in \text{Syl}_2(X)$, then by [17] $P$ is self-centralizing for $r = 23, 29, 31, 37$, and 43. The Sylow 11-subgroups of $X$ are non-abelian, so by Table 1 we have $|T_0|$ dividing $3^3 \cdot 5 \cdot 7 \cdot 11^2$. This contradicts the inequality above.

(11.15) $X \not\cong \text{He}$.

Proof. Suppose $X \cong \text{He}$. From Table 1 of [2] it follows that $X$ contains a klein group $R$ such that $E(C_X(R))$ is a covering group of $\text{PSL}(3, 4)$ with center $R$, and $N_X(R)$ is transitive on $R$. An easy argument using [18] shows that a faithful projective $K(E(C_X(R)))$-module has dimension at least $4 \cdot 3 = 12$. So (11.5) yields $n \geq 4$, hence $|T_0| \geq \frac{1}{2} \cdot 12^4$.

Since $N_X(R)$ has a section isomorphic to $\text{PGL}(3, 4)$, $X$ has non-abelian Sylow 3-subgroups. Therefore, $3^3 \nmid |T_0|$. The Sylow 17-subgroups of $X$ are self-centralizing, so $|T_0|$ is prime to 17. From Table (11.1) and the above inequality we conclude that $O_5(T_0) \subseteq \text{Syl}_5(X)$ or $O_5(T_0) \subseteq \text{Syl}_7(X)$. Since 5 and 7 are divisors of $|C_X(R)|$, we may assume (by (11.4)(iii)) that $R \leq N(T_0)$. But then $T_0 = \langle C_{T_0}(r) | 1 \neq r \in R \rangle$, whereas $C_{T_0}(r) \leq N(R)$ for each $r \in R^\*$ and $|N_X(R)|$ is not divisible by $|T_0|$. This is a contradiction.

(11.16) $X \not\cong \text{Ly}$.

Proof. Suppose $X \cong \text{Ly}$. Let $M$ be a module as given in (11.5). If $p \neq 5$, then the containment $G_2(5) \leq X$ (see Lyons [19]) and [18] implies $\dim(M) \geq 120$. Hence $n \geq 8$, equality possible only if $\bar{G} = E_8(K)$ and $d = 1$. It follows that $|T_0| \geq 12^8$. But (11.4)(iii) implies $|T_0|^2 \leq |X|$, and this is impossible. Therefore, $p = 5$ and $q \geq 25$. 

Since \( p = 5 \), \( |T_0| \) is prime to 5. As \( |T_0| \geq 48 \), and Sylow \( r \)-subgroups of \( X \) are self-centralizing for \( r = 31, 37, 67 \) (see (3.3) of [19] or use Sylow's theorem), we conclude \( |T_0| \) divides \( 3^7 \cdot 7 \cdot 11 \). By [19] there is an involution \( j \in X \) with \( C_X(t) \approx \hat{A}_{11} \), the covering group of \( A_{11} \). We may regard \( C_X(t) \) as acting on \( M \), and write \( M = M_1 \oplus M_2 \), where \( M_1 = C_X(t) \) and \( M_2 = [M, t] \). Each of \( M_1 \) and \( M_2 \) is non-trivial and considering the action of a Frobenius subgroup of \( C_X(t) \) of order 55 we have \( \dim(M) \geq 5 \) for \( i = 1, 2 \). Hence \( \dim(M) \geq 10 \), so (11.5) implies \( n \geq 4 \), with equality only if \( \overline{G} = F_4(K) \) and \( d = 2 \). Therefore, \( |T_0| \geq \frac{1}{2} 24^4 = 3^4 \cdot 2^{11} \). From the previous divisibility condition we have \( |T_0| = 3^7 \cdot 7 \cdot 11 \), whereas \( X \) does not have abelian Sylow 3-subgroups. This is a contradiction.

(11.17) \( X \cong \text{ON} \).

**Proof.** Suppose false. We quote O'Nan [20] for the following fact about \( X \). Namely, a Sylow 3-subgroup, \( A \), of \( X \) is elementary abelian of order \( 3^4 \), its normalizer being an extension of \( Q_8 \circ D_8 \) by \( D_{10} \). It follows that each orbit of \( N(A) \) on \( A^* \) has size a multiple of 40. Let \( \hat{X} \) be the covering group of \( X \) that acts on \( M \). If \( \hat{A} \) is abelian, then the above and Clifford's theorem implies \( \dim(M) \geq 40 \). Thus, \( n \geq 7 \) and \( |T_0| \geq \frac{1}{8} \cdot 12^7 \). Otherwise, \( \hat{A} \) contains an extraspecial subgroup of order \( 3^5 \). Here, \( \dim(M) \geq 9 \) and \( 3 | |Z(\hat{G}_0)| \). The proof of (11.5), shows \( n = 6 \) with equality only if \( \overline{G} = E_6(K) \). Hence, \( |T_0| \geq \frac{1}{2} \cdot 12^6 \). But (11.4)(iii) implies \( |T_0|^2 \leq |X| \), and this contradicts Table (11.1).

(11.18) \( X \cong F_{22}, F_{23}, \) or \( F_{24}' \).

**Proof.** Suppose \( X \cong F_{22}, F_{23}, \) or \( F_{24}' \). Let \( M \) be a module as in (11.5). We first obtain lower bounds for \( \dim(M) \). Suppose \( X \cong F_{22} \) or \( F_{23} \). By Table 1 of [2] there is an involution \( j \in X \) such that \( C_X(j) = C_X(j)' \) and \( C_X(j)/\langle j \rangle \approx U_6(2) \) or \( F_{22} \), respectively. Let \( \hat{D} \) be the derived group of the preimage of \( C_X(j) \) in a covering group of \( X \) that affords \( M \). By Griess [14], [15] we see that \( \hat{D} \) contains an involution \( \hat{j} \) such that \( \hat{j}Z(\hat{D}) = j \) (in fact \( \hat{D} \cong C_X(j) \) if \( X \cong F_{23} \)).

Write \( M = C_M(\hat{j}) \oplus [M, \hat{j}] \). Clearly \( [M, \hat{j}] \) affords a non-trivial module for \( \hat{D} \). This is also true of \( C_M(\hat{j}) \). For if \( \hat{D} \) is trivial on \( C_M(\hat{j}) \), then choosing \( j \neq j^x \in C_X(j) \) we find that \( j^x \) can be chosen so that \( j^x = \hat{j} \), and this is impossible. If \( X \cong F_{22} \), then by [18] we have \( C_M(\hat{j}) \) and \( [M, \hat{j}] \) each of dimension at least 21, and so \( \dim(M) \geq 42 \). If \( X \cong F_{23} \), we have \( C_M(\hat{j}) \) and \( [M, \hat{j}] \) modules for \( \hat{D} \), each non-trivial as before. Hence \( \dim(M) \geq 84 \). This also holds if \( X \cong F_{24}' \), since \( F_{23} \leq F_{24}' \).
If \( X \cong F_{22} \), then by (11.5) \( n \geq 7 \), with equality possible only if \( \overline{G} = E_7(K) \). Therefore, \( |T_0| \geq \frac{1}{2} \cdot (12)^7 \). On the other hand, (11.4)(iii) implies \( |T_0|^2 \leq X \), and this is a contradiction.

Now suppose \( X \cong F_{23} \) or \( F_{24} \). If \( X \) is a classical group then by (11.5) \( n \geq \frac{1}{2} (84 - 1) > 41 \). Hence \( |T_0| > \frac{1}{42} \cdot (12)^{41} \), contradicting \( |T_0|^2 \leq |X| \). So \( X \) is not a classical group, which forces \( G = E_8(K) \). From Table (11.1) we have \( 23 \mid |X| \). However, 23 does not divide \( |E_8(q)| \) for \( 13 \leq q < 47 \) with \( q \neq 23 \) (one checks this by noting that neither \( \Phi_{11}(q) \) nor \( \Phi_{22}(q) \) divides \( |E_8(q)| \)). On the other hand 17 does not divide \( |E_8(23)| \). Therefore, \( |T_0| > \frac{1}{2} (46)^8 \), contradicting \( |T_0|^2 \leq |X| \).

(11.19) \( X \cong G_2 \).

Suppose \( X \cong G_2 \). Table 1 of [2] shows that some involution in \( X \) has centralizer containing an extension of an elementary abelian group, \( A \), of order \( 2^4 \) by \( GL(4,2) \) (natural action). It follows from Clifford's theorem that \( \dim(M) \geq 15 \), where \( M \) is as in (11.5) (an easy argument shows that we can regard \( A \) as acting on \( M \)). By (11.5), \( n \geq 7 \) if \( \overline{G} \) is a classical group; otherwise \( n \geq 4 \). Suppose \( \overline{G} \cong F_4(K) \). By Table 1 of [2] we see that \( X \) contains an elementary abelian group, \( E \), of order \( 2^{10} \). By (5.16) of [25], \( E \) normalizes a maximal torus of \( \overline{G} \), and this forces the Weyl group of \( F_4(K) \) to have an elementary abelian subgroup of order at least \( 2^6 \). This is false. So \( \overline{G} \neq F_4(K) \) and we conclude from (11.5) that \( n \geq 6 \), equality only if \( \overline{G} = E_6(K) \). It follows that \( |T_0| \geq \frac{1}{2} \cdot 2^6 \). Sylow's theorem implies that subgroups of \( X \) having order 11 or 23 are self-centralizing, so Table (11.1) implies \( |T_0| \mid |3^6 5^3 7| \). This is a contradiction.

(11.20) \( X \cong G_2 \).

Proof. Suppose \( X \cong G_2 \). We first claim that \( n \geq 8 \), equality possible only for \( \overline{G} = E_6(K) \). Table 1 of [2] shows that there is an involution \( j \in X \) such that \( C_X(j) \cong A \times R \cong G_2(4) \times (Z_2 \times Z_2) \) and \( R^\# \) consists of conjugates of \( j \). As in (8.6) and (8.8) of [2] there exists a conjugate \( R^x \) of \( R \) such that \( R^x \leq AR \), \( R^x \cap A = 1 \), and \( R^x \) projects to the center of a Sylow 2-subgroup of \( A \).

Let \( V \) be the Lie algebra of \( \overline{G} \) and write \( V = [V, R] \oplus C_R(R) \). Then \([V, R] = V_1 \oplus V_2 \oplus V_3 \), where \( V_i = [V, R] \cap C(j_i) \) and \( R^\# = \{ j_1, j_2, j_3 \} \). As in (11.11) we have \([A, V_i] \neq 1 \) for \( i = 1, 2, 3 \), so by [18] \( \dim(V) \geq \dim[V, R] \geq 3 \cdot \dim(V_i) \geq 3 \cdot 60 = 180 \). The claim follows. Consequently, \( |T_0| \geq (q - 1)^8 \geq 12^8 \).
By (11.4)(iii), $|T_0|^2 \leq X$, which by the above inequality is impossible for \( q > 13 \). Hence, \( q = 13 \). Now 13 has order 11 (modulo 23) and this forces \( n \geq 10 \). But then, $|T_0| \geq \frac{1}{11}(12)^{10}$, again contradicting $|T_0|^2 \leq X$.

(11.21) \( X \cong F_5 \).

**Proof.** Suppose \( X \cong F_5 \) and let \( V \) be the Lie algebra of \( \bar{G} \). If \( p \neq 5 \), then use (§4, II) of Harada [16] to conclude that \( X \) contains an extraspecial group of order \( 5^5 \) and an element inducing \( \mathbb{Z}_4 \) on the center of the extraspecial group. Elementary arguments imply that \( \dim(V) \geq 100 \). Hence \( n \geq 7 \), equality possible only for \( \bar{G} = E_7(K) \) or \( C_7(K) \). We then have $|T_0| \geq \frac{1}{2}(12)^7$, contradicting $|T_0|^2 \leq X$.

Suppose \( p = 5 \). Then \( q = 5^a \) for \( a \geq 2 \) and from (11.1) we have $|T_0|$ dividing $3^6 \cdot 7 \cdot 11 \cdot 19$. Since 19 \( ||G_0| \), a primitive divisor argument shows that if \( q = 25 \), then \( \Phi_d(25) \) divides \( |G_0| \), for \( d \) a multiple of 9. But then \( \varphi(d_1) \geq \varphi(9) = 6 \) and \( |T_0| \geq \frac{1}{d}(24)^6 \), a contradiction. For \( q = 5^3 \) use the prime 11 to obtain a contradiction, etc.

(11.22) \( X \cong F_3 \).

**Proof.** Suppose \( X \cong F_3 \). Then \( X \) contains a non-split extension of an elementary abelian group of order \( 2^5 \), by \( \text{SL}(5, 2) \) (Thompson [27]). This group has trivial multiplier, so (11.5) implies that \( n \geq 7 \), with equality possible only if \( \bar{G} = E_7(K) \). Hence $|T_0| \geq \frac{1}{2}(q - 1)^7$.

If \( q \geq 23 \), then this contradicts $|T_0|^2 \leq |X|$ . For the remaining cases argue as follows. Since 31 divides \( |X| \), 31 divides \( |G_0| \). So there is an integer \( d_1 \) such that \( 31 \mid \Phi_d(q) \mid |G_0| \). One checks that \( d_1 \) is a multiple of 30, 30 or 15, according to \( q = 13, 17, \) or 19. It follows that \( n \geq 8 \), equality possible only if \( G_0 = E_8(q) \). Hence $|T_0| \geq (q - 1)^8$ and this contradicts $|T_0|^2 \leq |X|$ .

We have now considered all possibilities for \( X \), completing the proof of (10.1).

12. Some consequences of (10.1). In this section we derive some consequences of (10.1) and (10.2). Throughout this section assume that \( p > 3 \) and \( q > 11 \).

**Theorem (12.1).** The map \( \bar{X} \to \bar{X}_q \) is a bijection between the set of closed, connected, \( \sigma \)-invariant subgroups of \( \bar{G} \) containing a maximal torus and the set of subgroups of \( G \) generated by maximal tori of \( G \).
The above theorem will be a consequence of the next result which gives additional information. In particular, the inverse of the map $\overline{X} \to \overline{X}_\sigma$ is described. We first need some notation.

For a subset $\Delta$ of $\Sigma$, let $\overline{G}(\Delta, \overline{T}) = \langle \overline{U}_\alpha \mid \alpha \in \Delta \rangle$. If $\Delta = \Delta^\sigma$, then $\overline{G}(\Delta, \overline{T})$ is $\sigma$-invariant and we set $G(\Delta, T) = O^\sigma(\overline{G}(\Delta, \overline{T})_\sigma)$. For $\overline{X} \leq \overline{G}$, let $\overline{\Sigma}(\overline{X}, \overline{T}) = \{ \alpha \in \Sigma \mid \overline{U}_\alpha \leq \overline{X} \}$, and for $X \leq G$, set $\overline{\Sigma}(X, T) = \bigcup_{X \leq X} \overline{\Sigma}_i$. If the maximal torus $\overline{T}$ is understood we abbreviate the above to $G(\Delta)$, $G(\Delta)$, $\overline{\Sigma}(X)$, and $\overline{\Sigma}(X)$, respectively.

We say that a subset $\Delta$ of $\overline{\Sigma}$ is $\overline{T}$-closed if $\Delta = \Delta^\sigma = \overline{\Sigma}(G(\Delta))$. This agrees with the concept introduced in §10. A final notation. For $Y \leq G$, let $\overline{G}(Y, T) = \overline{G}(\overline{\Sigma}(Y)) \cdot \overline{T}$, abbreviated to $\overline{G}(Y)$ when $T$ is understood. We can now state

**Theorem (12.2).** Let $T_0 \leq Y \leq G$. Then

(i) $\overline{\Sigma}(Y)$ is the unique $\overline{T}$-closed subset of $\overline{\Sigma}$ satisfying $G(\Delta)T_0 \leq Y$.

(ii) $\overline{G}(Y)$ is independent of $T_0$. That is, if $T_1$ is a maximal torus of $G$ with $T_1 \cap G_0 \leq Y$, then $\overline{G}(Y, T_1) = \overline{G}(Y, T)$.

(iii) $Y \leq N(\overline{G}(Y))$, $T$ normalizes $Y$, and $G(\Delta)T \leq YT$.

(iv) If $Y \geq T$ and if $Y$ is generated by maximal tori of $G$, then $\overline{G}(Y, T)$ is the unique closed, connected, $\sigma$-invariant subgroup of $G$ containing a maximal torus of $\overline{G}$ and having fixed point set $Y$.

(v) If $\overline{X} = \overline{X}^\sigma = \overline{X}^0 \geq \overline{T}$, then $X = \langle T^X \rangle$, where $X = \overline{X}_\sigma$.

It is clear that (12.1) follows from (12.2) and that the inverse of the map $\overline{X} \to \overline{X}_\sigma$ is the map $Y \to \overline{G}(Y)$. The next several results aim at the proof of (12.2). First we characterize $\overline{T}$-closed subsets of $\overline{\Sigma}$ at the $G$-level.

**Theorem (12.3).** A $\sigma$-invariant subset $\Delta$ of $\overline{\Sigma}$ is $\overline{T}$-closed if and only if $\overline{\Sigma}(G(\Delta)) = \Delta$.

**Proof.** Suppose $\Delta = \Delta^\sigma$. First assume that $\overline{\Sigma}(G(\Delta)) = \Delta$. Clearly $\Delta \subseteq \overline{\Sigma}(G(\Delta))$. If $\Delta$ is not closed, then there is some $\langle \sigma \rangle$-orbit $\Sigma_i$ of $\overline{\Sigma}$ such that $\overline{X}_i \leq \overline{G}(\Delta)$ and $\Sigma_i \nsubseteq \Delta$. However, $\overline{X}_i \leq \overline{G}(\Delta)$ implies $X_i \leq G(\Delta)$, so the assumption gives $\Sigma_i \subseteq \Delta$, a contradiction. Therefore, $\Delta$ is closed.

Now assume that $\Delta$ is closed. Hence, $\overline{\Sigma}(G(\Delta)) = \Delta$ and we must prove $\overline{\Sigma}(G(\Delta)) = \Delta$. Clearly, $\Delta \subseteq \overline{\Sigma}(G(\Delta))$, so it will suffice to take $X_i \leq G(\Delta)$ and show $\overline{X}_i \leq \overline{G}(\Delta)$. If $X_i$ is a $p$-group, then by (3.9) of [4] there is a $\sigma$-invariant parabolic subgroup $\overline{P}$ of $\overline{G}(\Delta)\overline{T}$ such that $X_i \leq R_u(\overline{P})$ and $T_0 \leq \overline{P}$. By (6.4) we also have $\overline{T} \leq \overline{P}$. Therefore, $R_u(\overline{P})$ is a product of $\overline{T}$-root subgroups of $\overline{G}$ and using (6.9), we have $\overline{X}_i \leq R_u(\overline{P}) \leq \overline{G}(\Delta)$, as required.
Suppose \( X_t \) is a group of Lie type and let \( T_x \) be a Cartan subgroup \( G(\Delta) \) (a \( p \)-complement in a Sylow \( p \)-normalizer). Then \( T_2 = C_{G(\Delta)\gamma}(T_t) \) is a maximal torus of \( G_0 \) (by (9.1)), and we set \( \overline{T}_2 = C_{\overline{G}(\Delta)}^0 \), a maximal torus of \( \overline{G} \). Then \( T_2 \leq G(\Delta)T_0 \leq \overline{G}(\Delta)\overline{T} \) and we claim that \( T_2 \) normalizes a \( \sigma \)-invariant maximal torus of \( \overline{G}(\Delta)\overline{T} \). To see this first use (5.16) of \([25]\) to get \( T_2 \leq N(A_\gamma) \), where \( A_\gamma \) is a \( \sigma \)-invariant maximal torus of \( \overline{G}(\Delta) \). Then set \( A = A_\gamma Z(\overline{G}(\Delta))^0 \) and check that \( A \) is a maximal torus of \( \overline{G}(\Delta)\overline{T} \). This proves the claim, so by (6.3) and (2.8) we must have \( A = \overline{T}_2 \).

Set \( Y = G(\Delta) \) and note that \( G(\Delta) < Y \). Since \( G(\Delta) < G(\Delta) \) and since each \( T_2 \)-root subgroup contained in \( G(\Delta) \) is a \( \sigma \)-group, the argument in the second paragraph of the proof shows that \( Y(T_2) = Y(T_0) \). By definition, \( X_\gamma \leq Y(T_0) \), so we now have \( G(\Delta) = Y(T_0) \). By (10.2) and (9.1) we have \( Y(T_0) = Y(T_2) \). If \( D \) is any \( T_2 \)-root subgroup of \( G \), then \( \langle T_2^D \rangle = DT_2 \). As \( \langle T_2^Y \rangle \leq Y(T_0)T_0 \), we conclude that \( Y(T_2^Y) \leq Y(T_0) \), and hence \( Y(T_0) = Y(T_2) \). From (9.1) we also have \( \overline{Y}(T_2) = \overline{Y}(T_0) \) and \( \overline{Y}(T_0)\overline{T} = \overline{Y}(T_2)\overline{T}_2 \). The Frattini argument yields \( Y = Y(T_2)N_{Y}(T_2) \).
Since $\bar{G}(\Delta) = \bar{Y}(T_0)$, we have $\bar{G}(Y) = \bar{Y}(T_0)\bar{T} = \bar{Y}(T_2)\bar{T}_2$. By (2.5) $\bar{G}(Y)_\sigma = (\bar{G}(\Delta)\bar{T})_\sigma = G(\Delta)T$ and also $\bar{G}(Y)_\sigma = Y(T_0)T = Y(T_2)T_3$, where $T_3 = (\bar{T}_2)_\sigma$.

As $Y(T_2) \leq \bar{Y}(T_2)$ and $N_Y(T_2) \leq N(\bar{Y}(T_2)) \cap N(\bar{T}_2)$, we necessarily have $Y = Y(T_2)N_Y(T_2) \leq N(\bar{G}(Y))$, proving (i). By the above, $Y \leq N(\bar{G}(Y)) = N(G(\Delta)T)$, so this will prove (iii), once (ii) is proved.

Now, $Y \leq N(\bar{G}(Y))$, so $Y$ normalizes $\bar{G}(Y)_\sigma = Y(T_0)T = Y(T_2)T_3$. The group $N_Y(T_2)$ also normalizes $T_3$, so $[N_Y(T_2), T_3] \leq T_3 \cap G_0 = T_2$, and $[N_Y(T_2), Y(T_0)T_3] \leq Y(T_0)T_2 = Y(T_0)T_0$. Letting $\sim$ denote images modulo $Y(T_0)$ we use the above to conclude $[Y^-, T^-] = [N_Y(T_2)^-, T^-] = [N_Y(T_2)^-, T_3^-] \leq T_0^- \leq Y^-$. This proves (ii) and completes the proof of (12.5).

(12.6) Let $T_0 \leq Y \leq G$. If $T_1$ is any maximal torus of $G$ with $T_2 = T_1 \cap G_0 \leq Y$, then $\bar{G}(Y, T_1) = G(Y, T)$.

Proof. Let $\Delta = \bar{\Sigma}(Y, T)$, so $G(\Delta)T_0 \leq Y$ by (12.4). Also, $G(\Delta) = Y(T_0)$, by (10.1). By (10.2), $Y(T_0)/O_p(Y) = E(Y/O_p(Y))$, so $Y(T_0) = Y(T_2)$. Also, $Y(T_0) = G(\Delta, T)$ and $Y(T_2) = G(\Delta, T_2)$, where $\Delta_1 = \bar{\Sigma}(Y, T_1)$. By (12.5)(iii) $Y$ normalizes $\bar{G}(Y, T) = G(\Delta)\bar{T}$. In particular, $T_2 \leq N(\bar{G}(\Delta)\bar{T})$.

We claim that $\bar{T}_1 \leq \bar{G}(\Delta)\bar{T}$, where $\bar{T}_1 = C_{\bar{G}}(T_1)^0$. Let $\bar{Q} = R_\sigma(\bar{G}(\Delta)\bar{T})$ and $\bar{Z}/\bar{Q}$ the connected center of $\bar{G}(\Delta)\bar{T}/\bar{Q}$. So $\bar{G}(\Delta)\bar{T}/\bar{Z}$ is semisimple and (5.16) of [25] shows that $T_2$ normalizes a $\sigma$-invariant maximal torus of $\bar{G}(\Delta)\bar{T}/\bar{Z}$. So there is a $\sigma$-invariant maximal torus $\bar{A}$ of $\bar{G}$, with $\bar{A} \leq \bar{G}(\Delta)\bar{T}$ and $T_2 \leq N(A\bar{Q})$. Let $A = A_\sigma$ and $Q = \bar{Q}_\sigma$. Then $T_2$ normalizes $(A\bar{Q})_\sigma = AQ$ and $T_2 AQ/Q$ is a solvable $p'$-group. By Hall's theorem $T_2$ is contained in a Hall $p'$-group of $T_2 AQ$, so $T_2$ normalizes $A^x$, for some $x \in Q$. But $A^x$ is a maximal torus of $G$, whence $T_2 = A^x \cap G_0$ by (6.3). Then (2.6) gives $A^x = \bar{T}_1$, and the claim is proved. Therefore, $\bar{G}(\Delta)\bar{T} = \bar{G}(\Delta)\bar{T}_1$.

Let $\Omega = \bar{\Sigma}(\bar{G}(\Delta), \bar{T}_1)$ so that $\bar{G}(\Delta) = \bar{G}(\Omega, \bar{T}_1)$. Clearly $\Omega$ is $\bar{T}_1$-closed and $G(\Omega, T_1) = G(\Delta)$. By (2.5)(iv), $G(\Omega, T_1)T_2 = G(\Delta)T_0 \leq Y$, so (12.4) forces $\Omega = \bar{\Sigma}(Y, T_1)$. At this stage we have $\bar{G}(Y, T_1) = \bar{G}(Y, T)$, as desired.

(12.7) Assume that $Y$ is generated by $G$ conjugates of maximal tori of $G$. Then $Y = \bar{X}_\sigma$ for a unique closed connected, $\sigma$-invariant subgroup $\bar{X}$ of $\bar{G}$ such that $\bar{X}$ contains a maximal torus of $\bar{G}$.

Proof. Let $T_1, T_2$ be maximal tori of $G$ with $T_1 \leq Y \leq T_2$. Let $\Delta_1 = \bar{\Sigma}(Y, T_i)$ for $i = 1, 2$. Then $G(\Delta_i, T_i)T_i \leq Y$ for $i = 1, 2$ (see
(12.5)(iii)). By (12.6), \( \overline{G}(Y, T_1) = \overline{G}(Y, T_2) \), so taking fixed points (and applying (2.5)) we have \( G(\Delta_1, T_1)T_1 = G(\Delta_2, T_2)T_2 \). Fixing \( T_1 \) and letting \( T_2 \) vary over all maximal tori of \( G \) contained in \( Y \), we conclude that \( Y = G(\Delta_1, T_1)T_1 \) and this gives the existence of \( \overline{X} \).

Now suppose \( Y = \overline{X}_\sigma \), with \( \overline{X} \) closed, connected, \( \sigma \)-invariant, and containing a maximal torus \( \overline{T}_1 \) of \( G \). We may take \( \overline{T}_1 \) to be \( \sigma \)-invariant. Set \( T_1 = (\overline{T}_1)_\sigma \). Let \( \Delta_1 = \overline{\Sigma}(\overline{X}, T_1) \), so \( \overline{X} = \overline{G}(\Delta_1)\overline{T}_1 \). Hence \( G(\Delta_1) \leq Y \) and \( Y/G(\Delta_1) \) is a \( p' \)-group. This implies that each \( T_1 \)-root subgroup of \( G \) contained in \( Y \) is actually in \( G(\Delta_1) \). That is, \( \overline{\Sigma}(G(\Delta_1)) = \overline{\Sigma}(Y, T_1) \). But \( \Delta_1 \) is \( \overline{T}_1 \)-closed. Thus (12.3) shows that \( \Delta_1 = \overline{\Sigma}(Y, T_1) \) and then \( \overline{X} = \overline{G}(Y, T_1) \), proving uniqueness.

At this point all parts of Theorem (12.2) have been established, with the exception of (v). Suppose \( \overline{X} \) is as in (12.2)(v) and let \( X = \overline{X}_\sigma \). By (10.1), \( X(T_0)T_0 = \langle T_0^X \rangle \leq X \). Also, \( X(T_0) = O^p(\overline{X}(T_0)\sigma) \geq O^p(\overline{X}_\sigma) \). Since \( X = O^p(\overline{X}_\sigma)T \), we have the result.

We close this section with some results on generation. From now on, \( \overline{T} \) is fixed. We thus delete mention of \( \overline{T} \) and \( T \) from the earlier notation and say \( \Delta \subseteq \overline{\Sigma} \) is closed if it is \( \overline{T} \)-closed.

(12.8) Let \( \Omega_1, \ldots, \Omega_k \) be closed subsets of \( \overline{\Sigma} \). Then
\[
\langle G(\Omega_1), \ldots, G(\Omega_k) \rangle = O^p(\langle \overline{G}(\Omega_1), \ldots, \overline{G}(\Omega_k) \rangle_\sigma).
\]

Proof. Let \( \Omega = \overline{\Sigma}(\langle \overline{G}(\Omega_1), \ldots, \overline{G}(\Omega_k) \rangle) \). Then \( \overline{G}(\Omega) \geq \overline{G}(\Omega_i) \) for \( i = 1, \ldots, k \) and \( \Omega \) is closed. Let \( Y = \langle G(\Omega_1), \ldots, G(\Omega_k) \rangle \) and set \( \Delta = \overline{\Sigma}(Y) \). By (12.4) we have \( G(\Delta)T_0 \leq YT_0 \) and it follows that \( Y = G(\Delta) \).

Suppose \( \overline{\Sigma}_j \subseteq \Omega_i \) for some \( i \) and \( j \). Then \( \overline{X}_j \leq \overline{G}(\Omega_j) \) and \( X_j \leq G(\Omega_j) \leq Y \), proving \( \overline{\Sigma}_j \subseteq \overline{\Sigma}(Y) = \Delta \). We conclude \( \Omega_i \subseteq \Delta \) for \( i = 1, \ldots, k \). Therefore, \( \overline{G}(\Delta) \geq \langle \overline{G}(\Omega_1), \ldots, \overline{G}(\Omega_k) \rangle \). Since \( \Delta \) is closed (by (12.4)), we have \( \Omega \subseteq \Delta \). Hence, \( G(\Omega) \leq G(\Delta) \). On the other hand, \( \overline{G}(\Omega) \geq \overline{G}(\Omega_i) \) for \( i = 1, \ldots, k \), so \( G(\Omega) \geq \langle G(\Omega_1), \ldots, G(\Omega_k) \rangle = Y = G(\Delta) \). This proves \( G(\Omega) = G(\Delta) = Y \), which proves the result.

The following results extend (6.10) to arbitrary collections of \( T \)-root subgroups.

THEOREM (12.9). Let \( X_{i_1}, \ldots, X_{i_k} \) be \( T \)-root subgroups of \( G \). Then
\[
\langle X_{i_1}, \ldots, X_{i_k} \rangle = O^p(\langle \overline{X}_{i_1}, \ldots, \overline{X}_{i_k} \rangle_\sigma).
\]
Proof. Set \( Y = \langle X_{i_1}, \ldots, X_{i_k} \rangle \) and \( \Delta = \Sigma(Y) \). Apply (12.4) to \( YT_0 \) and conclude that \( Y = G(\Delta) \) and \( \Delta \) is closed. For \( j = 1, \ldots, k \), \( \Sigma_{i_j} \subseteq \Delta \), so \( \bar{X}_{i_j} \leq \bar{G}(\Delta) \). Therefore, \( O^\nu(\langle \bar{X}_{i_1}, \ldots, \bar{X}_{i_k} \rangle) \leq O^\nu(\bar{G}(\Delta)) = G(\Delta) = Y = \langle X_{i_1}, \ldots, X_{i_k} \rangle \). Since the other containment is obvious, the proof is complete.

**Theorem (12.10).** Let \( S \) be an arbitrary set of \( p' \)-elements of \( G \). Then

(i) If \( \bar{G} \) is simply connected, then \( \langle C_{G_1}(a) : s \in S \rangle = G_1 \cap \langle C_{\bar{G}}(s) : s \in S \rangle \).

(ii) If \( G \) is simply connected, then \( G_1 = \langle C_{G_1}(s) : s \in S \rangle \) if and only if \( \bar{G} = \langle C_{\bar{G}}(s) : s \in S \rangle \).

(iii) If \( \bar{G} = \langle C_{\bar{G}}(s)^0 : s \in S \rangle \), then \( G_1 = \langle C_{G_1}(s) : s \in S \rangle \).

(iv) If \( S \subseteq T \) with \( T \) a maximal torus of \( G \), then \( G_1 = \langle E(C_{G_1}(s)) : s \in S \rangle \) if and only if \( \bar{G} = \langle E(C_{\bar{G}}(s)) : s \in S \rangle \).

Proof. Set \( X = \langle C_{G_1}(s) : s \in S \rangle \) and \( \bar{X} = \langle C_{\bar{G}}(s)^0 : s \in S \rangle \). Fix \( s \in S \) and \( T \) a maximal torus of \( G \) with \( s \in T \). Then \( T_0 = T \cap G_0 \) and \( T_1 = T \cap G_1 \) are maximal tori of \( G_0 \) and \( G_1 \), respectively. Let \( \bar{T} \) be the unique maximal torus of \( \bar{G} \) containing \( T \). Then \( T_0 \leq T_1 \leq X \) and \( \bar{T} \leq \bar{X} \).

Let \( \bar{C} = C_{\bar{G}}(s)^0 \) and \( C = \bar{C}_o \). By (10.1) we have \( X(T_0) \leq X \). Also \( T \) normalizes \( X(T_0) \). Let \( \bar{X} \) be the unique \( \sigma \)-invariant, connected subgroup of \( \bar{G} \) satisfying \( \bar{T} \leq \bar{X} \) and \( \bar{X}_o = X(T_0)T \) (use (12.1)). Then (12.2)(iv) implies \( \bar{X} = \bar{G}(X, T) \). Since \( C_{\bar{G}}(s) = C_{\bar{G}_1}(s)T \), we have \( \bar{G}(C, T) \leq \bar{G}(X, T) = \bar{X} \).

On the other hand, \( \bar{G}(C, T) = \bar{C} \) (this follows from (12.1) as both have fixed point set under \( \sigma \) equal to \( C \)). So \( \bar{C} \leq \bar{X} \) and letting \( s \) vary we obtain \( \bar{X} \leq \bar{X} \).

From \( \bar{X} \leq \bar{X} \) and \( \bar{X}_o = X(T_0)T \), we immediately have (iii). Suppose \( \bar{G} \) is simply connected. Then \( C_{\bar{G}}(s) = C_{\bar{G}}(s)^0 \) for each \( s \in S \). In particular \( \bar{X}_o \geq \langle C_{\bar{G}}(s) : s \in S \rangle \geq X \). Letting \( s \), \( T \) be as before we have \( \bar{X}_o \geq \langle X, T \rangle \geq X(T_0)T = \bar{X}_o \geq \bar{X}_o \). Consequently, \( \bar{X}_o = \bar{X}_o \) and (12.1) yields \( \bar{X} = \bar{X} \). Also \( \langle X, T \rangle = X(T_0)T = XT \). Therefore, \( \bar{X} \cap G_1 = \bar{X} \cap G_1 = \bar{X}_o \cap G_1 = XT \cap G_1 = X(T \cap G_1) = XT_1 \leq X \), proving (i). If \( X = G_1 \), then \( XT = G \), forcing \( \bar{X} = \bar{G} \). Hence \( \bar{X} = \bar{G} \). Combining this with (iii) we have (ii).

Finally, suppose \( S \subseteq T \) for \( T \) a maximal torus of \( G \). For purposes of proving (iv) we may assume \( \bar{G} \) is simply connected. For \( s \in S \), \( C_{\bar{G}}(s) = E(C_{\bar{G}}(s))T \) and \( \bar{C}_{\bar{G}}(s) = E(C_{\bar{G}}(s))T \) (by (2.9) and its proof). Hence \( \bar{X} = \langle E(C_{\bar{G}}(s)) : s \in S \rangle \bar{T} \bar{T}_1 \). From the previous paragraph \( \bar{X} = \bar{X} \), so \( \bar{X}_o = \bar{X}_o = XT \). (iv) now follows from (12.1).
(12.11) Let $S_1, \ldots, S_k$ be subsets of $T$. Suppose that for each $\alpha \in \Sigma^+$, there is some $i \in \{1, \ldots, k\}$ with $S_i \subseteq C(\overline{U}_\alpha)$. Then $G_0 = \langle E(C_{G_0}(S_i)) \mid i = 1, \ldots, k \rangle$.

Proof. For $1 \leq i \leq k$, $C_G(S_i)^0$ is a reductive group and $(C_G(S_i)^0)^\lor$ a semi-simple group. Hence $E(C_{G_0}(S_i)) \geq O^p((C_G(S_i)^0)^\lor)$. By (12.4), $E(C_{G_0}(S_i)) = G(\Omega_i)$ for a unique closed subset $\Omega_i$ of $\Sigma$ (if $E(C_{G_0}(S_i)) = 1$, set $\Omega_i = \emptyset$ and $G(\Omega_i) = 1$).

By (12.8), $\langle G(\Omega_1), \ldots, G(\Omega_k) \rangle = O^p(\langle \overline{G}(\Omega_1), \ldots, \overline{G}(\Omega_k) \rangle_0)$. If $\alpha \in \Sigma^+$ with $S_i \subseteq C(\overline{U}_\alpha)$, then $S_i$ centralizes $\langle \overline{U}_{\pm \alpha} \rangle \leq (C_G(S_i)^0)^\lor$. So if $\alpha \in \Sigma_j$, we have $\overline{X}_j, \overline{X}_j^* \leq (C_G(S_j)^0)^\lor$ and then $X_j, X_j^* \leq G(\Omega_j)$. Since $\Omega_i$ is closed, we conclude $\Sigma_j, \Sigma_j^* \subseteq \Omega_i$. These remarks and our assumption show that $\langle \overline{G}(\Omega_1), \ldots, \overline{G}(\Omega_k) \rangle = G$, and the result follows.

(12.12) Let $T$ be a maximal torus of $G$ and $R \leq T$. Then $G_0 = \langle E(C_{G_0}(R)) \mid R_1 \leq R$ and $R/R_1$ cyclic$\rangle$.

Proof. Let $R \leq T$ and $\alpha \in \Sigma^+$. Then $R$ induces a cyclic group on $\overline{U}_\alpha$, hence $R_1 = C_R(\overline{U}_\alpha)$ has cyclic quotient group. So (12.12) follows from (12.11).

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