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THE SELBERG TRACE FORMULA II: PARTITION, REDUCTION, TRUNCATION

M. SCOTT OSBORNE AND GARTH WARNER

Let G be a reductive Lie group; let Γ be a non-uniform lattice in G. Here we shall lay the analytic and geometric foundations on which the derivation of the Selberg trace formula for the pair (G, Γ) will eventually be based.

CONTENTS

1. Introduction. This is the second in a projected series of papers in which we plan to come to grips with the Selberg trace formula, the ultimate objective being a reasonably explicit expression. We shall take as the basic reference and point of departure our memoir [**3.a**] to which we refer the reader for a complete discussion of the foundations of the theory, as well as additional background material. It will be recalled that the first paper in this series (cf. [**3.b**]) was devoted to a discussion of these questions in the special case when the rank of the ambient lattice was unity. Philosophically heuristic, the essential plan of attack, incorporating most of the basic ideas, can be found there already. We would not be stretching matters much by saying that our chief concern in this paper and its successors is to take a given point from the rank-one picture and push it through in general, leading eventually to a grand compilation.

The theory centers on a reductive Lie group G and a non-uniform lattice Γ in G, both satisfying the usual conditions, the ultimate object of study being $L^2(G/\Gamma)$. Since we have amply dealt with what one knows (and what one wants to know) about $L^2(G/\Gamma)$ elsewhere, there is nothing to be gained by repeating this theme here. Instead, we shall content ourselves with a brief indication of the highlights of the present paper.

Section 2, while in a sense preliminary and seemingly even peripheral, actually makes its presence felt, directly or indirectly, throughout the entire work, the main result in this circle of ideas being the Combinatorial Lemma of Langlands. A first application is made in §3 where we establish an important extension of the by now classical reduction theory, focusing on an *exact* partition of G/Γ using all the Γ-cuspidal split parabolic subgroups of G (not just the Γ-percuspidals...). §4 is technical in character, collecting a number of estimates which are used in the later going. In §5 we introduce the definition of the truncation operator and formulate its fundamental properties, the corresponding proofs being deferred until §§6–7. Additional facts about the truncation operator, somewhat formal in nature, are to be found in §8. In §9 we obtain an inner product formula for two truncated Eisenstein series associated with cusp forms, this formula then providing a link to the next paper in this series but finding application also in §10, where we use it in a characterization of the truncation operator.

Some suggestions for reading may be helpful. To begin with, it is definitely necessary to acquire a reasonable familiarity with §§2–3. After a quick perusal of §4, one could then turn to §5 which contains precise statements but no proofs, they being presented in §§6–7. Setting aside their study, it would be possible to pass on to §8 for additional orientation, thence to §10, the latter having the character of a summary, thus providing motivation for the role of the truncation operator in general and for the role of the inner product formula (§9) in particular. Additional remarks can be found at the beginning of each section.

In conclusion, we would like to acknowledge our indebtedness to the geometric insights of Langlands **[2.a]** and Arthur **[1.a, 1.b]**. It was Langlands who was the first to recognize the significance of 'Combinatorial Lemmata' and Arthur who pioneered in their use.

2. Partitionings of euclidean space. The purpose of this section is the development of a series of geometric facts, admittedly intricate, which, however, are at the basis of everything that follows. Chief among them is the Combinatorial Lemma of Langlands (Proposition 2.5 infra). In the setting of parabolic subgroups alone, some of our results have been obtained by Arthur **[1.a]**; the approach below, though, is frequently quite different and, of course, the situation is more general.

The investigation centers on the following data:

(1) A finite dimensional inner product space

$$(V, (?, ?))$$

of dimension l, say;

(2) A basis $\{\lambda_1, \ldots, \lambda_l\}$ of V subject to the condition

$$(\lambda_i, \lambda_j) \leq 0 \qquad (i \neq j).$$

We remark that (2) is suggested by the theory of 'root systems'; the possibility that the λ_i are mutually orthogonal is not excluded, naturally.

Let $\{\lambda^1,\ldots,\lambda^l\}$ be the basis of V dual to $\{\lambda_1,\ldots,\lambda_l\}$ — then it is a well-known and familiar fact that

$$(\lambda^i,\lambda^j)\geq 0 \qquad (i\neq j).$$

Suppose now that F is a subset of $\{1,\ldots,l\}$. Let $V(F)$ denote the subspace of V spanned by the λ_i $(i\in F)$, $P(F)$ the orthogonal projection of V onto $V(F)$; let V_F be the orthogonal complement of $V(F)$ in V, P_F the orthogonal projection of V onto V_F. Put

$$\lambda_i^F=\begin{cases}\lambda_i & \text{if } i\in F\\ P_F\lambda_i & \text{if } i\notin F.\end{cases}$$

Then $\{\lambda_1^F,\ldots,\lambda_l^F\}$ is a basis of V with associated dual basis $\{\lambda^1{}_F,\ldots,\lambda^l{}_F\}$. One knows that

$$(\lambda_i^F,\lambda_j^F)\leq 0\leq(\lambda^i{}_F,\lambda^j{}_F) \qquad (i\neq j),$$

implying, therefore, a reproduction of data. In this connection, observe that

$$\lambda^i{}_F=\begin{cases}P(F)\lambda^i & \text{if } i\in F\\ \lambda^i & \text{if } i\notin F.\end{cases}$$

Let

$$\begin{cases}\mathcal{C}=\{H\in V\colon(\lambda_i,H)>0\ \forall i\}\\ \mathcal{D}=\{H\in V\colon(\lambda^i,H)>0\ \forall i\}.\end{cases}$$

It is customary to refer to $\mathcal{C}$ as the positive chamber in V, to $\mathcal{D}$ as the positive cone in V. Note that $\mathcal{D}\supset\mathcal{C}$, the inclusion being, in general, strict. There are pointwise descriptions of $\mathcal{C}$ and $\mathcal{D}$, viz.:

$$\begin{cases}\mathcal{C}=\left\{\sum_{i=1}^{l}t^i\lambda^i\colon t^i>0\ \forall i\right\}\\ \mathcal{D}=\left\{\sum_{i=1}^{l}t_i\lambda_i\colon t_i>0\ \forall i\right\}.\end{cases}$$

No hyperplane of the form

$$\{H\colon(\lambda_i^F,H)=0\},\qquad\{H\colon(\lambda^i{}_F,H)=0\}$$

meets $\mathcal{C}$ since

$$H \in \mathcal{C} \Rightarrow \begin{cases} (\lambda_i^F, H) > 0 \\ (\lambda_F^i, H) > 0. \end{cases}$$

Let $\mathcal{S}$ be a subset of V — then we write $\operatorname{Pos}(\mathcal{S})$ for the interior of

$$\{H \in V: (\sigma, H) > 0 \ \forall \sigma \in \mathcal{S}\}.$$

Plainly

$$\mathcal{S}'' \supset \mathcal{S}' \Rightarrow \operatorname{Pos}(\mathcal{S}') \supset \operatorname{Pos}(\mathcal{S}'').$$

It is also clear that

$$\operatorname{Pos}(\mathcal{C}) = \mathcal{D} \quad \text{and} \quad \operatorname{Pos}(\mathcal{D}) = \mathcal{C}.$$

If again F is a subset of $\{1,\ldots,l\}$, then

$$\mathcal{B}_F = \{\lambda_i: i \in F\} \cup \{\lambda^i: i \notin F\}$$

is a basis of V, the corresponding dual basis being

$$\mathcal{B}^F = \{\lambda_i^F: i \notin F\} \cup \{\lambda_F^i: i \in F\}.$$

Claim:

$$\begin{cases} \mathcal{C} \subset \operatorname{Pos}(\mathcal{B}_F) \subset \mathcal{D} \\ \mathcal{C} \subset \operatorname{Pos}(\mathcal{B}^F) \subset \mathcal{D}. \end{cases}$$

Indeed, that $\mathcal{C}$ is contained in both $\operatorname{Pos}(\mathcal{B}_F)$ and $\operatorname{Pos}(\mathcal{B}^F)$ is a consequence of a remark supra. On the other hand,

$$\mathcal{D} = \operatorname{Pos}(\mathcal{C}) \supset \begin{cases} \operatorname{Pos}(\operatorname{Pos}(\mathcal{B}^F)) = \operatorname{Pos}(\mathcal{B}_F) \\ \operatorname{Pos}(\operatorname{Pos}(\mathcal{B}_F)) = \operatorname{Pos}(\mathcal{B}^F). \end{cases}$$

Hence the claim.

LEMMA 2.1. *Let F_1, F_2 be subsets of $\{1,\ldots,l\}$ — then*

$$\sum_{\{F:\, F_1 \subset F \subset F_2\}} (-1)^{\#(F-F_1)} = \begin{cases} 1 & \text{if } F_1 = F_2 \\ 0 & \text{if } F_1 \neq F_2. \end{cases}$$

Proof. Since the assertion is obvious if $F_1 = F_2$ or $F_1 \not\subset F_2$, let us assume that $F_1 \neq F_2$ and $F_1 \subset F_2$. We have then

$$\begin{aligned} 0 &= (1-1)^{\#(F_2-F_1)} \\ &= \sum_{i=0}^{\#(F_2-F_1)} \binom{\#(F_2-F_1)}{i} (-1)^i \\ &= \sum_{\{F:\, F\subset F_2-F_1\}} (-1)^{\#(F)} \\ &= \sum_{\{F:\, F_1\subset F\subset F_2\}} (-1)^{\#(F-F_1)}, \end{aligned}$$

as desired. □

Let F_1, F_2 be subsets of $\{1,\ldots,l\}$ with $F_1 \subset F_2$. We shall then agree to write

$$\chi_{F_1,F_2}$$

for the characteristic function of the set

$$\left\{H \in V\colon \left(\lambda_i^{F_1}, H\right) > 0 \; (i \in F_2 - F_1)\right\}$$

and, dually,

$$\chi^{F_1,F_2}$$

for the characteristic function of the set

$$\left\{H \in V\colon \left(\lambda_{F_2}^i, H\right) > 0 \; (i \in F_2 - F_1)\right\}.$$

The abbreviations $\chi_{1,2}$ or $\chi^{1,2}$ will be employed when no confusion is possible. In the special case when $F_1 = \varnothing$, we use the notation

$$\begin{cases} \chi_{*,F_2} \\ \chi^{*,F_2}; \end{cases}$$

in the special case when $F_2 = \{1,\ldots,l\}$, we use the notation

$$\begin{cases} \chi_{F_1,*} \\ \chi^{F_1,*}. \end{cases}$$

Form now the following function on V,

$$\sigma_{F_1}^{F_2}(H) = \sum_{\{F:\, F\supset F_2\}} (-1)^{\#(F-F_2)} \chi_{1,F}(H)\cdot\chi^{F,*}(H),$$

about which we can say the following.

PROPOSITION 2.2. $\sigma_{F_1}^{F_2}$ *is the characteristic function of the set* $\mathcal{S}_{F_1}^{F_2}$ *of all* $H \in V$ *such that*:

(i) $(\lambda_i^{F_1}, H) > 0 \ \forall\, i \in F_2 - F_1$;
(ii) $(\lambda_i^{F_1}, H) \leq 0 \ \forall\, i \notin F_2$;
(iii) $(\lambda_{F_1}^{i}, H) > 0 \ \forall\, i \notin F_2$.

Proof. To begin with, observe that the

$$\lambda_i^{F_1}(i \in F_2 - F_1), \begin{cases} \lambda_i^{F_1} \\ \lambda_{F_1}^{i} \end{cases} \quad (i \notin F_2),$$

all belong to V_{F_1}. The value of $\sigma_{F_1}^{F_2}$ at a particular H depends, therefore, only on its projection onto V_{F_1}. We can assume, then, without any loss of generality, that $F_1 = \varnothing$, $F_2 = F_0$ (say). This said, fix an $H_0 \in V$. Let

$$F_0(H_0) = F_0 \cup \{i \notin F_0 \colon (\lambda^i, H_0) \leq 0\}.$$

Thus

$$\chi^{F,*}(H_0) = \begin{cases} 1 & \text{if} F \supset F_0(H_0) \\ 0 & \text{if } F \not\supset F_0(H_0) \end{cases}$$

and so

$$\sigma_{\varnothing}^{F_0}(H_0) = \sum_{\{F\colon F \supset F_0(H_0)\}} (-1)^{\#(F-F_0)} \chi_{*,F}(H_0).$$

Let

$$F_0^+(H_0) = \{i \colon (\lambda_i, H_0) > 0\}.$$

Thus

$$\chi_{*,F}(H_0) = \begin{cases} 1 & \text{if } F \subset F_0^+(H_0) \\ 0 & \text{if } F \not\subset F_0^+(H_0) \end{cases}$$

and so

$$\begin{aligned} &F_0(H_0) \not\subset F_0^+(H_0) \\ &\qquad \Rightarrow \sigma_{\varnothing}^{F_0}(H_0) = 0, \\ &F_0(H_0) \subset F_0^+(H_0) \\ &\qquad \Rightarrow \sigma_{\varnothing}^{F_0}(H_0) = \sum_{\{F\colon F_0(H_0) \subset F \subset F_0^+(H_0)\}} (-1)^{\#(F-F_0)}, \end{aligned}$$

or still (cf. Lemma 2.1)

$$\begin{aligned} &F_0(H_0) \neq F_0^+(H_0) \\ &\qquad \Rightarrow \sigma_\varnothing^{F_0}(H_0) = 0, \\ &F_0(H_0) = F_0^+(H_0) \\ &\qquad \Rightarrow \sigma_\varnothing^{F_0}(H_0) = (-1)^{\#(F_0(H_0)-F_0)}. \end{aligned}$$

To complete the proof, suppose first that H_0 is actually in $\mathcal{S}_\varnothing^{F_0}$ — then

$$F_0 = \begin{cases} F_0(H_0) \\ F_0^+(H_0), \end{cases}$$

hence, by the above, $\sigma_\varnothing^{F_0}(H_0) = 1$. As for the other direction, it is a question of showing that $H_0 \notin \mathcal{S}_\varnothing^{F_0} \Rightarrow \sigma_\varnothing^{F_0}(H_0) = 0$ or, equivalently, that $\sigma_\varnothing^{F_0}(H_0) \neq 0 \Rightarrow H_0 \in \mathcal{S}_\varnothing^{F_0}$. Supposing the latter to be the case, H_0 must belong to the set of all $H \in V$ such that

$$\begin{cases} (\lambda_i, H) > 0 & \forall\, i \in F_0(H_0) \\ (\lambda_i, H) \leq 0 & \forall\, i \notin F_0(H_0) \\ (\lambda^i, H) > 0 & \forall\, i \notin F_0(H_0), \end{cases}$$

a subset of $\mathfrak{B}_{F_0(H_0)}$ which, in turn, is contained in $\mathfrak{I}$. In other words: $\sigma_\varnothing^{F_0}(H_0) \neq 0 \Rightarrow H_0 \in \mathfrak{I}$. This implies that $F_0(H_0) = F_0$ so that, in fact, $H_0 \in \mathcal{S}_\varnothing^{F_0}$, as was to be shown. □

It is a corollary that

$$\sigma_F^F = 0$$

for all $F \neq \{1,\ldots,l\}$.

REMARK. On the basis of the preceding argument, one can see without difficulty that

$$\sum_F \sigma_\varnothing^F = \chi_{\mathfrak{I}}.$$

Given $H \in V$, write $H = H(F) + H_F$ where $H(F) \in V(F)$, $H_F \in V_F$.

PROPOSITION 2.3. *Let $T \in \mathfrak{C}$. Suppose that $H \in T + \mathcal{S}_\varnothing^F$ — then*

$$(\lambda_i, H(F)) > 0 \quad \forall\, i \in F.$$

Moreover, there exists a positive constant C_F, depending only on F, such that

$$\|H\| \leq C_F(1 + \|T\|)(1 + \|H(F)\|).$$

The initial assertion is easy enough. For

$$i \in F \Rightarrow (\lambda_i, H(F)) = (\lambda_i, H - T) + (\lambda_i, T) - (\lambda_i, H_F)$$

and the right-hand side is certainly positive. The final assertion, however, is a little more complicated. We shall preface its proof with a lemma.

LEMMA 2.4. *Fix* $H_0 \in \mathcal{D}$ — *then*

$$\|H_0\| = \sup\{\|H\|: H \in \mathcal{D} \cap H_0 - \mathcal{C}^-\}.$$

Proof. Since $H_0 \in \mathcal{D} \cap H_0 - \mathcal{C}^-$, we have

$$\|H_0\| \leq \sup\{\|H\|: H \in \mathcal{D} \cap H_0 - \mathcal{C}^-\}.$$

On the other hand, if $H \in \mathcal{D} \cap H_0 - \mathcal{C}^-$, then

$$\begin{aligned}\|H_0\|^2 &= (H_0, H_0) = (H + (H_0 - H), H + (H_0 - H)) \\ &= \|H\|^2 + \|H_0 - H\|^2 + 2(H, H_0 - H) \geq \|H\|^2.\end{aligned}$$

Hence the lemma. □

In passing, let us note that if $H_0 \notin \mathcal{D}$, then $\mathcal{D} \cap H_0 - \mathcal{C}^- = \varnothing$.

Now introduce, in the obvious way, the positive chamber $\mathcal{C}_F$ and the positive cone $\mathcal{D}_F$ in V_F. Define a linear operator

$$A_F: V \to V_F$$

by the rule

$$A_F(H) = \sum_{i \notin F} (\lambda_i, H) \cdot \lambda^i \qquad (H \in V).$$

For all $i \notin F$, we evidently have

$$(\lambda_i^F, A_F(H)) = (\lambda_i, H).$$

To finish the proof of our proposition, suppose that H is as there, i.e. $H \in T + \mathcal{S}_\varnothing^F$ where $T \in \mathcal{C}$ — then it follows from the definitions that for all $i \notin F$,

$$\begin{aligned}(\lambda_i^F, H_F) &= (\lambda_i, H - T) + (\lambda_i, T) - (\lambda_i, H(F)) \\ &\leq (\lambda_i, T) - (\lambda_i, H(F)), \\ (\lambda^i_F, H_F) &= (\lambda^i, H - T) + (\lambda^i, T) \\ &> 0,\end{aligned}$$

so

$$H_F \in \mathcal{D}_F \cap [A_F(T - H(F)) - \mathcal{C}_F^-].$$

Taking into account Lemma 2.4, we then find that

$$\begin{aligned}
\|H\| &\le \|H(F)\| + \|H_F\| \\
&\le \|H(F)\| + \|A_F(T - H(F))\| \\
&\le \|H(F)\| + \|A_F(T)\| + \|A_F(H(F))\| \\
&\le \|A_F\|_{\mathrm{OP}}\|T\| + (1 + \|A_F\|_{\mathrm{OP}})\|H(F)\| \\
&\le (1 + \|A_F\|_{\mathrm{OP}})(\|T\| + \|H(F)\|) \\
&\le (1 + \|A_F\|_{\mathrm{OP}})(1 + \|T\| + \|H(F)\| + \|T\| \cdot \|H(F)\|) \\
&= (1 + \|A_F\|_{\mathrm{OP}})(1 + \|T\|)(1 + \|H(F)\|) \\
&= C_F(1 + \|T\|)(1 + \|H(F)\|),
\end{aligned}$$

C_F being, by definition, $1 + \|A_F\|_{\mathrm{OP}}$. The proof is therefore complete.

Our next task will be to formulate and prove the Combinatorial Lemma of Langlands. To this end, let F_0, F_1, F_2 be subsets of $\{1,\ldots,l\}$ with $F_1 \subset F_2$ and $F_0 \subset F_2 - F_1$. If $F_1 \subset F \subset F_2$, call

$$\tau_{F_1,F}(F_0 : ?)$$

the characteristic function of the set $\mathfrak{T}_{F_1,F}(F_0)$ of all $H \in V$ such that $\forall\, i \in F - F_1$:

$$\begin{cases} i \in F_0 \Rightarrow (\lambda^i_F, H) > 0 \\ i \notin F_0 \Rightarrow (\lambda^i_F, H) \le 0. \end{cases}$$

Note that the τ-function does not, in reality, depend on F_2, it being merely a fixed set of reference.

PROPOSITION 2.5. *For all $H \in V$,*

$$\sum_{\{F:\, F_1 \subset F \subset F_2\}} (-1)^{\#(F \cap F_0)} \tau_{F_1,F}(F_0 : H)\chi_{F,2}(H)$$

is equal to

$$\begin{cases} 1 & \text{if } F_0 = \varnothing \\ 0 & \text{if } F_0 \neq \varnothing. \end{cases}$$

A result of this type was first stated without proof by Langlands [**2.a**], who also introduced the term 'Combinatorial Lemma'. Arthur [**1.a**] has recently established a related version. The present formulation is simpler and, at the same time, more general.

We shall need two lemmas. The first is a straightforward technicality; the second, while a formal consequence of the first, will serve to reduce the proof of Proposition 2.5 to a special case.

LEMMA 2.6. *Suppose that* $F_1 \subset F_2$ — *then the set*

$$\{\lambda_i^{F_1}: i \in F_2 - F_1\}$$

is a basis of $V_{F_1} \cap V(F_2)$ *with associated dual basis*

$$\{\lambda^i_{F_2}: i \in F_2 - F_1\}.$$

Proof. Since F_2 contains F_1, $V(F_1)$ is contained in $V(F_2)$. Let $i \in F_2 - F_1$ — then, by definition, $\lambda_i^{F_1} \in V_{F_1}$. On the other hand, $\lambda_i - \lambda_i^{F_1} \in V(F_1) \subset V(F_2)$, $\lambda_i \in V(F_2)$, so $\lambda_i^{F_1} \in V(F_2)$. Therefore, by dimension, the set

$$\{\lambda_i^{F_1}: i \in F_2 - F_1\}$$

is a basis of $V_{F_1} \cap V(F_2)$. As for the assertion regarding the dual basis, let again $i \in F_2 - F_1$ — then, by definition, $\lambda^i_{F_2} \in V(F_2)$. On the other hand,

$$(\lambda^i_{F_2}, \lambda_j^{F_2}) = 0 \quad \forall j \in F_1.$$

But $\lambda_j^{F_2} = \lambda_j$ ($j \in F_1$), the latter spanning $V(F_1)$, so $\lambda^i_{F_2} \in V_{F_1}$. Therefore, by dimension, the set

$$\{\lambda^i_{F_2}: i \in F_2 - F_1\}$$

is a basis of $V_{F_1} \cap V(F_2)$. Finally, $\lambda^i_{F_2}$ being the orthogonal projection of $\lambda^i_{F_1}$ onto $V(F_2)$, the difference $\lambda^i_{F_2} - \lambda^i_{F_1}$ is orthogonal to every $\lambda_j^{F_1}$ ($j \in F_2 - F_1$) (since they lie in $V(F_2)$). Thus, $\forall\, i, j \in F_2 - F_1$,

$$(\lambda_i^{F_1}, \lambda^j_{F_2}) = (\lambda_i^{F_1}, \lambda^j_{F_1}) = \delta_{ij},$$

proving that the bases are in fact dual to one another. □

LEMMA 2.7. *Suppose that* $F_1 \subset F \subset F_2$ — *then*

(1) $\forall\, i \in F - F_1$, *the orthogonal projection of* $\lambda^i_{F_2}$ *on* $\operatorname{span}\{\lambda_j^{F_1}: j \in F - F_1\}$ *is* λ^i_F;

(2) $\forall\, i \in F_2 - F$, *the orthogonal projection of* $\lambda_i^{F_1}$ *on the orthogonal complement of* $\operatorname{span}\{\lambda_j^{F_1}: j \in F - F_1\}$ *in* $V_{F_1} \cap V(F_2)$ *is* λ_i^F.

Proof. Making a change in the notation, it follows from the preceding lemma that

$$\begin{cases} i \in F - F_1 \Rightarrow \lambda^i_F \in V_{F_1} \cap V(F) \subset V_{F_1} \cap V(F_2) \\ i \in F_2 - F \Rightarrow \lambda_i^F \in V_F \cap V(F_2) \subset V_{F_1} \cap V(F_2), \end{cases}$$

so the relevant λ^i_F and λ_i^F appearing in (1) and (2) all do lie in $V_{F_1} \cap V(F_2)$. Furthermore,

$$\begin{cases} i \in F - F_1 \Rightarrow \lambda^i_F \in V_{F_1} \cap V(F) = \operatorname{span}\{\lambda_j^{F_1} : j \in F - F_1\} \\ i \in F_2 - F \Rightarrow \lambda_i^F \in V_F \cap V(F_2) \perp V_{F_1} \cap V(F). \end{cases}$$

There remains only to show, therefore, that

$$\begin{cases} i \in F - F_1 \Rightarrow \lambda^i_{F_2} - \lambda^i_F \perp \operatorname{span}\{\lambda_j^{F_1} : j \in F - F_1\} \\ i \in F_2 - F \Rightarrow \lambda_i^F - \lambda_i^{F_1} \in \operatorname{span}\{\lambda_j^{F_1} : j \in F - F_1\}. \end{cases}$$

We have, however,

$$i \in F - F_1 \Rightarrow \begin{cases} \lambda^i_{F_2} - \lambda^i \in V_{F_2} \subset V_F \\ \lambda^i_F - \lambda^i \in V_F, \end{cases}$$

so

$$\lambda^i_{F_2} - \lambda^i_F \in V_F \perp V_{F_1} \cap V(F) = \operatorname{span}\{\lambda_j^{F_1} : j \in F - F_1\},$$

while

$$i \in F_2 - F \Rightarrow \begin{cases} \lambda_i^F - \lambda_i \in V(F) \\ \lambda_i^{F_1} - \lambda_i \in V(F_1) \subset V(F), \end{cases}$$

so

$$\lambda_i^F - \lambda_i^{F_1} \in V_{F_1} \cap V(F) = \operatorname{span}\{\lambda_j^{F_1} : j \in F - F_1\},$$

completing the proof. □

Turning back to Proposition 2.5, the preceding lemma implies that the value of the sum in question at a particular $H \in V$ depends only on its projection onto $V_{F_1} \cap V(F_2)$. Upon making the replacement of data

$$\begin{cases} V \to V_{F_1} \cap V(F_2) \\ \{\lambda_i : 1 \le i \le l\} \to \{\lambda_i^{F_1} : i \in F_2 - F_1\}, \end{cases}$$

we thereby reduce our proof to the special case when $F_1 = \varnothing$, $F_2 = \{1, \ldots, l\}$. Let us agree to write $\mathfrak{L}$ for the set $\{1, \ldots, l\}$, $\mathcal{P}_{\mathfrak{L}}$ for the power set of $\mathfrak{L}$. Abbreviating

$$\tau_{\varnothing, F}(F_0 : ?) \quad \text{to} \quad \tau_{*, F}(F_0 : ?),$$

we must show that for all $H \in V$,

$$\sum_{F \in \mathcal{P}_{\mathfrak{L}}} (-1)^{\#(F \cap F_0)} \tau_{*,F}(F_0 : H) \chi_{F,*}(H)$$

is equal to

$$\begin{cases} 1 & \text{if } F_0 = \varnothing \\ 0 & \text{if } F_0 \neq \varnothing. \end{cases}$$

This will be done by induction on the dimension of V. The case $\dim(V) = 1$ is clear enough. Inductively, then, we may assume that $\forall F_2 \subset \mathfrak{L}$, $F_2 \neq \mathfrak{L}$,

$$\sum_{\{F:\, F \subset F_2\}} (-1)^{\#(F \cap F_0)} \tau_{*,F}(F_0 : H) \chi_{F,2}(H)$$

is equal to

$$\begin{cases} 1 & \text{if } F_0 \cap F_2 = \varnothing \\ 0 & \text{if } F_0 \cap F_2 \neq \varnothing. \end{cases}$$

This said, suppose that F is a proper subset of $\mathfrak{L}$, F_2 a subset of $\mathfrak{L}$ containing F — then

$$\#(F_2 - F) + \#(\mathfrak{L} - F) = \#(\mathfrak{L} - F_2) + 2(\#(F_2) - \#(F)).$$

Accordingly,

$$\begin{aligned} 0 &= (-1)^{\#(\mathfrak{L} - F)} \sigma_F^F(H) \\ &= \sum_{\{F_2:\, F_2 \supset F\}} (-1)^{\#(\mathfrak{L} - F_2)} \chi_{F,2}(H) \cdot \chi^{2,*}(H). \end{aligned}$$

But, when F is all of $\mathfrak{L}$,

$$1 = (-1)^{\#(\mathfrak{L} - F)} \sigma_F^F(H).$$

These remarks make it clear that if $c_F (F \in \mathcal{P}_{\mathfrak{L}})$ are complex numbers, then

$$\sum_{F \in \mathcal{P}_{\mathfrak{L}}} c_F \sum_{\{F_2:\, F_2 \supset F\}} (-1)^{\#(\mathfrak{L} - F_2)} \chi_{F,2}(H) \cdot \chi^{2,*}(H)$$

is equal to $c_{\mathfrak{L}}$. Specialize and take

$$c_F = (-1)^{\#(F \cap F_0)} \tau_{*,F}(F_0 : H).$$

We can then say that the sum over all $F \in \mathcal{P}_{\mathfrak{L}}$ of

$$(-1)^{\#(F \cap F_0)} (-1)^{\#(\mathfrak{L} - F_2)} \tau_{*,F}(F_0 : H) \chi_{F,2}(H) \cdot \chi^{2,*}(H)$$

summed over all F_2 containing a given F is the same as

$$(-1)^{\#(F_0)}\tau_{*,\mathfrak{L}}(F_0 : H).$$

Now reverse the order of summation, splitting off the term whose value we are attempting to calculate, viz. the one corresponding to $F_2 = \mathfrak{L}$. In this way, we find that

$$(-1)^{\#(F_0)}\tau_{*,\mathfrak{L}}(F_0 : H)$$

is given by

$$\sum_{F \in \mathcal{P}_{\mathfrak{L}}} (-1)^{\#(F \cap F_0)}\tau_{*,F}(F_0 : H)\chi_{F,*}(H)$$

plus the sum over all $F_2 \neq \mathfrak{L}$ of

$$(-1)^{\#(\mathfrak{L}-F_2)}\chi^{2,*}(H)$$

times

$$\sum_{\{F:\, F \subset F_2\}} (-1)^{\#(F \cap F_0)}\tau_{*,F}(F_0 : H)\chi_{F,2}(H),$$

the last sum being, thanks to the induction hypothesis, 1 or 0, depending on whether $F_0 \cap F_2 = \varnothing$ or $F_0 \cap F_2 \neq \varnothing$. There are then two possibilities.

$\underline{F_0 = \varnothing}$. In this case,

$$\sum_{F \in \mathcal{P}_{\mathfrak{L}}} (-1)^{\#(F \cap F_0)}\tau_{*,F}(F_0 : H)\chi_{F,*}(H)$$

is equal to

$$1 + \tau_{*,\mathfrak{L}}(\varnothing : H) - \sum_{F_2 \in \mathcal{P}_{\mathfrak{L}}} (-1)^{\#(\mathfrak{L}-F_2)}\chi^{2,*}(H).$$

$\underline{F_0 \neq \varnothing}$. In this case,

$$\sum_{F \in \mathcal{P}_{\mathfrak{L}}} (-1)^{\#(F \cap F_0)}\tau_{*,F}(F_0 : H)\chi_{F,*}(H)$$

is equal to

$$(-1)^{\#(F_0)}\tau_{*,\mathfrak{L}}(F_0 : H) - \sum_{\{F_2:\, F_2 \subset \mathfrak{L}-F_0\}} (-1)^{\#(\mathfrak{L}-F_2)}\chi^{2,*}(H).$$

Everything thus comes down to the following lemma.

LEMMA 2.8. *Let* $F_0 \in \mathcal{P}_{\mathfrak{L}}$ — *then*

$$(-1)^{\#(F_0)}\tau_{*,\mathfrak{L}}(F_0 : H)$$

and

$$\sum_{\{F_2:\, F_2 \subset \mathfrak{L} - F_0\}} (-1)^{\#(\mathfrak{L} - F_2)}\chi^{2,*}(H)$$

are equal.

Proof. Since

$$\#(\mathfrak{L}) + \#(\mathfrak{L} - F_0) + \#(\mathfrak{L} - F_2) = \#((\mathfrak{L} - F_0) - F_2) + 2(\#(\mathfrak{L})),$$

it will be enough to show that

$$\tau_{*,\mathfrak{L}}(\mathfrak{L} - F_0 : H) = \sum_{\{F_2:\, F_2 \subset F_0\}} (-1)^{\#(F_0 - F_2)}\chi^{2,*}(H).$$

By definition, $\tau_{*,\mathfrak{L}}(\mathfrak{L} - F_0 : ?)$ is the characteristic function of the set $\mathfrak{T}_{*,\mathfrak{L}}(\mathfrak{L} - F_0)$ of all $H \in V$ such that $\forall\, i \in \mathfrak{L}$:

$$\begin{cases} i \in \mathfrak{L} - F_0 \Rightarrow (\lambda^i, H) > 0 \\ i \notin \mathfrak{L} - F_0 \Rightarrow (\lambda^i, H) \le 0. \end{cases}$$

On the other hand, $\chi^{2,*}$ is, by definition, the characteristic function of the set

$$\{H \in V\colon (\lambda^i, H) > 0\ (i \notin F_2)\}.$$

Now fix an $H_0 \in V$. Put

$$F_{2,0}(H_0) = \{i \in F_0\colon (\lambda^i, H_0) \le 0\}.$$

Since $F_2 \subset F_0$, $\chi^{2,*}(H_0) = 0$ unless $F_{2,0}(H_0) \subset F_2$. Therefore

$$\sum_{\{F_2:\, F_2 \subset F_0\}} (-1)^{\#(F_0 - F_2)}\chi^{2,*}(H_0)$$

$$= \sum_{\{F_2:\, F_{2,0}(H_0) \subset F_2 \subset F_0\}} (-1)^{\#(F_0 - F_2)} \begin{cases} 1 & \text{if } (\lambda^i, H_0) > 0\ \forall\, i \notin F_0 \\ 0 & \text{otherwise.} \end{cases}$$

Owing to Lemma 2.1,

$$\sum_{\{F_2:\, F_{2,0}(H_0) \subset F_2 \subset F_0\}} (-1)^{\#(F_0 - F_2)} = \begin{cases} 1 & \text{if } F_{2,0}(H_0) = F_0 \\ 0 & \text{if } F_{2,0}(H_0) \ne F_0. \end{cases}$$

In other words,

$$\sum_{\{F_2 : F_2 \subset F_0\}} (-1)^{\#(F_0 - F_2)} \chi^{2,*}(H_0)$$

is equal to the product of

$$\begin{cases} 1 & \text{if } (\lambda^i, H_0) > 0 \ \forall\, i \notin F_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{cases} 1 & \text{if } (\lambda^i, H_0) \leq 0 \ \forall\, i \in F_0 \\ 0 & \text{otherwise,} \end{cases}$$

that is, to

$$\tau_{*,\mathfrak{L}}(\mathfrak{L} - F_0 : H_0),$$

as desired. □

The fact that

$$\sum_{F \in \mathcal{P}_{\mathfrak{L}}} \tau_{*,F}(\varnothing : H)\chi_{F,*}(H) = 1$$

for all $H \in V$ leads to a partitioning of V into disjoint subsets which will be exploited in particular cases in the next section. Indeed, $\tau_{*,F}(\varnothing : ?)\chi_{F,*}(?)$ is the characteristic function of the set $V\langle F \rangle$ of all H in V such that

$$\begin{cases} (\lambda^i_F, H) \leq 0 & \forall\, i \in F \\ (\lambda^F_i, H) > 0 & \forall\, i \notin F. \end{cases}$$

Consequently (cf. supra)

$$V = \coprod_{F \in \mathcal{P}_{\mathfrak{L}}} V\langle F \rangle.$$

Our next objective will be to obtain a decomposition of $V\langle F \rangle$, this time as a direct sum. Let

$$\begin{cases} \mathcal{C}(F) = \text{positive chamber in } V(F) \\ \mathcal{D}(F) = \text{positive cone in } V(F); \end{cases}$$

let

$$\begin{cases} \mathcal{C}_F = \text{positive chamber in } V_F \\ \mathcal{D}_F = \text{positive cone in } V_F. \end{cases}$$

The reader will recall that the second pair of entities figured earlier in the proof of Proposition 2.3.

PROPOSITION 2.9. *Let $F \in \mathcal{P}_{\mathfrak{L}}$ — then*

$$V\langle F\rangle = (-\mathcal{D}(F))^{\check{}} \oplus \mathcal{C}_F.$$

Proof. Let $H \in V\langle F\rangle$. Write $H = H(F) + H_F$ where $H(F) \in V(F)$, $H_F \in V_F$ — then we have to prove that

$$\begin{cases} H(F) \in (-\mathcal{D}(F))^{\check{}} \\ H_F \in \mathcal{C}_F. \end{cases}$$

Let $i \in F$ — then

$$\left(\lambda^i_F, H\right) = \left(P(F)\lambda^i, H\right) = \left(\lambda^i_F, H(F)\right).$$

But

$$\begin{aligned} H \in V\langle F\rangle \\ &\Rightarrow \left(\lambda^i_F, H\right) \le 0 \\ &\Rightarrow \left(\lambda^i_F, H(F)\right) \le 0 \\ &\Rightarrow H(F) \in (-\mathcal{D}(F))^{\check{}}. \end{aligned}$$

Let $i \notin F$ — then

$$\left(\lambda^F_i, H\right) = \left(P_F\lambda_i, H\right) = \left(\lambda^F_i, H_F\right).$$

But

$$\begin{aligned} H \in V\langle F\rangle \\ &\Rightarrow \left(\lambda^F_i, H\right) > 0 \\ &\Rightarrow \left(\lambda^F_i, H_F\right) > 0 \\ &\Rightarrow H_F \in \mathcal{C}_F. \end{aligned}$$

As the argument is evidently reversible, the proposition is established. □

We shall also need a description of the intersection of $\mathcal{C}$ with a translate of $V\langle F\rangle$ by an element of $\mathcal{C}$.

PROPOSITION 2.10. *Let $H \in \mathcal{C}$, say $H = H(F) + H_F$ where $H(F) \in V(F)$, $H_F \in V_F$ — then*

$$(H + V\langle F\rangle) \cap \mathcal{C}$$

is equal to

$$\left(H(F) + (-\mathcal{D}(F))^{\check{}}\right) \cap \mathcal{C}(F) \oplus \left(H_F + \mathcal{C}_F\right) \qquad \left(F \in \mathcal{P}_{\mathfrak{L}}\right).$$

There is a simple generality which must be dealt with first.

LEMMA 2.11. *Fix an element $H_0 \in \mathcal{C}$. Suppose that $H \in V$ has the following properties*:

(i) $(\lambda_i, H) > 0 \ \forall\, i \in F$;

(ii) $H - H_0 \in V\langle F\rangle$.

Then

$$(\lambda_i, H) > (\lambda_i, H_0) > 0 \quad \forall\, i \notin F.$$

[In consequence, therefore: $H \in \mathcal{C}$.]

Proof. Suppose that $i \notin F$— then $\lambda_i - \lambda_i^F \in V(F)$ so there exist constants $c_{ij} \leq 0$ such that

$$\lambda_i = \lambda_i^F + \sum_{j \in F} c_{ij} \lambda_F^j.$$

Taking into account (i) and (ii), we then find that

$$\begin{aligned}(\lambda_i, H) &= (\lambda_i^F, H) + \sum_{j \in F} c_{ij}(\lambda_F^j, H)\\ &> (\lambda_i^F, H_0) + \sum_{j \in F} c_{ij}(\lambda_F^j, H)\\ &\geq (\lambda_i^F, H_0) + \sum_{j \in F} c_{ij}(\lambda_F^j, H_0)\\ &= (\lambda_i, H_0) > 0.\end{aligned}$$

Hence the lemma. □

Proof of Proposition 2.10. It is clear that

$$(H + V\langle F\rangle) \cap \mathcal{C}$$

is contained in

$$\left(H(F) + (-\mathcal{D}(F))^{\hat{}}\right) \cap \mathcal{C}(F) \oplus \left(H_F + \mathcal{C}_F\right).$$

To go the other way, consider an element

$$H(F) + H^0(F) + H_F + H_F^0$$

where

$$H^0(F) \in (-\mathcal{D}(F))^{\hat{}}, \qquad H_F^0 \in \mathcal{C}_F$$

and

$$H(F) + H^0(F) \in \mathcal{C}(F).$$

Put $H^0 = H^0(F) + H_F^0$ — then $H^0 \in V\langle F\rangle$ (cf. Proposition 2.9). We have only to show, therefore, that $H + H^0 \in \mathcal{C}$. For this purpose, we shall use the preceding lemma. Let $i \in F$ — then

$$(\lambda_i, H + H^0) = (\lambda_i, H(F) + H^0(F)) > 0.$$

On the other hand, $(H + H^0) - H = H^0 \in V\langle F\rangle$, so $H + H^0 \in \mathcal{C}$, this being the case of H. □

3. Reduction theory. The purpose of this section will be to first establish the assumptions and notation and to recall the main points from reduction theory. This done, we shall then have to break new ground by formulating and proving a rather delicate refinement of the fundamental theorem of reduction. In this connection, the Combinatorial Lemma of Langlands plays an important role. It should be remarked that Langlands himself had made a start on the theorem in question (cf. [**2.b**]) but did not pursue the matter beyond the 'one dimensional' case (which is relatively simple to deal with directly). Even so, this weak version had applications. Langlands used it to make certain important estimates in the theory of Eisenstein series while Harish-Chandra used it to prove the Maass-Selberg relations. We shall need the full strength of the theorem to define and develop the properties of the truncation operator as well as to handle questions related to it. Let us also mention that Arthur [**1.a**] has obtained an adelic analogue of our result but, as always, the setting there is, for structural reasons, considerably less complex than the one to be found here.

It will be necessary to suppose that the reader has some acquaintance with our memoir [**3.a**] to which we refer for details and elaboration insofar as the background material set forth below is concerned.

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$. We shall assume that G is admissible in that it satisfies the following conditions:

(i) The adjoint group of G is contained in the adjoint group of the complexification of $\mathfrak{g}$;

(ii) The analytic subgroup of G associated with the derived algebra of $\mathfrak{g}$ has finite center;

(iii) The identity component of G is of finite index in G.

The above assumptions on G are, of course, those generally imposed by Harish-Chandra. One then introduces in the usual way:

K — a maximal compact subgroup of G;

θ — an involutive automorphism of G with fixed point set K;

B — a real nondegenerate symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ such that:

$$\begin{cases} B(\mathrm{Ad}(x)X_1, \mathrm{Ad}(x)X_2) = B(X_1, X_2) & (x \in G;\ X_1, X_2 \in \mathfrak{g}) \\ B(\theta X_1, \theta X_2) = B(X_1, X_2) & (X_1, X_2 \in \mathfrak{g}) \\ -B(X, \theta X) > 0 & (X \in \mathfrak{g}). \end{cases}$$

In particular, the bilinear form

$$(X_1, X_2)_\theta = -B(X_1, \theta X_2) \qquad (X_1, X_2 \in \mathfrak{g})$$

equips $\mathfrak{g}$ with the structure of a real Hilbert space.

Let now Γ be a lattice in G subject to the fundamental assumption imposed by us in **[3.a]** — then one may associate with Γ a certain collection of split parabolic subgroups (P, S) of G, said to be Γ-cuspidal, the minimal elements for the relation of succession then being termed Γ-percuspidal. Given a pair (P, S) with split component A and corresponding centralizer L, introduce, as usual, the associated admissible closed reductive subgroup M of G so that $L = M \cdot A$ with $M \cap A = \{1\}$. Denoting by N the unipotent radical of P, the Langlands decomposition of P per the split component A is given by $P = M \cdot A \cdot N$. In passing, recall that $S = M \cdot N$, hence that S is a closed normal subgroup of P which is uniquely determined by P and A. M, being an admissible reductive Lie group, has the same general properties as G, hence the symbols K_M, θ_M, B_M are to be assigned the obvious interpretations. We shall often identify, without specific comment, M with S/N; K_M is then identified with the image of $K \cap S$ in S/N. Put $\Gamma_M = M \cap \Gamma \cdot N$ — then Γ_M is a discrete subgroup of M and, in fact, is actually a lattice in M which is uniform iff P is Γ-percuspidal. The pair (M, Γ_M) thus satisfies the same general conditions as the pair (G, Γ), a point crucial for inductive arguments. One should also note that A is not uniquely determined by the pair (P, S). In fact, the conjugates nAn^{-1} $(n \in N)$ constitute the set of split components of (P, S). Among the split components of (P, S) there is one and only one which is θ-stable. We shall refer to it as the special split component of (P, S). The rank of (P, S) is, by definition, the dimension of a split component.

Let $E(G, \Gamma)$ be the set of Γ-percuspidal split parabolic subgroups of G — then, modulo Γ-conjugacy, there are but finitely many elements of $E(G, \Gamma)$. Furthermore, any two elements of $E(G, \Gamma)$ are strongly conjugate, thus, in particular, have the same rank. Let $(P, S) \in E(G, \Gamma)$ with split component A; let $W(A)$ be the Weyl group of A, i.e. the quotient of the normalizer of A in G by the centralizer of A in G — then, while the disjoint union

$$\bigcup_{w \in W(A)} PwP$$

need not fill out G so that the Bruhat lemma is not literally valid, nevertheless it is true that

$$\Gamma \subset \bigcup_{w \in W(A)} PwP.$$

By the rank of Γ, we understand the rank of any element of $E(G, \Gamma)$. In particular: $\operatorname{rank}(\Gamma) = 0$ iff Γ is uniform in G. Accordingly, we shall suppose henceforth that the rank of Γ is not less than one.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A — then

$$\begin{cases} \Sigma_P(\mathfrak{g}, \mathfrak{a}) \\ \Sigma_P^0(\mathfrak{g}, \mathfrak{a}) \end{cases}$$

stand for the roots, respectively simple roots, of $(P, S; A)$. Given $\lambda \in \Sigma_P(\mathfrak{g}, \mathfrak{a})$, let $\xi_\lambda\colon A \to \mathbf{R}^+$ be the associated quasi-character of A. For any $t > 0$, put

$$A_\lambda[t] = \{a \in A\colon \xi_\lambda(a) \leq t\}$$

and then set

$$A[t] = \bigcap_{\lambda \in \Sigma_P^0(\mathfrak{g}, \mathfrak{a})} A_\lambda[t].$$

If ω be a compact neighborhood of 1 in S, then

$$\mathfrak{S}_{t,\omega} = K \cdot A[t] \cdot \omega$$

is called a Siegel domain in G (relative to $(P, S; A)$). It is a standard simple fact that

$$\bigcup_{a \in A[t]} a\omega a^{-1}$$

is relatively compact.

We shall now formulate the fundamental theorem of reduction, as spelled out in [3.a]. Let r_0 be the number of Γ-inequivalent cusps. Fix an element (P_0, S_0) in $E(G, \Gamma)$ — then one can choose elements k_{i_0} in K $(k_1 = 1)$ such that the conjugates $P_{i_0} = k_{i_0} P_0 k_{i_0}^{-1}$ $(i_0 = 1, \ldots, r_0)$ form a complete set of representatives for the Γ-conjugacy classes in $E(G, \Gamma)$. Let $P_0 = M_0 \cdot A_0 \cdot N_0$ be the Langlands decomposition of P_0 per the special split component A_0 — then each P_{i_0} admits a Langlands decomposition $P_{i_0} = M_{i_0} \cdot A_{i_0} \cdot N_{i_0}$ where

$$\begin{cases} M_{i_0} = k_{i_0} M_0 k_{i_0}^{-1} \\ A_{i_0} = k_{i_0} A_0 k_{i_0}^{-1} \\ N_{i_0} = k_{i_0} N_0 k_{i_0}^{-1}, \end{cases}$$

A_{i_0} being the special split component of P_{i_0} $(i_0 = 1, \ldots, r_0)$. Put

$$\kappa_{i_0} = k_{i_0}^{-1}, \quad \mathfrak{s}_0 = \{\kappa_{i_0}\colon 1 \leq i_0 \leq r_0\}.$$

THEOREM 3.1. *There exists a Siegel domain* $\mathfrak{S}_{t_0,\omega_0}$ *relative to* $(P_0, S_0; A_0)$ *such that the set*

$$\mathfrak{S}_0 = \mathfrak{S}_{t_0,\omega_0} \cdot \mathfrak{s}_0$$

has the following properties

(i) $\mathfrak{S}_0 \cdot \Gamma = G$;

(ii) $\#(\{\gamma \in \Gamma: \mathfrak{S}_0\gamma \cap \mathfrak{S}_0 \neq \varnothing\}) < +\infty$.

Moreover, there exists ${}_0t < t_0$ *such that if* $\gamma \in \Gamma$:

(iii) $\mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}\gamma \cap \mathfrak{S}_{{}_0t,\omega_0}\kappa_{j_0} = \varnothing \ (i_0 \neq j_0)$;

(iv) $\mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}\gamma \cap \mathfrak{S}_{{}_0t,\omega_0}\kappa_{i_0} \neq \varnothing \Rightarrow \gamma \in \Gamma \cap P_{i_0}$.

[Tacitly, we suppose ω_0 is chosen in such a way that the K-conjugates $\omega_{i_0} = k_{i_0}\omega_0 k_{i_0}^{-1}$ contain a fundamental domain for the action of $S_{i_0} \cap \Gamma$ on S_{i_0} $(i_0 = 1, \ldots, r_0)$.]

To even state our refinement of the fundamental theorem of reduction requires a fair amount of preparation which will now be undertaken. It is perhaps appropriate to remark that in the event that rank$(\Gamma) = 1$, one need not proceed further: In that special case, the required result follows directly from the theorem supra (cf. [**3.b**]).

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A, $P = M \cdot A \cdot N$ the corresponding Langlands decomposition of P. Suppose that (P_0, S_0) is a dominated predecessor of (P, S) — then one can associate with (P_0, S_0) a Γ_M-cuspidal split parabolic group $(P_0^\dagger, S_0^\dagger)$ of M given by

$$\begin{cases} P_0^\dagger = P_0 \cap S/N \\ S_0^\dagger = S_0/N. \end{cases}$$

The correspondence

$$(P', S') \leftrightarrow ('P, 'S)$$

where

$$\begin{cases} 'P = P' \cap S/N \\ 'S = S'/N, \end{cases}$$

is one-to-one between the set of Γ-cuspidal split parabolic subgroups of G which are dominated predecessors of (P, S) and the set of Γ_M-cuspidal split parabolic subgroups of M. This correspondence preserves per-cuspidality. If

$$(P, S; A) \succcurlyeq (P', S'; A'),$$

there are Langlands decompositions

$$P' = M' \cdot A' \cdot N', \quad 'P = 'M \cdot 'A \cdot 'N$$

characterized by the relations

$$\begin{cases} M' = 'M, & A' = 'A \cdot A, & N' = 'N \cdot N \\ 'M = M', & 'A = M \cap A', & 'N = M \cap N'. \end{cases}$$

On the other hand, one may attach to each subset F of $\Sigma_P^0(\mathfrak{g}, \mathfrak{a})$ a Γ-cuspidal split parabolic subgroup (P_F, S_F) of G with split component A_F such that

$$(P_F, S_F; A_F) \succcurlyeq (P, S; A).$$

The map

$$F \mapsto (P_F, S_F; A_F)$$

sets up a bijection between the subsets of $\Sigma_P^0(\mathfrak{g}, \mathfrak{a})$ and the dominant successors of (P, S) per the initial link A. Let $l = \text{rank}(P, S)$ — then the $2^l(P_F, S_F; A_F)$ so obtained comprise the 'standard picture' over $(P, S; A)$.

Let (P_1, S_1), (P_2, S_2) be two Γ-cuspidal split parabolic subgroups of G with split components A_1 and A_2. Call $W(A_2, A_1)$ the set of all bijections w: $A_1 \to A_2$ induced by an inner automorphism of G — then $W(A_2, A_1)$ is a finite set. (P_1, S_1) and (P_2, S_2) are said to be associate if $W(A_2, A_1)$ is not empty.

The relation of association breaks up the Γ-cuspidal split parabolic subgroups of G into equivalence classes. Fix one such, say $\mathcal{C}$. Let $\mathcal{C}_i$ be a G-conjugacy class in $\mathcal{C}$. Let

$$\begin{cases} (P_1, S_1; A_1) \\ (P_2, S_2; A_2) \end{cases}$$

be members of $\mathcal{C}_i$. We then define an element

$$I(P_2 \mid A_2 : P_1 \mid A_1) \in W(A_2, A_1)$$

as follows (cf. [**3.a**]). Select x in G with the property that

$$x(P_1, S_1; A_1)x^{-1} = (P_2, S_2; A_2).$$

Put

$$I(P_2 \mid A_2 : P_1 \mid A_1) = \text{Int}(x) \mid A_1,$$

a definition independent of the choice of x. There are certain elementary properties inherent in this construction, e.g. transitivity. Less elementary but still easy are the conditions of descent.

SUBLEMMA. *Let* $\mathcal{C}_1$, $\mathcal{C}_2$ *be association classes of* Γ*-cuspidal split parabolic subgroups of* G, $\mathcal{C}_{i_1}$, $\mathcal{C}_{i_2}$ *G-conjugacy classes in* $\mathcal{C}_1$, $\mathcal{C}_2$. *Let*

$$\begin{cases} (P_1', S_1'; A_1') \\ (P_1'', S_1''; A_1'') \end{cases} \in \mathcal{C}_{i_1}, \qquad \begin{cases} (P_2', S_2'; A_2') \\ (P_2'', S_2''; A_2'') \end{cases} \in \mathcal{C}_{i_2}$$

with

$$\begin{cases} (P_1', S_1'; A_1') \succcurlyeq (P_2', S_2'; A_2') \\ (P_1'', S_1''; A_1'') \succcurlyeq (P_2'', S_2''; A_2''). \end{cases}$$

Then

$$I(P_2''\,|\,A_2'' : P_2'\,|\,A_2')\,|\,A_1' = I(P_1''\,|\,A_1'' : P_1'\,|\,A_1').$$

SUBLEMMA. *Let* $\mathcal{C}_1$, $\mathcal{C}_2$ *be association classes of* Γ*-cuspidal split parabolic subgroups of* G, $\mathcal{C}_{i_1}$, $\mathcal{C}_{i_2}$ *G-conjugacy classes in* $\mathcal{C}_1$, $\mathcal{C}_2$. *Let*

$$(P_1, S_1; A_1) \in \mathcal{C}_{i_1}, \qquad \begin{cases} (P_2', S_2'; A_2') \\ (P_2'', S_2''; A_2'') \end{cases} \in \mathcal{C}_{i_2}$$

with

$$\begin{cases} (P_1, S_1, A_1) \succcurlyeq (P_2', S_2'; A_2') \\ (P_1, S_1; A_1) \succcurlyeq (P_2'', S_2''; A_2''). \end{cases}$$

Then

$$I(P_2''\,|\,A_2'' : P_2'\,|\,A_2')\,|\,'A_2 = I(''P_2\,|\,''A_2 : 'P_2\,|\,'A_2).$$

Both of these facts will be used without comment in what follows.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A — then $G = K \cdot P$ and $P = A \cdot S$. Let $x \in G$ — then x admits a decomposition

$$x = k_x a_x s_x$$

where $k_x \in K$, $a_x \in A$, $s_x \in S$. The factor a_x is unique, thus determines an element $H_x \in \mathfrak{a}$ such that

$$a_x = \exp(H_x).$$

It will sometimes be convenient to write $H(x)$ in place of H_x or even, when P and A need to be emphasized, $H_{P|A}(x)$. If Λ is a linear function on $\mathfrak{a}$ (possibly complex valued), then Λ determines a quasi-character ξ_Λ on A. We shall often write a_x^Λ in place of $\xi_\Lambda(a_x)$.

Fix a set of representatives

$$\{(P_m^{\max}, S_m^{\max})\}$$

for the Γ-conjugacy classes of maximal Γ-cuspidal split parabolic subgroups of G. Let $A_m^{\max}$ be the special split component of $(P_m^{\max}, S_m^{\max})$. Put

$$\mathfrak{a} = \bigoplus_m \mathfrak{a}_m^{\max}.$$

Given a Γ-cuspidal split parabolic subgroup (P, S) of G with special split component A, our first task will be to define a map

$$I_P\colon \mathfrak{a} \to \mathfrak{a}.$$

This is done as follows. Let $l = \operatorname{rank}(P, S)$. Denote by $(P_1, S_1; A_1), \ldots, (P_l, S_l; A_l)$ the maximal Γ-cuspidal split parabolic subgroups of G sitting in the standard picture over $(P, S; A)$. If $\{\lambda_1, \ldots, \lambda_l\}$ are the simple roots of $(P, S; A)$, then it can be supposed that they are ordered in such a way that

$$\mathfrak{a}_\mu = \bigcap_{\nu \neq \mu} \operatorname{Ker}(\lambda_\nu).$$

Since

$$\mathfrak{a} = \bigoplus_\mu \mathfrak{a}_\mu,$$

given $\mathbf{H} \in \mathfrak{a}$, $I_P(\mathbf{H})$ is determined when its orthogonal projection onto each $\mathfrak{a}_\mu$ is specified. There exist elements $\gamma_1, \ldots, \gamma_l$ in Γ and indices $m(1), \ldots, m(l)$ such that

$$\gamma_\mu P_\mu \gamma_\mu^{-1} = P_{m(\mu)}^{\max}.$$

This said, we then require that the μ-component of $I_P(\mathbf{H})$ be the vector

$$I\big(P_\mu \,|\, A_\mu : P_{m(\mu)}^{\max} \,|\, A_{m(\mu)}^{\max}\big)\mathbf{H}_{m(\mu)} + H_{P_\mu|A_\mu}(\gamma_\mu).$$

We explicitly observe that our definition does in fact make sense. For if γ_μ', γ_μ'' are two conjugators, then

$$\begin{aligned} & (\gamma_\mu')^{-1}\gamma_\mu'' \in \Gamma \cap P_\mu \subset S_\mu \\ \Rightarrow\ & H_{P_\mu|A_\mu}(\gamma_\mu') = H_{P_\mu|A_\mu}(\gamma_\mu''). \end{aligned}$$

Let us also note that

$$I_{P_\mu}(\mathbf{H}) = I_P(\mathbf{H})_\mu \quad \forall\, \mu,$$

that is, the μ-component of $I_P(\mathbf{H})$ is precisely $I_{P_\mu}(\mathbf{H})$.

There is a simple formula for I_P in terms of the root data associated with $(P, S; A)$. Under the usual identification of $\mathfrak{a}$ with its dual $\check{\mathfrak{a}}$, let $H_1, \ldots, H_l$ be the elements corresponding to $\lambda_1, \ldots, \lambda_l$ — then the span of

$$\{H_1, \ldots, H_{\mu-1}, H_{\mu+1}, \ldots, H_l\}$$

is the orthogonal complement of $\mathfrak{a}_\mu$. Consequently,

$$I_P(\mathbf{H}) = \sum_{\mu=1}^{l} \langle I_{P_\mu}(\mathbf{H}), \lambda^\mu \rangle H_\mu \qquad (\mathbf{H} \in \mathfrak{a}).$$

Here, of course, $\lambda^1, \ldots, \lambda^l$ is the ambient dual basis.

Let $(P, S; A)$ be a Γ-cuspidal split parabolic subgroup of G with split component A — then, as always

$$\begin{cases} \mathcal{C}_P(\mathfrak{a}) \\ \mathcal{D}_P(\mathfrak{a}) \end{cases}$$

denote, respectively, the positive chamber or positive cone of $(P, S; A)$.

We shall now introduce an important definition. Given $\mathbf{H}_1$, $\mathbf{H}_2$ in $\mathfrak{a}$, write

$$\mathbf{H}_1 < \mathbf{H}_2$$

if for every Γ-cuspidal split parabolic subgroup (P, S) of G with special split component A it is true that

$$I_P(\mathbf{H}_2) \in I_P(\mathbf{H}_1) + \mathcal{C}_P(\mathfrak{a}).$$

This relation partially orders $\mathfrak{a}$.

LEMMA 3.2. *Let* $\mathbf{H}_1, \mathbf{H}_2 \in \mathfrak{a}$ — *then*

$$\mathbf{H}_1 < \mathbf{H}_2 \text{ iff } I_{P_{i_0}}(\mathbf{H}_2) \in I_{P_{i_0}}(\mathbf{H}_1) + \mathcal{C}_{P_{i_0}}(\mathfrak{a}_{i_0}) \qquad (i_0 = 1, \ldots, r_0).$$

[Note: The point, therefore, is that one has only to check the partial ordering on the fixed set of Γ-percuspidals, a finite set of conditions.]

Proof. We need only show, of course, that the stated condition implies the asserted relation. So fix a Γ-cuspidal split parabolic subgroup (P, S) of G with special split component A — then there is an index i and a $\gamma_i \in \Gamma$ such that $P^i = \gamma_i P \gamma_i^{-1} \supset P_{i_0}$. If $l = \operatorname{rank}(P, S)$, let

$$\begin{cases} (P_1, S_1; A_1), \ldots, (P_l, S_l; A_l) \\ (P_1^i, S_1^i; A_1^i), \ldots, (P_l^i, S_l^i; A_l^i) \end{cases}$$

be the maximal Γ-cuspidal split parabolic subgroups of G sitting in the standard picture over $(P, S; A)$, $(P^{\iota}, S^{\iota}; A^{i})$ — then it is clear that

$$I_P(\mathbf{H}_2) - I_P(\mathbf{H}_1) \in \mathcal{C}_P(\mathfrak{a}) \quad \text{iff} \quad I_{P^{\iota}}(\mathbf{H}_2) - I_{P^{\iota}}(\mathbf{H}_1) \in \mathcal{C}_{P^{\iota}}(\mathfrak{a}^{\iota}).$$

But

$$I_{P^{\iota}}(\mathbf{H}_2) - I_{P^{\iota}}(\mathbf{H}_1) \in \mathcal{C}_{P^{\iota}}(\mathfrak{a}^{\iota})$$

holds since, by hypothesis,

$$I_{P_{\iota_0}}(\mathbf{H}_2) - I_{P_{\iota_0}}(\mathbf{H}_1) \in \mathcal{C}_{P_{\iota_0}}(\mathfrak{a}_{\iota_0})$$

and, as is well-known and easy to verify, the orthogonal projection of $\mathcal{C}_{P_{\iota_0}}(\mathfrak{a}_{\iota_0})$ onto $\mathfrak{a}^{i}$ is exactly $\mathcal{C}_{P^{\iota}}(\mathfrak{a}^{i})$. □

The role of $\mathfrak{a}$ in the later going will be that of a parameter space. To say that '?' is true for $\mathbf{H}$ sufficiently regular means that there exists an $\mathbf{H}_0$ such that for all $\mathbf{H} < \mathbf{H}_0$, '?' obtains. In this connection, note that $\mathfrak{a}$ contains a one-parameter cofinal set tending to $-\infty$, viz. $\{t\mathbf{H}_\rho: t < 0\}$, $\mathbf{H}_\rho$ the element constructed in the obvious way from the $\rho_m^{\max}$ canonically attached to $(P_m^{\max}, S_m^{\max}; A_m^{\max})$.

Let

$$\begin{cases} (P_1, S_1; A_1) \\ (P_2, S_2; A_2) \end{cases}$$

be Γ-cuspidal split parabolic subgroups of G with special split components A_1, A_2. Assume that P_1 and P_2 are in addition Γ-conjugate, say $P_1 = \gamma P_2 \gamma^{-1}$ $(\gamma \in \Gamma)$. We then define a map

$$I_\Gamma(P_2 : P_1) : \mathfrak{a}_1 \to \mathfrak{a}_2$$

as follows. Given $H_1 \in \mathfrak{a}_1$, put

$$I_\Gamma(P_2 : P_1)(H_1) = I(P_2 \,|\, A_2 : P_1 \,|\, A_1)(H_1) + H_{P_2|A_2}(\gamma).$$

We have suppressed A_1 and A_2 from the notation since, being special, they are unique. It is clear that $I_\Gamma(P_2 : P_1)$ is well defined, that is, independent of the choice of conjugators. One has

$$I_\Gamma(P_2 : P_1)(H_1' + H_1'') = I_\Gamma(P_2 : P_1)(H_1') + I(P_2 \,|\, A_2 : P_1 \,|\, A_1)(H_1'')$$

for all $H_1', H_1'' \in \mathfrak{a}_1$. There are also the expected elementary properties, e.g. transitivity and descent, whose statements and proofs need not be considered explicitly. One point, however, should be noted.

LEMMA 3.3. *Let* $H_1 \in \mathfrak{a}_1$ — *then*

$$K \cdot \exp(H_1) \cdot S_1 \cdot \Gamma = K \cdot \exp(I_\Gamma(P_2 : P_1)(H_1)) \cdot S_2 \cdot \Gamma.$$

Proof. Suppose that $P_1 = \gamma P_2 \gamma^{-1}$ ($\gamma \in \Gamma$). Write

$$\gamma = k \exp\left(H_{P_2|A_2}(\gamma) \right) s_2.$$

We then have

$$\begin{aligned} K \cdot \exp(I_\Gamma(P_2 : P_1)(H_1)) \cdot S_2 \cdot \Gamma & \\ &= K \cdot \exp(I_\Gamma(P_2 : P_1)(H_1)) \cdot s_2 S_2 \gamma^{-1} \cdot \Gamma \\ &= K \cdot \exp(I_\Gamma(P_2 : P_1)(H_1)) \cdot s_2 \gamma^{-1} \cdot S_1 \cdot \Gamma \\ &= K \cdot \exp(\mathrm{Ad}(k^{-1}) H_1) \exp\left(H_{P_2|A_2}(\gamma) \right) \cdot s_2 \gamma^{-1} \cdot S_1 \cdot \Gamma \\ &= K k^{-1} \cdot \exp(H_1) \cdot k \exp\left(H_{P_2|A_2}(\gamma) \right) s_2 \cdot \gamma^{-1} \cdot S_1 \cdot \Gamma \\ &= K \cdot \exp(H_1) \cdot S_1 \cdot \Gamma, \end{aligned}$$

as desired. □

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with special split component A — then, as has been observed above, the map

$$I_P : \mathfrak{a} \to \mathfrak{a}$$

can be written in the form

$$I_P(\mathbf{H}) = \sum_{\mu=1}^{l} \left\langle I_{P_\mu}(\mathbf{H}), \lambda^\mu \right\rangle H_\mu \qquad (\mathbf{H} \in \mathfrak{a}).$$

But

$$I_{P_\mu}(\mathbf{H}) = I\left(P_\mu \,|\, A_\mu : P^{\max}_{m(\mu)} \,|\, A^{\max}_{m(\mu)} \right) \mathbf{H}_{m(\mu)} + H_{P_\mu|A_\mu}(\gamma_\mu)$$

or still

$$I_{P_\mu}(\mathbf{H}) = I_\Gamma\left(P_\mu : P^{\max}_{m(\mu)} \right) \mathbf{H}_{m(\mu)}$$

from which it follows that

$$I_P(\mathbf{H}) = \sum_{\mu=1}^{l} \left\langle I_\Gamma\left(P_\mu : P^{\max}_{m(\mu)} \right) \mathbf{H}_{m(\mu)}, \lambda^\mu \right\rangle H_\mu$$

for all $\mathbf{H} \in \mathfrak{a}$.

There is a small matter of consistency which should be mentioned. Let

$$\begin{cases}(P_1, S_1; A_1)\\ (P_2, S_2; A_2)\end{cases}$$

be Γ-cuspidal split parabolic subgroups of G with special split components A_1, A_2. If P_1 and P_2 are in addition Γ-conjugate, say $P_1 = \gamma P_2 \gamma^{-1}$ $(\gamma \in \Gamma)$, then, as can be checked without difficulty, the triangle

$$\begin{array}{ccccc} \mathfrak{a} & \overset{I_{P_2}}{\rightarrow} & \mathfrak{a}_2 & & \\ & I_{P_1}\searrow & \uparrow & I_\Gamma(P_2 : P_1) \\ & & \mathfrak{a}_1 & & \end{array}$$

is commutative.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with special split component A. Let $l = \text{rank}(P, S)$ — then $l = \#(\Sigma_P^0(\mathfrak{g}, \mathfrak{a}))$ or still, $l = \#(\mathfrak{L})$, $\mathfrak{L} = \{1, \ldots, l\}$. If

$$\begin{cases} V = \check{\mathfrak{a}} \\ (?, ?) = \text{inner product on } \check{\mathfrak{a}} \text{ derived from the Killing form,}\end{cases}$$

then $(\lambda_i, \lambda_j) \leq 0$ $(i \neq j)$, so the general set-up in §2 is realized by the situation at hand. It will, however, be more convenient for us to work in $\mathfrak{a}$ rather than in its dual $\check{\mathfrak{a}}$, which can, of course, be achieved by making the obvious transcriptions. To reestablish our notations, given $F \in \mathcal{P}_{\mathfrak{L}}$, let $\mathfrak{a}\langle F\rangle$ be the set of all H in $\mathfrak{a}$ such that

$$\begin{cases} \langle H, \lambda^i_F\rangle \leq 0 & \forall\, i \in F \\ \langle H, \lambda_i^F\rangle > 0 & \forall\, i \notin F. \end{cases}$$

Then

$$\mathfrak{a} = \coprod_{F \in \mathcal{P}_{\mathfrak{L}}} \mathfrak{a}\langle F\rangle,$$

one of the main consequences of the Combinatorial Lemma of Langlands. There are two other points which should be recalled (cf. Propositions 2.9 and 2.10). Fix $F \in \mathcal{P}_{\mathfrak{L}}$ — then:

(1) $\mathfrak{a}\langle F\rangle = (-\mathcal{D}(F))^{-} \oplus \mathcal{C}_F$;

(2) $\forall\, H \in \mathcal{C}$, $H = H(F) + H_F (H(F) \in \mathfrak{a}(F),\ H_F \in \mathfrak{a}_F)$, $(H + \mathfrak{a}\langle F\rangle) \cap \mathcal{C}$ is equal to

$$\big(H(F) + (-\mathcal{D}(F))^{-}\big) \cap \mathcal{C}(F) \oplus \big(H_F + \mathcal{C}_F\big).$$

The symbols $\mathcal{C}(F)$, $\mathcal{D}(F)$, $\mathcal{C}_F$, $\mathcal{D}_F$, as well as $\mathcal{C}$ and $\mathcal{D}$, carry the meanings assigned to them in §2. They also admit an interpretation in terms of parabolic subgroups. Thus:

$$\begin{cases} \mathcal{C} = \mathcal{C}_P(\mathfrak{a}) \\ \mathcal{D} = \mathcal{D}_P(\mathfrak{a}). \end{cases}$$

Moreover, each $F \in \mathcal{F}_{\mathfrak{L}}$ determines a triple $(P_F, S_F; A_F)$ such that

$$(P_F, S_F; A_F) \succcurlyeq (P, S; A).$$

Write $P_F = M_F \cdot A_F \cdot N_F$— then the Lie algebra of A_F is $\mathfrak{a}_F$ and, via the daggering procedure, M_F contains a parabolic subgroup $P(F) = M(F) \cdot A(F) \cdot N(F)$, the Lie algebra of $A(F)$ being $\mathfrak{a}(F)$. In addition,

$$M = M(F), \quad A = A(F) \cdot A_F, \quad N = N(F) \cdot N_F.$$

All this implies, therefore, that

$$\begin{cases} \mathcal{C}(F) = \mathcal{C}_{P(F)}(\mathfrak{a}(F)) = \mathcal{C}_{P^{\dagger_F}}(\mathfrak{a}^{\dagger_F}) \\ \mathcal{D}(F) = \mathcal{D}_{P(F)}(\mathfrak{a}(F)) = \mathcal{D}_{P^{\dagger_F}}(\mathfrak{a}^{\dagger_F}) \end{cases}$$

$$\begin{cases} \mathcal{C}_F = \mathcal{C}_{P_F}(\mathfrak{a}_F) \\ \mathcal{D}_F = \mathcal{D}_{P_F}(\mathfrak{a}_F) \end{cases}$$

The theorem infra depends upon some choices which we shall now make. Let

$$\{(P_i, S_i): 1 \leq i \leq r\}$$

be a set of representatives for the Γ-conjugacy classes of Γ-cuspidal split parabolic subgroups of G. It will be supposed that the Γ-percuspidal split parabolic subgroups of G in this set are exactly the (P_{i_0}, S_{i_0}) $(1 \leq i_0 \leq r_0)$ appearing in the theorem supra. Given (P_i, S_i), fix a set

$$\{(P_{i:\iota_0}, S_{i:\iota_0}): 1 \leq \iota_0 \leq r_i\}$$

of representatives for the $\Gamma \cap P_i$-conjugacy classes of Γ-percuspidal split parabolic subgroups of G which are dominated predecessors of (P_i, S_i). In terms of the special split components A_i and $A_{i:\iota_0}$, we have

$$(P_i, S_i; A_i) \succcurlyeq (P_{i:\iota_0}, S_{i:\iota_0}; A_{i:\iota_0}).$$

Each pair (i, ι_0) determines a unique index $i_0(i, \iota_0)$, $1 \leq i_0(i, \iota_0) \leq r_0$, such that $P_{i:\iota_0}$ is Γ-conjugate to $P_{i_0(i,\iota_0)}$. On the other hand, if $1 \leq i_0 \leq r_0$

and $F \subset \Sigma^0_{P_{\iota_0}}(\mathfrak{g}, \mathfrak{a}_{i_0})$, then there exist unique indices $i(i_0, F)$ and $\iota_0(i_0, F)$ such that for some $\gamma \in \Gamma$

$$\begin{cases} \gamma(P_{\iota_0})_F\gamma^{-1} = P_{i(\iota_0, F)} \\ \gamma P_{\iota_0}\gamma^{-1} = P_{i(i_0, F):\iota_0(i_0, F)}. \end{cases}$$

In this way there is determined a bijective map

$$i \times \iota_0, \quad (i_0, F) \mapsto (i(i_0, F), \quad \iota_0(i_0, F))$$

from the disjoint union of the power sets of the $\Sigma^0_{P_{\iota_0}}(\mathfrak{g}, \mathfrak{a}_{i_0})$ to $\{(i, \iota_0): 1 \le i \le r, 1 \le \iota_0 \le r_i\}$.

Fix an index i, $1 \le i \le r$, and an index ι_0, $1 \le \iota_0 \le r_i$ — then

$$(P_\iota, S_i; A_i) \succcurlyeq (P_{\iota:\iota_0}, S_{i:\iota_0}; A_{i:\iota_0})$$

determining, therefore, a parabolic subgroup

$$P^\dagger_{\iota:\iota_0} = M^\dagger_{i:\iota_0} \cdot A^\dagger_{i:\iota_0} \cdot N^\dagger_{i:\iota_0}$$

of M_i. There is an orthogonal decomposition

$$\mathfrak{a}_{i:\iota_0} = \mathfrak{a}^\dagger_{i:\iota_0} \oplus \mathfrak{a}_\iota$$

and a commutative triangle

$$\begin{array}{ccc} \mathfrak{a} & \overset{I_{P_{\iota:\iota_0}}}{\rightarrow} & \mathfrak{a}_{i:\iota_0} \\ & I_{P_i}\searrow & \downarrow\perp \\ & & \mathfrak{a}_i \end{array}$$

One would also like to say that there exists a commutative triangle

$$\begin{array}{ccc} \mathfrak{a} & \overset{I_{P_{\iota:\iota_0}}}{\rightarrow} & \mathfrak{a}_{i:\iota_0} \\ & I_{P^\dagger_{i:\iota_0}}\searrow & \downarrow\perp \\ & & \mathfrak{a}^\dagger_{i:\iota_0} \end{array}$$

This, however, is not really a meaningful assertion since $P^\dagger_{\iota:\iota_0}$ is not a parabolic subgroup of G. We shall therefore simply define $I_{P^\dagger_{\iota:\iota_0}}$ by the requirement that it be the composition of the two indicated arrows.

Keeping to the preceding notations, given $\mathbf{H} \in \mathfrak{a}$, denote by $A_\iota(\mathbf{H})$ the exponentiation to A_i of the subset $\mathfrak{a}_i(\mathbf{H})$ of $\mathfrak{a}_\iota$ defined by

$$\{H \in \mathfrak{a}_i: \lambda(H) < \lambda(I_{P_\iota}(\mathbf{H}))\ \forall\, \lambda \in \Sigma^0_{P_\iota}(\mathfrak{g}, \mathfrak{a}_\iota)\}$$

or still

$$\mathfrak{a}_i(\mathbf{H}) = I_{P_i}(\mathbf{H}) - \mathcal{C}_{P_i}(\mathfrak{a}_i).$$

Note that

$$\begin{aligned} \mathbf{H}_2 > \mathbf{H}_1 &\Rightarrow I_{P_i}(\mathbf{H}_2) - I_{P_i}(\mathbf{H}_1) \in \mathcal{C}_{P_i}(\mathfrak{a}_i) \\ &\Rightarrow A_i(\mathbf{H}_2) \supset A_i(\mathbf{H}_1). \end{aligned}$$

Given $\mathbf{H}_0 \in \mathfrak{a}$, $\mathbf{H}_0 > \mathbf{H}$, denote by $A^\dagger_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0)$ the exponentiation to $A^\dagger_{i:\iota_0}$ of the subset $\mathfrak{a}^\dagger_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0)$ of $\mathfrak{a}^\dagger_{i:\iota_0}$ defined by

$$\begin{aligned} \Big\{H \in \mathfrak{a}^\dagger_{i:\iota_0}: H \in \Big(I_{P^\dagger_{i:\iota_0}}(\mathbf{H}) + \mathcal{D}_{P^\dagger_{i:\iota_0}}\big(\mathfrak{a}^\dagger_{i:\iota_0}\big)^-\Big) \\ \cap \Big(I_{P^\dagger_{i:\iota_0}}(\mathbf{H}_0) - \mathcal{C}_{P^\dagger_{i:\iota_0}}\big(\mathfrak{a}^\dagger_{i:\iota_0}\big)\Big)\Big\}. \end{aligned}$$

Note that $A^\dagger_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0)$, while not compact, is at least relatively compact.

Fix a compact neighborhood $\omega_{i:\iota_0}$ of 1 in $S_{i:\iota_0}$ containing a fundamental domain for the action of $S_{i:\iota_0} \cap \Gamma$ on $S_{i:\iota_0}$.

There is one final convention to be made before we state the main result of this section. Let us agree that the symbol $\ll \mathbf{0}$ when applied to an element of $\mathfrak{a}$ means that this element is sufficiently regular whereas the symbol $\gg \mathbf{0}$ when applied to an element of $\mathfrak{a}$ means that the negative of this element is sufficiently regular.

THEOREM 3.4. *Let* $\mathbf{H}_0 \in \mathfrak{a}$, $\mathbf{H}_0 \gg \mathbf{0}$ — *then, for all* $\mathbf{H} < \mathbf{H}_0$,

$$G = \bigcup_{i=1}^{r} \bigcup_{\gamma_i \in \Gamma/\Gamma \cap P_i} \bigcup_{\iota_0 = 1}^{r_i} K \cdot A^\dagger_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) \cdot A_i(\mathbf{H}) \cdot \omega_{i:\iota_0} \cdot (\Gamma \cap P_i)\gamma_i^{-1}.$$

Moreover, for fixed $\mathbf{H}_0$, *the outer two unions are actually disjoint provided* $\mathbf{H} \ll \mathbf{0}$.

This theorem implies that G/Γ admits a partitioning indexed by the (P_i, S_i). Thus let $C_i(\mathbf{H}:\mathbf{H}_0)$ be the Γ-saturation of

$$\bigcup_{\iota_0=1}^{r_i} K \cdot A^\dagger_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) \cdot A_i(H) \cdot \omega_{i:\iota_0}.$$

Then: $\mathbf{H}_0 \gg \mathbf{0}$, $\mathbf{H} < \mathbf{H}_0$, $\mathbf{H} \ll \mathbf{0} \Rightarrow$

$$G/\Gamma = \coprod_{i=1}^{r} \pi(C_i(\mathbf{H}:\mathbf{H}_0)),$$

$\pi\colon G \to G/\Gamma$ the natural projection.

The fact that one can select $\mathbf{H}_0 \gg \mathbf{0}$ so as to ensure that G is covered by the sets in question for all $\mathbf{H} < \mathbf{H}_0$ is a fairly direct consequence of the fundamental theorem of reduction (Theorem 3.1) and the corollaries to the Combinatorial Lemma of Langlands. We shall therefore deal with it first, postponing for the time being the disjointness argument.

Write, after Theorem 3.1,

$$G = \bigcup_{i_0=1}^{r_0} \mathfrak{S}_{t_0,\omega_0} \kappa_{\iota_0} \cdot \Gamma.$$

We then demand that $\mathbf{H}_0$ be so chosen that

$$\inf_{\lambda \in \Sigma^0_{P_{\iota_0}}(\mathfrak{g}, \mathfrak{a}_{\iota_0})} \lambda\big(I_{P_{\iota_0}}(\mathbf{H}_0)\big) > \log t_0 \qquad (1 \le i_0 \le r_0).$$

Supposing that $\mathbf{H} < \mathbf{H}_0$, fix the index i_0 and a subset F of $\Sigma^0_{P_{\iota_0}}(\mathfrak{g}, \mathfrak{a}_{i_0})$. Denote by $A_{\iota_0}(\mathbf{H}:\mathbf{H}_0)\langle F\rangle$ the exponentiation to A_{i_0} of the subset $\mathfrak{a}_{i_0}(\mathbf{H}:\mathbf{H}_0)\langle F\rangle$ of $\mathfrak{a}_{i_0}$ defined by

$$\big\{H \in \mathfrak{a}_{i_0} : H \in \big(I_{P_{\iota_0}}(\mathbf{H}) - \mathfrak{a}_{\iota_0}\langle F\rangle\big) \cap \big(I_{P_{\iota_0}}(\mathbf{H}_0) - \mathcal{C}_{P_{\iota_0}}(\mathfrak{a}_{\iota_0})\big)\big\}.$$

Because

$$\mathfrak{a}_{i_0} = \bigcup_F \mathfrak{a}_{i_0}\langle F\rangle,$$

it follows that

$$I_{P_{\iota_0}}(\mathbf{H}_0) - \mathcal{C}_{P_{\iota_0}}(\mathfrak{a}_{i_0}) = \bigcup_F \mathfrak{a}_{i_0}(\mathbf{H}:\mathbf{H}_0)\langle F\rangle.$$

In view of the way in which $\mathbf{H}_0$ has been selected, it can thus be said that

$$A_{i_0}[t_0] \subset \bigcup_F A_{i_0}(\mathbf{H}:\mathbf{H}_0)\langle F\rangle.$$

Consequently, in order to establish the covering contention, we need only show that

$$K \cdot A_{\iota_0}(\mathbf{H}:\mathbf{H}_0)\langle F\rangle \cdot \omega_{i_0} \cdot \Gamma$$

is equal to

$$K \cdot A^{\dagger}_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) \cdot A_i(\mathbf{H}) \cdot \omega_{i:\iota_0} \cdot \Gamma$$

where

$$\begin{cases} i = i(i_0, F) \\ \iota_0 = \iota_0(i_0, F). \end{cases}$$

Taking into account the definitions, Lemma 3.3, and the relation

$$I_\Gamma(P_{i:\iota_0}: P_{i_0}) \circ I_{P_{\iota_0}} = I_{P_{\iota:\iota_0}},$$

we thereby reduce our problem to verifying that

$$\left(I_{P^\dagger_{\iota:\iota_0}}(\mathbf{H}) + \mathcal{D}_{P^\dagger_{\iota:\iota_0}}\left(\mathfrak{a}^\dagger_{i:\iota_0}\right)^-\right) \cap \left(I_{P^\dagger_{\iota:\iota_0}}(\mathbf{H}_0) - \mathcal{C}_{P^\dagger_{\iota:\iota_0}}\left(\mathfrak{a}^\dagger_{i:\iota_0}\right)\right)$$
$$+\left(I_{P_\iota}(\mathbf{H}) - \mathcal{C}_{P_\iota}(\mathfrak{a}_i)\right)$$

is equal to

$$\left(I_{P_{\iota:\iota_0}}(\mathbf{H}) - \mathfrak{a}_{i:\iota_0}\langle F\rangle\right) \cap \left(I_{P_{\iota:\iota_0}}(\mathbf{H}_0) - \mathcal{C}_{P_{\iota:\iota_0}}(\mathfrak{a}_{i:\iota_0})\right)$$

or still that the sum of

$$\left(I_{P^\dagger_{\iota:\iota_0}}(\mathbf{H}_0) - I_{P^\dagger_{\iota:\iota_0}}(\mathbf{H}) - \mathcal{D}_{P^\dagger_{\iota:\iota_0}}\left(\mathfrak{a}^\dagger_{i:\iota_0}\right)^-\right) \cap \mathcal{C}_{P^\dagger_{\iota:\iota_0}}\left(\mathfrak{a}^\dagger_{i:\iota_0}\right)$$

and

$$\left(I_{P_\iota}(\mathbf{H}_0) - I_{P_\iota}(\mathbf{H}) + \mathcal{C}_{P_\iota}(\mathfrak{a}_i)\right)$$

is equal to

$$\left(I_{P_{\iota:\iota_0}}(\mathbf{H}_0) - I_{P_{\iota:\iota_0}}(\mathbf{H}) + \mathfrak{a}_{i:\iota_0}\langle F\rangle\right) \cap \mathcal{C}_{P_{\iota:\iota_0}}(\mathfrak{a}_{i:\iota_0}).$$

To this end, put

$$H = I_{P_{\iota:\iota_0}}(\mathbf{H}_0) - I_{P_{\iota:\iota_0}}(\mathbf{H}) \in \mathcal{C}_{P_{\iota:\iota_0}}$$

so that, in the notations of Proposition 2.10,

$$\begin{cases} H(F) = I_{P^\dagger_{\iota:\iota_0}}(\mathbf{H}_0) - I_{P^\dagger_{\iota:\iota_0}}(\mathbf{H}) \\ H_F = I_{P_\iota}(\mathbf{H}_0) - I_{P_\iota}(\mathbf{H}). \end{cases}$$

There remains only to cite the proposition itself.

As for the assertion of separation, it is somewhat more difficult. We shall return to it after establishing the necessary preparation.

LEMMA 3.5. *Let*

$$\begin{cases} (P', S'; A') \\ (P'', S''; A'') \end{cases}$$

be Γ-percuspidal split parabolic subgroups of G with split components A', A''. Let

$$\begin{cases} \mathfrak{S}'_{t,\omega'} = K \cdot A'[t] \cdot \omega' \\ \mathfrak{S}''_{t,\omega''} = K \cdot A''[t] \cdot \omega'' \end{cases}$$

be Siegel domains in G associated with $P' \ldots$, $P'' \ldots$. Suppose that $\{x_n\}$ is a sequence in

$$\mathfrak{S}'_{t,\omega'} \cdot \Gamma \cap \mathfrak{S}''_{t,\omega''},$$

say

$$x_n = k'_n a'_n s'_n \gamma_n = k''_n a''_n s''_n.$$

Let

$$F = \{\lambda'' \in \Sigma^0_{P''}(\mathfrak{g}, \mathfrak{a}''): \xi_{\lambda''}(a''_n) \nrightarrow 0\}.$$

Then

$$\gamma_n P''_F \gamma_n^{-1} \supset P' \quad \forall\, n \gg 0.$$

Proof. The proof is a variant on a theme which has been employed frequently in [**3.a**, §2]. Accordingly, there is nothing to be gained by setting down every detail of the present argument which, in brief, runs as follows. Any element of $\Sigma_{P''_F}(\mathfrak{g}, \mathfrak{a}''_F)$ is the restriction to $\mathfrak{a}''_F$ of an element of $\Sigma_{P''}(\mathfrak{g}, \mathfrak{a}'')$ whose expression, as a linear combination of elements from $\Sigma^0_{P''}(\mathfrak{g}, \mathfrak{a}'')$, must contain some $\lambda'' \notin F$ in a nontrivial manner. Therefore

$$\lim_{n \to \infty} a''_n n''_F a''^{-1}_n = 1 \quad \forall\, n''_F \in N''_F.$$

Let now $\gamma''_F \in N''_F \cap \Gamma$ — then

$$\lim_{n \to \infty} a''_n \gamma''_F a_n''^{-1} = 1$$

$$\Rightarrow$$

$$\lim_{n \to \infty} (k''_n a''_n s''_n) \gamma''_F (k''_n a''_n s''_n)^{-1} = 1$$

$$\Rightarrow$$

$$\lim_{n \to \infty} (k'_n a'_n s'_n) \gamma_n \gamma''_F \gamma_n^{-1} (k'_n a'_n s'_n)^{-1} = 1$$

$$\Rightarrow$$

$$\lim_{n \to \infty} a'_n (\gamma_n \gamma''_F \gamma_n^{-1}) a'^{-1}_n = 1.$$

From this it follows that $\gamma_n \gamma''_F \gamma_n^{-1}$ is eventually in N' (see [**3.a**]). Due to the arbitrariness of γ''_F, upon taking a set of generators for $N''_F \cap \Gamma$, we then conclude that

$$\gamma_n N''_F \gamma_n^{-1} \subset N' \quad \forall\, n \gg 0$$

which in turn implies that

$$\gamma_n P_F'' \gamma_n^{-1} \supset P' \quad \forall\, n \gg 0,$$

as was to be shown. □

PROPOSITION 3.6. *Let*

$$\begin{cases} (P', S'; A') \\ (P'', S''; A'') \end{cases}$$

be Γ-percuspidal split parabolic subgroups of G with split components A', A''. Let

$$\begin{cases} \mathfrak{S}'_{t,\omega'} = K \cdot A'[t] \cdot \omega' \\ \mathfrak{S}''_{t,\omega''} = K \cdot A''[t] \cdot \omega'' \end{cases}$$

be Siegel domains in G associated with $P' \ldots$, $P'' \ldots$. Suppose that $\{x_n\}$ is a sequence in

$$\mathfrak{S}'_{t,\omega'} \cdot \Gamma \cap \mathfrak{S}''_{t,\omega''},$$

say

$$x_n = k_n' a_n' s_n' \gamma_n = k_n'' a_n'' s_n''.$$

Let F denote either

$$\{\lambda' \in \Sigma_{P'}^0(\mathfrak{g}, \mathfrak{a}') \colon \xi_{\lambda'}(a_n') \not\to 0\}$$

or

$$\{\lambda'' \in \Sigma_{P''}^0(\mathfrak{g}, \mathfrak{a}'') \colon \xi_{\lambda''}(a_n'') \not\to 0\}.$$

Then there exists an index n_0 such that

$$\begin{cases} \gamma_n P_F'' \gamma_n^{-1} \supset P' \\ \gamma_n^{-1} P_F' \gamma_n \supset P'' \end{cases} \quad \forall\, n \geq n_0.$$

Moreover,

$$P_F' \cap \gamma_{n_0} P_F'' \gamma_{n_0}^{-1}$$

is a Γ-cuspidal parabolic subgroup of G having the property that

$$\gamma_n = \delta_n \gamma_{n_0} (n \geq n_0) \Rightarrow \delta_n \in P_F' \cap \gamma_{n_0} P_F'' \gamma_{n_0}^{-1}.$$

Proof. The first assertion is an immediate consequence of the preceding lemma. This said, let

$$P = P_F' \cap \gamma_{n_0} P_F'' \gamma_{n_0}^{-1}.$$

Since

$$P' \subset \begin{cases} P'_F \\ \gamma_{n_0} P''_F \gamma_{n_0}^{-1}, \end{cases}$$

P is Γ-cuspidal (cf. [**3.a**]). If $n \geq n_0$, then $\gamma_n P''_F \gamma_n^{-1}$, $\gamma_{n_0} P''_F \gamma_{n_0}^{-1}$ both contain P', are conjugate, hence equal. Therefore δ_n normalizes $\gamma_{n_0} P''_F \gamma_{n_0}^{-1}$, thus belongs to $\gamma_{n_0} P''_F \gamma_{n_0}^{-1}$, the latter being self-normalizing. For similar reasons, δ_n also belongs to P'_F, completing the proof of the proposition. □

We are now in a position to finish the proof of our theorem. If we deny the disjointness contention, then there is overlap in the outer two unions no matter how much $\ll \mathbf{0}$ the parameter $\mathbf{H}$ is. It therefore follows that one can choose indices i', ι'_0 and i'', ι''_0 and a sequence $\{x_n\}$ in

$$K \cdot A_{i' : \iota'_0} \cdot \omega_{i' : \iota'_0} \cdot \Gamma \cap K \cdot A_{i'' : \iota''_0} \cdot \omega_{i'' : \iota''_0},$$

say

$$x_n = k'_n a'_n s'_n \gamma_n = k''_n a''_n s''_n,$$

where either

$$i' \neq i'' \text{ or } i' = i'' \quad \text{and} \quad \gamma_n \notin P_{i'} = P_{i''}$$

and

$$\begin{cases} a'_n \in A^{\dagger}_{i' : \iota'_0}(\mathbf{H}_n : \mathbf{H}_0) \cdot A_{i'}(\mathbf{H}_n) \subset A_{i' : \iota'_0}[t] \\ a''_n \in A^{\dagger}_{i'' : \iota''_0}(\mathbf{H}_n : \mathbf{H}_0) \cdot A_{i''}(\mathbf{H}_n) \subset A_{i'' : \iota''_0}[t] \end{cases} \quad (t \gg 0)$$

with

$$\mathbf{H}_n \to -\infty.$$

From all this, we shall derive a contradiction.

Let F' or F'' denote the subset of

$$\Sigma^0_{P_{i'} . \iota'_0}(\mathfrak{g}, \mathfrak{a}_{i' : \iota'_0}) \quad \text{or} \quad \Sigma^0_{P_{i''} . \iota''_0}(\mathfrak{g}, \mathfrak{a}_{\iota'' : \iota''_0})$$

determining

$$P_{i'} \quad \text{or} \quad P_{i''},$$

that is,

$$\begin{cases} P_{i'} = \left(P_{i' : \iota'_0}\right)_{F'} \\ P_{i''} = \left(P_{i'' : \iota''_0}\right)_{F''}. \end{cases}$$

Then

$$\mathbf{H}_n \to -\infty \Rightarrow \begin{cases} \xi_{\lambda'}(a_n') \to 0 & \forall\, \lambda' \notin F' \\ \xi_{\lambda''}(a_n'') \to 0 & \forall\, \lambda'' \notin F''. \end{cases}$$

Owing to Proposition 3.6, it can be supposed that

$$\gamma_n = \delta_n \gamma_0, \qquad \delta_n \in P_{i'} \cap \gamma_0 P_{i''} \gamma_0^{-1}$$

provided, of course, that the sequence be restricted to beyond a certain point. Put

$$\begin{cases} P_{i_0} = \gamma_0 P_{i'' : \iota_0''} \gamma_0^{-1} \\ P_0 = \gamma_0 P_{i''} \gamma_0^{-1}. \end{cases}$$

Let F_0 denote the subset of $\Sigma_{P_{i_0}}^0(\mathfrak{g}, \mathfrak{a}_{i_0})$ corresponding to P_0. Proceeding as in the proof of Lemma 3.3, fix the index n and write

$$\begin{aligned} k_n'' a_n'' s_n'' &= k_0 a_0 s_0 \gamma_0 \\ &\in K \cdot A_{i_0}(\mathbf{H}_n : \mathbf{H}_0)\langle F_0 \rangle \cdot \omega_{i_0} \cdot \Gamma. \end{aligned}$$

We have then

$$k_n' a_n' s_n' \delta_n = k_0 a_0 s_0.$$

For brevity, set $P = P_{i'} \cap \gamma_0 P_{i''} \gamma_0^{-1}$ — then P is a Γ-cuspidal parabolic subgroup of G (cf. Proposition 3.6). Let A be its special split component — then the sought for contradiction will arise from consideration of the A-components of $k_n' a_n' s_n' \delta_n$ and $k_0 a_0 s_0$ which, a priori, must be the same. To this end, we first remark that

$$S_{i' : \iota_0'} \subset S$$

as follows from Proposition 3.6 (S having the customary connotation per P). Because

$$\delta_n \in \Gamma \cap P = \Gamma \cap S,$$

we conclude that the A-component of $k_n' a_n' s_n' \delta_n$ is the same as the A-component of a_n' alone; similarly, the A-component of $k_0 a_0 s_0$ is that of a_0 alone. We can assume that $i' \neq i''$. For the other possibility, viz. $i' = i''$ and $\gamma_n \notin P_{i'} = P_{i''}$, is immediately untenable implying, as it does, that γ_n must normalize $P_{i'} = P_{i''}$, an evident contradiction. The supposition that $i' \neq i''$ carries with it the consequence that $P_{i'}$ and $P_{i''}$ are not Γ-conjugate, hence that $P_{i'} \neq P_0$. Let

$$\begin{cases} H_n' = I_{P_{i'' : \iota_0''}}(\mathbf{H}_n) - \log a_n' \in \mathfrak{a}_{i' : \iota_0'}\langle F' \rangle \\ H_n^0 = I_{P_{i_0}}(\mathbf{H}_n) - \log a_0 \in \mathfrak{a}_{i_0}\langle F_0 \rangle. \end{cases}$$

Then both H_n' and H_n^0 have the same projection onto $\mathfrak{a}$, call it H_n. We shall force a contradiction by showing that H_n lies, of necessity, in two mutually disjoint subsets of $\mathfrak{a}$. For the purpose of keeping things straight, it may be helpful to note that

$$\begin{array}{ccc} P_0 & & \\ & \supset & \\ \nparallel & & P \supset P_{\iota' : \iota_0'}. \\ & \supset & \\ P_{\iota'} & & \end{array}$$

Because $H_n' \in \mathfrak{a}_{\iota' : \iota_0'}\langle F' \rangle$ we have, in the notations of §2,

$$\tau_{*,F'}(\varnothing : H_n')\chi_{F',*}(H_n') = 1.$$

Choose F so that

$$P = (P_{\iota' : \iota_0'})_F.$$

Then

$$\begin{aligned} P_{\iota'} \supset P &\Rightarrow F' \supset F \\ &\Rightarrow \tau_{F,F'}(\varnothing : H_n')\chi_{F',*}(H_n') = 1. \end{aligned}$$

Write

$$P_{\iota'} = P_{F(P_{\iota'})}.$$

Thanks to Lemma 2.7 and subsequent remarks, we can thus say that H_n, being the projection of H_n' onto $\mathfrak{a}$, lies in $\mathfrak{a}\langle F(P_{\iota'})\rangle$. For entirely analogous reasons, if

$$P_0 = P_{F(P_0)},$$

then H_n, being the projection of H_n^0 onto $\mathfrak{a}$, lies in $\mathfrak{a}\langle F(P_0)\rangle$. But

$$\begin{aligned} P_{\iota'} \neq P_0 &\Rightarrow F(P_{\iota'}) \neq F(P_0) \\ &\Rightarrow \mathfrak{a}\langle F(P_{\iota'})\rangle \cap \mathfrak{a}\langle F(P_0)\rangle = \varnothing. \end{aligned}$$

The contradiction is therefore manifest.

There is an extension of Theorem 3.4 which will eventually be needed. In essence, the problem is this. Given a Γ-cuspidal split parabolic subgroup (P, S) of G with special split component A, obtain a decomposition of $G/\Gamma \cap P$ from that of M/Γ_M, the latter being provided for already by the theorem itself (applied to (M, Γ_M)).

Fix a set $\{(P_m, S_m)\}$ of Γ-cuspidal split parabolic subgroups of G which are dominated predecessors of (P, S) and with the property that

$\{(P_m^\dagger, S_m^\dagger)\}$ is a set of representatives for the Γ_M-conjugacy classes of maximal Γ_M-cuspidal split parabolic subgroups of M. In terms of the special split components A and A_m, we have

$$(P, S; A) \succcurlyeq (P_m, S_m; A_m).$$

In addition, there is an orthogonal decomposition

$$\mathfrak{a}_m = \mathfrak{a}_m^\dagger \oplus \mathfrak{a}.$$

According to our notational principles, we now put

$$\mathfrak{a}_M = \bigoplus_m \mathfrak{a}_m^\dagger.$$

Then it is the elements of $\mathfrak{a}_M$ which figure as parameters in the partition of M or M/Γ_M.

Let

$$\{(P_i', S_i'): 1 \leq i \leq r_M\}$$

be a set of Γ-cuspidal split parabolic subgroups of G which are dominated predecessors of (P, S) and with the property that

$$\{({}'P_i, {}'S_i): 1 \leq i \leq r_M\}$$

is a set of representatives for the Γ_M-conjugacy classes of Γ_M-cuspidal split parabolic subgroups of M. Agreeing to employ self-explanatory notations, the parabolic data reads:

$$\begin{cases} P \supset P_i' \supset P_{i:\iota_0}' \\ M \supset {}'P_i \supset {}'P_{i:\iota_0}. \end{cases}$$

Here we had best remind ourselves that the correspondence $?' \leftrightarrow {}'?$ preserves percuspidality. The associated Euclidean data is then:

$$\begin{cases} \mathfrak{a}_{i:\iota_0}' = \mathfrak{a}_{i:\iota_0}^\dagger \oplus \mathfrak{a}_i' \\ \quad \mathfrak{a}_i' = {}'\mathfrak{a}_i \oplus \mathfrak{a}, \end{cases}$$

$$\begin{cases} \mathfrak{a}_{i:\iota_0}' = {}'\mathfrak{a}_{i:\iota_0} \oplus \mathfrak{a} \\ \quad {}'\mathfrak{a}_{i:\iota_0} = {}'\mathfrak{a}_{i:\iota_0}^\dagger \oplus {}'\mathfrak{a}_i. \end{cases}$$

In particular, therefore,

$$\mathfrak{a}_{i:\iota_0}^\dagger = {}'\mathfrak{a}_{i:\iota_0}^\dagger.$$

These points settled, Theorem 3.4 can be stated in terms of (M, Γ_M) in the following way. Let $\mathbf{H}_0(M), \mathbf{H}(M) \in \mathfrak{a}_M$ — then $\mathbf{H}_0(M) \gg \mathbf{0}$, $\mathbf{H}(M) < \mathbf{H}_0(M)$, $\mathbf{H}(M) \ll \mathbf{0}$ imply that

$$M = \bigcup_{i=1}^{r_M} \bigcup_{\delta_i \in \Gamma_M/\Gamma_M \cap 'P_i} \bigcup_{\iota_0=1}^{r_{\circ M}} K_M \cdot 'A^{\dagger}_{i:\iota_0}(\mathbf{H}(M):\mathbf{H}_0(M)) \cdot 'A_i(\mathbf{H}(M)) \cdot '\omega_{i:\iota_0} \cdot (\Gamma_M \cap 'P_i)\delta_i^{-1},$$

the outer two unions being disjoint. Our objective will be to pull this decomposition back to G, so to speak.

Fix a compact neighborhood ω_i' of 1 in N_i' containing a fundamental domain for the action of $N_i' \cap \Gamma$ on N_i'.

PROPOSITION 3.7. *If*

$$M = \bigcup_{i=1}^{r_M} \bigcup_{\delta_i \in \Gamma_M/\Gamma_M \cap 'P_i} \bigcup_{\iota_0=1}^{r_{\circ M}} K_M \cdot 'A^{\dagger}_{i:\iota_0}(\mathbf{H}(M):\mathbf{H}_0(M)) \cdot 'A_i(\mathbf{H}(M)) \cdot '\omega_{i:\iota_0} \cdot (\Gamma_M \cap 'P_i)\delta_i^{-1},$$

then

$$G = \bigcup_{i=1}^{r_M} \bigcup_{\gamma_i \in \Gamma \cap P/\Gamma \cap P_i'} \bigcup_{\iota_0=1}^{r_{\circ M}} K \cdot A \cdot 'A^{\dagger}_{i:\iota_0}(\mathbf{H}(M):\mathbf{H}_0(M)) \cdot 'A_i(\mathbf{H}(M)) \cdot '\omega_{i:\iota_0} \cdot \omega_i' \cdot (\Gamma \cap P_i')\gamma_i^{-1}.$$

Furthermore, if the outer two unions giving M are disjoint, then the outer two unions giving G are disjoint.

Proof. We shall first show that the putative union does in fact cover G. Let $x \in G$. Write $x = k\,man$. Using our hypothesis, write in turn

$$m = k_M 'a^{\dagger}_{i:\iota_0} 'a_i 's_{i:\iota_0}\delta, \qquad \delta \in \Gamma_M.$$

Since $\Gamma_M = M \cap \Gamma \cdot N$, $\delta = \gamma\eta$ $(\gamma \in \Gamma \cap P, \eta \in N)$. Therefore

$$\begin{aligned} x &= k\,man \\ &= k\,amn \\ &= (k\,k_M)(a\,'a^{\dagger}_{i:\iota_0}{}'a_i)'s_{i:\iota_0}\gamma\eta n \\ &= (k\,k_M)(a\,'a^{\dagger}_{i:\iota_0}{}'a_i)'s_{i:\iota_0}(\gamma\eta n\gamma^{-1})\gamma. \end{aligned}$$

But

$$\gamma \in \Gamma \cap P \Rightarrow \gamma\eta n\gamma^{-1} \in N \subset N_i' = \omega_i' \cdot \Gamma \cap N_i'$$
$$\Rightarrow \gamma\eta n\gamma^{-1} = s_i'\gamma_i.$$

As it is a question of special split components, $K_M \subset K$. Thus

$$x = (k\, k_M)\big(a\,'a^{\dagger}_{i\,:\,\iota_0}{}'a_i\big)\big('s_{i\,:\,\iota_0}{}'s_i\big)(\gamma_i\gamma),$$

proving that we have covered G. Supposing now that the outer two unions giving M are disjoint, assume, to get a contradiction, that the outer two unions giving G are not disjoint. The indices determining the M-union are in a canonical one-to-one correspondence with the indices determining the G-union. If there exists an element x of G belonging to the sets associated with (i, γ_i) and (j, γ_j), say, then the M-component of x, viewed in $K_M \backslash M$, belongs to the sets associated with (i, δ_i) and (j, δ_j), an impossibility. □

In order to establish a connection with the corresponding parameters on G, that is the elements of $\mathfrak{a}$, we need to define a map

$$I_M : \mathfrak{a} \to \mathfrak{a}_M.$$

There is a commutative triangle

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{I_{P_m}} & \mathfrak{a}_m \\ & {\scriptstyle I_{P_m^\dagger}} \searrow & \downarrow \perp \\ & & \mathfrak{a}_m^\dagger \end{array}$$

if, as before, we agree that $I_{P_m^\dagger}$ is the composition of the other two arrows. This said, we then define I_M by requiring that

$$\mathrm{Proj}_m \circ I_M = I_{P_m^\dagger},$$

where $\mathrm{Proj}_m : \mathfrak{a}_M \to \mathfrak{a}_m$ is the orthogonal projection onto the mth component. I_M possesses the usual elementary properties, e.g. descent. In addition:

LEMMA 3.8. *I_M is $(\pm\infty)$-cofinal, i.e.,*

$$\forall\, \mathbf{H}(M) \in \mathfrak{a}_M\ \exists \mathbf{H}^+, \mathbf{H}^- \in \mathfrak{a}\ \text{st} \begin{cases} \mathbf{H}(M) < I_M(\mathbf{H}^+) \\ \mathbf{H}(M) > I_M(\mathbf{H}^-), \end{cases}$$

and order preserving, i.e.,

$$\forall\, \mathbf{H}_1, \mathbf{H}_2 \in \mathfrak{a} : \mathbf{H}_1 < \mathbf{H}_2 \Rightarrow I_M(\mathbf{H}_1) < I_M(\mathbf{H}_2).$$

[Both properties follow by descent. For instance consider the first. Since $\rho' = {}'\rho + \rho$ always, $I_M(\mathbf{H}_\rho) = \mathbf{H}_{\rho_M} \cdots$. Incidentally, it can be shown by example that I_M need not be surjective.]

On the basis of this lemma, we can therefore say that if $\mathbf{H}_0, \mathbf{H} \in \mathfrak{a}$ with $\mathbf{H}_0 \gg \mathbf{0}$, $\mathbf{H} < \mathbf{H}_0$, $\mathbf{H} \ll \mathbf{0}$, then

$$G = \bigcup_{i=1}^{r_M} \bigcup_{\gamma_i \in \Gamma \cap P/\Gamma \cap P_i'} \bigcup_{\iota_0=1}^{r_{\circ M}} K \cdot A \cdot A^{\dagger}_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) \cdot {}'A_\iota(I_M(\mathbf{H}))$$

$$\cdot {}'\omega_{i:\iota_0} \cdot \omega_i' \cdot (\Gamma \cap P_\iota')\gamma_i^{-1}.$$

Lest there be a misunderstanding, let us explicitly note that

$$A^{\dagger}_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) = {}'A^{\dagger}_{i:\iota_0}(I_M(\mathbf{H}):I_M(\mathbf{H}_0)).$$

It is now a simple matter to obtain a partitioning of $G/\Gamma \cap P$ indexed by the (P_i', S_i'). Thus let $C_i'(\mathbf{H}:\mathbf{H}_0)$ be the $\Gamma \cap P$-saturation of

$$\bigcup_{\iota_0=1}^{r_{\circ M}} K \cdot A \cdot A^{\dagger}_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) \cdot {}'A_\iota(I_M(\mathbf{H})) \cdot {}'\omega_{i:\iota_0} \cdot \omega_i'.$$

Then $\mathbf{H}_0 \gg \mathbf{0}$, $\mathbf{H} < \mathbf{H}_0$, $\mathbf{H} \ll \mathbf{0} \Rightarrow$

$$G/\Gamma \cap P = \coprod_{i=1}^{r_M} \pi_P(C_i'(\mathbf{H}:\mathbf{H}_0)),$$

$\pi_P: G \to G/\Gamma \cap P$ the natural projection.

For technical reasons, to be spelled out in detail later on, it will be necessary to establish an alternative description of the partitionings involving characteristic functions.

As a prelude to this, let us first make a few simple observations. Suppose that (P, S) is a Γ-cuspidal split parabolic subgroup of G with special split component A. Fix a Γ-percuspidal split parabolic subgroup (P_0, S_0) of G with special split component A_0 such that

$$(P, S; A) \succcurlyeq (P_0, S_0; A_0).$$

Assigning to the symbols

$$A_0^{\dagger}(\mathbf{H}:\mathbf{H}_0),\ A(\mathbf{H}),\ \omega_0,$$

the obvious interpretation, consider

$$K \cdot A_0^{\dagger}(\mathbf{H}:\mathbf{H}_0) \cdot A(\mathbf{H}) \cdot \omega_0 \cdot (\Gamma \cap P).$$

Since

(1) $K_M \subset K$,
(2) $K_M \cdot A(\mathbf{H}) = A(\mathbf{H}) \cdot K_M$,
(3) $A_0^{\dagger}(\mathbf{H}:\mathbf{H}_0) \cdot A(\mathbf{H}) = A(\mathbf{H}) \cdot A_0^{\dagger}(\mathbf{H}:\mathbf{H}_0)$,

our expression can be rewritten as

$$K \cdot A(\mathbf{H}) \cdot K_M \cdot A_0^\dagger(\mathbf{H}:\mathbf{H}_0) \cdot \omega_0 \cdot (\Gamma \cap P).$$

But

$$\begin{aligned}\omega_0 \cdot (\Gamma \cap P) & \\ &= \omega_0 \cdot (\Gamma \cap P_0) \cdot (\Gamma \cap P) \\ &= S_0 \cdot (\Gamma \cap P) \\ &= M_0 \cdot N_0^\dagger \cdot N \cdot (\Gamma \cap P) \\ &= M_0 \cdot N_0^\dagger \cdot \Gamma_M \cdot N,\end{aligned}$$

leading, therefore, to

$$K \cdot A(\mathbf{H}) \cdot K_M \cdot A_0^\dagger(\mathbf{H}:\mathbf{H}_0) \cdot S_0^\dagger \cdot \Gamma_M \cdot N.$$

Thanks to Lemma 3.3, the set

$$K_M \cdot A_0^\dagger(\mathbf{H}:\mathbf{H}_0) \cdot S_0^\dagger \cdot \Gamma_M$$

is invariant under Γ_M-conjugacy, i.e., is unchanged if $P_0^\dagger$ is replaced by a Γ_M-conjugate, or still, if P_0 is replaced by a $\Gamma \cap P$-conjugate. On the other hand, let $\gamma \in \Gamma/\Gamma \cap P$. Put

$$P_\gamma = \gamma P \gamma^{-1}, \qquad P_{0\gamma} = \gamma P_0 \gamma^{-1}.$$

Decomposing γ according to $G = K \cdot P_0$ and using definitions, we then find that

$$\begin{aligned}K \cdot A_0^\dagger(\mathbf{H}:\mathbf{H}_0) \cdot A(\mathbf{H}) \cdot S_0 \cdot (\Gamma \cap P)\gamma^{-1} & \\ = K \cdot A_{0\gamma}^\dagger(\mathbf{H}:\mathbf{H}_0) \cdot A_\gamma(\mathbf{H}) \cdot S_{0\gamma} \cdot \left(\Gamma \cap P_\gamma\right),\end{aligned}$$

exhibiting, thereby, the variance of our data with Γ-conjugacy. Finally, write

$$M(\mathbf{H}:\mathbf{H}_0)$$

for the union over all $P_0 \preccurlyeq P$, P_0 Γ-percuspidal, of the

$$K_M \cdot A_0^\dagger(\mathbf{H}:\mathbf{H}_0) \cdot S_0^\dagger \cdot \Gamma_M,$$

the union being effectively finite in that it can be taken over a set of representatives for the $\Gamma \cap P$-conjugacy classes of Γ-percuspidal split parabolic subgroups of G which are dominated predecessors of (P, S). We may thus attach to P the set

$$K \cdot A(\mathbf{H}) \cdot M(\mathbf{H}:\mathbf{H}_0) \cdot N,$$

the role of which will be explicated momentarily.

In the notations of Theorem 3.4, we have

$$G = \bigcup_{i=1}^{r} \bigcup_{\gamma_i \in \Gamma/\Gamma \cap P_i} \bigcup_{\iota_0 = 1}^{r_i} K \cdot A^{\dagger}_{i:\iota_0}(\mathbf{H}:\mathbf{H}_0) \cdot A_i(\mathbf{H}) \cdot \omega_{i:\iota_0} \cdot (\Gamma \cap P_i)\gamma_i^{-1}.$$

The outer two unions are actually disjoint provided that $\mathbf{H} \ll \mathbf{0}$, as we suppose. Let $\mathcal{C}_\Gamma$ be the set of all Γ-cuspidal split parabolic subgroups of G — then

$$\mathcal{C}_\Gamma = \bigcup_{i=1}^{r} \bigcup_{\gamma_i \in \Gamma/\Gamma \cap P_i} \{\gamma_i P_i \gamma_i^{-1}\},$$

implying that the remarks above can be translated to read

$$G = \coprod_{P \in \mathcal{C}_\Gamma} K \cdot A(\mathbf{H}) \cdot M(\mathbf{H}:\mathbf{H}_0) \cdot N.$$

This is the 'G/Γ-decomposition'.

To obtain a '$G/\Gamma \cap P$-decomposition', we start from

$$M = \coprod_{'P \in \mathcal{C}_{\Gamma_M}} K_M \cdot {}'A(\mathbf{H}(M)) \cdot {}'M(\mathbf{H}(M):\mathbf{H}_0(M)) \cdot {}'N,$$

the immediately preceding result applied to the pair (M, Γ_M). To pass from M to G, multiply

$$K_M \cdot {}'A(\mathbf{H}(M)) \cdot {}'M(\mathbf{H}(M):\mathbf{H}_0(M)) \cdot {}'N$$

on the left by $K \cdot A$ and on the right by N (cf. Proposition 3.7). Denoting by

$$\mathrm{Dom}_\Gamma(P)$$

the set of all Γ-cuspidal split parabolic subgroups of G which are dominated predecessors of (P, S), so that

$$\mathcal{C}_{\Gamma_M} = \{'P : P' \in \mathrm{Dom}_\Gamma(P)\},$$

we get, correspondingly,

$$G = \coprod_{P' \in \mathrm{Dom}_\Gamma(P)} K \cdot A \cdot {}'A(\mathbf{H}(M)) \cdot {}'M(\mathbf{H}(M):\mathbf{H}_0(M)) \cdot N'$$

or still, in terms of the $\mathfrak{a}$-parameters rather than the $\mathfrak{a}_M$-parameters (cf. supra),

$$G = \coprod_{P' \in \mathrm{Dom}_\Gamma(P)} K \cdot A \cdot {}'A(I_M(\mathbf{H})) \cdot M'(\mathbf{H}:\mathbf{H}_0) \cdot N',$$

it being the case that

$$M'(\mathbf{H}:\mathbf{H}_0) = {}'M(\mathbf{H}(M):\mathbf{H}_0(M)).$$

Given $x \in G$, write, with $G = K \cdot P'$ $(P' = M' \cdot A' \cdot N')$, $x = k_x m'_x a'_x n'_x$ — then m'_x, while not unique in M', is unique in $K_{M'} \backslash M'$. Let

$$F_{P'}(\mathbf{H} : \mathbf{H}_0 : x) = \begin{cases} 1 & \text{if } m_x \in M'(\mathbf{H} : \mathbf{H}_0) \\ 0 & \text{if } m_x \notin M'(\mathbf{H} : \mathbf{H}_0). \end{cases}$$

Per the domination

$$(P, S; A) \succcurlyeq (P', S'; A'),$$

determine F' by $P = P'_{F'}$ — then (cf. §2) $\chi_{*,F'}$ is the characteristic function of the set

$$\{H' \in \mathfrak{a}' \colon \langle H', \lambda_i \rangle > 0 \ (i \in F')\}.$$

LEMMA 3.9. *The characteristic function of*

$$K \cdot A \cdot {}'A(I_M(\mathbf{H})) \cdot M'(\mathbf{H} : \mathbf{H}_0) \cdot N'$$

is

$$F_{P'}(\mathbf{H} : \mathbf{H}_0 : ?) \cdot \chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(?)\big)$$

Proof. An element x of G belongs to

$$K \cdot A \cdot {}'A(I_M(\mathbf{H})) \cdot M'(\mathbf{H} : \mathbf{H}_0) \cdot N'$$

iff

$$\begin{cases} m_x \in M'(\mathbf{H} : \mathbf{H}_0) \\ a_x \in A \cdot {}'A(I_M(\mathbf{H})). \end{cases}$$

But

$$a_x \in A \cdot {}'A(I_M(\mathbf{H}))$$

iff the projection onto $'\mathfrak{a}$ of $H_{P'|A'}(x)$ is in

$$I_{'P}(I_M(\mathbf{H})) - \mathcal{C}_{'P}('\mathfrak{a})$$

which is true iff the projection onto $'\mathfrak{a}$ of $I_{P'}(\mathbf{H}) - H_{P'|A'}(x)$ is in $\mathcal{C}_{'P}('\mathfrak{a})$, that is, iff

$$\chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(x)\big) = 1.$$

Hence the lemma. □

There is therefore a corollary, viz.:

$$\forall P \in \mathcal{C}_\Gamma, \quad \exists \, \mathbf{H}_0, \mathbf{H}_{00} \in \mathfrak{a}, \quad \mathbf{H}_{00} < \mathbf{H}_0,$$

such that

$$\forall\, \mathbf{H} < \mathbf{H}_{00}$$

$$1_G = \sum_{P' \in \mathrm{Dom}_\Gamma(P)} F_{P'}(\mathbf{H} : \mathbf{H}_0 : ?) \cdot \chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(?)\big),$$

1_G the characteristic function of G.

Ostensibly, $\mathbf{H}_0$ and $\mathbf{H}_{00}$ depend on the choice of P. That a uniform selection is possible is contained in the following reinforcement of the corollary.

PROPOSITION 3.10. *There exist* $\mathbf{H}_0, \mathbf{H}_{00} \in \mathfrak{a}$, $\mathbf{H}_{00} < \mathbf{H}_0$, *such that for all* $P \in \mathcal{C}_\Gamma$

$$\mathbf{H} < \mathbf{H}_{00}$$

$$\Rightarrow 1_G = \sum_{P' \in \mathrm{Dom}_\Gamma(P)} F_{P'}(\mathbf{H} : \mathbf{H}_0 : ?) \cdot \chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(?)\big).$$

Here is the point. There are finitely many Γ-conjugacy classes of Γ-cuspidal split parabolic subgroups of G. So, if we can show that $\mathbf{H}_0$ and $\mathbf{H}_{00}$ depend only on the particular Γ-conjugacy class to which a given $P \in \mathcal{C}_\Gamma$ belongs, then the proof will be complete.

Fix $P \in \mathcal{C}_\Gamma$:

$$P = P'_{F'} \qquad (P' \in \mathrm{Dom}_\Gamma(P)).$$

Let $\gamma \in \Gamma$. Put $P_\gamma = \gamma P \gamma^{-1}$ — then

$$\mathrm{Dom}_\Gamma(P_\gamma) = \gamma\, \mathrm{Dom}_\Gamma(P)\gamma^{-1} \qquad \big(P'_\gamma = \gamma P' \gamma^{-1}\big),$$

say

$$P_\gamma = P'_{F'_\gamma}.$$

LEMMA 3.11. *In the above notations,*

$$\chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(x\gamma)\big) = \chi_{*,F'_\gamma}\big(I_{P'_\gamma}(\mathbf{H}) - H_{P'_\gamma|A'_\gamma}(x)\big)$$

for all $x \in G$.

LEMMA 3.12. *In the above notations,*

$$F_{P'}(\mathbf{H} : \mathbf{H}_0 : x\gamma) = F_{P'_\gamma}(\mathbf{H} : \mathbf{H}_0 : x)$$

for all $x \in G$.

Admit these conclusions — then we would have

$$\begin{aligned} 1_G &= \sum_{P' \in \mathrm{Dom}_\Gamma(P)} F_{P'}(\mathbf{H}:\mathbf{H}_0:x)\cdot\chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(x)\big) \\ &= \sum_{P' \in \mathrm{Dom}_\Gamma(P)} F_{P'}(\mathbf{H}:\mathbf{H}_0:x\gamma)\cdot\chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(x\gamma)\big) \\ &= \sum_{P'_\gamma \in \mathrm{Dom}_\Gamma(P_\gamma)} F_{P'_\gamma}(\mathbf{H}:\mathbf{H}_0:x)\cdot\chi_{*,F'_\gamma}\Big(I_{P'_\gamma}(\mathbf{H}) - H_{P'_\gamma|A'_\gamma}(x)\Big), \end{aligned}$$

as desired.

Proof of Lemma 3.11. The K-component of γ per the decomposition $G = K\cdot P'$ takes the special split component A' of P' to the special split component A'_γ of P'_γ. Noting that

$$H_{P'|A'}(x\gamma) = I_\Gamma(P':P'_\gamma)\Big(H_{P'_\gamma|A'_\gamma}(x)\Big),$$

the definitions then imply that

$$\begin{aligned} &\chi_{*,F'}\big(I_{P'}(\mathbf{H}) - H_{P'|A'}(x\gamma)\big) \\ &\qquad = \chi_{*,F'}\Big(I_\Gamma(P':P'_\gamma)\big(I_{P'_\gamma}(\mathbf{H})\big) - I_\Gamma(P':P'_\gamma)\Big(H_{P'_\gamma|A'_\gamma}(x)\Big)\Big) \\ &\qquad = \chi_{*,F'}\Big(I(P'\,|\,A':P'_\gamma\,|\,A'_\gamma)\Big[I_{P'_\gamma}(\mathbf{H}) - H_{P'_\gamma|A'_\gamma}(x)\Big]\Big) \\ &\qquad = \chi_{*,F'_\gamma}\Big(I_{P'_\gamma}(\mathbf{H}) - H_{P'_\gamma|A'_\gamma}(x)\Big), \end{aligned}$$

the contention of the lemma. □

Proof of Lemma 3.12. There is evidently no loss of generality in supposing for our proof that $P' = P$. It is then a question of showing that

$$F_P(\mathbf{H}:\mathbf{H}_0:x\gamma) = F_{P_\gamma}(\mathbf{H}:\mathbf{H}_0:x)$$

for all $x \in G$. Let

$$\begin{cases} \gamma = k_\gamma m_\gamma a_\gamma n_\gamma & (G = K\cdot P(P = M\cdot A\cdot N)) \\ x = k_x m_x a_x n_x & \big(G = K\cdot P_\gamma\big(P_\gamma = M_\gamma\cdot A_\gamma\cdot N_\gamma\big)\big). \end{cases}$$

Then

$$x\gamma = k_x k_\gamma\big(k_\gamma^{-1} m_x k_\gamma m_\gamma\big)\big(k_\gamma^{-1} a_x k_\gamma a_\gamma\big) n$$

where

$$n = m_\gamma^{-1} a_\gamma^{-1} \left(k_\gamma^{-1} n_x k_\gamma \right) a_\gamma m_\gamma n_\gamma \in N,$$

the M-component of $x\gamma$ being, therefore,

$$k_\gamma^{-1} m_x k_\gamma m_\gamma .$$

We must prove, accordingly, that

$$\begin{aligned} & k_\gamma^{-1} m_x k_\gamma m_\gamma \in M(\mathbf{H} : \mathbf{H}_0) \\ \Leftrightarrow & \\ & m_x \in M_\gamma(\mathbf{H} : \mathbf{H}_0). \end{aligned}$$

For reasons of symmetry, we need only deal explicitly with '$\Leftarrow$'. So assume that

$$m_x \in M_\gamma(\mathbf{H} : \mathbf{H}_0).$$

Then there exists a Γ-percuspidal $P_0 \preccurlyeq P$ such that

$$m_x \in K_{M_\gamma} \cdot A_{0\gamma}^\dagger(\mathbf{H} : \mathbf{H}_0) \cdot S_{0\gamma}^\dagger \cdot \Gamma_{M_\gamma},$$

with the understanding, of course, that

$$P_{0\gamma} = \gamma P_0 \gamma^{-1}.$$

We claim that

$$k_\gamma^{-1} m_x k_\gamma m_\gamma \in K_M \cdot A_0^\dagger(\mathbf{H} : \mathbf{H}_0) \cdot S_0^\dagger \cdot \Gamma_M .$$

Let us begin the verification by decomposing m_x into a product of four terms,

$$m_x = *_1 \cdot *_2 \cdot *_3 \cdot *_4,$$

to get

$$k_\gamma^{-1} m_x k_\gamma = k_0 a_0^\dagger s_0^\dagger \delta_0$$

where

$$k_0 = k_\gamma^{-1} *_1 k_\gamma, \ldots, \delta_0 = k_\gamma^{-1} *_4 k_\gamma .$$

Because k_γ conjugates K_M to K_{M_γ} and $S_0^\dagger$ to $S_{0\gamma}^\dagger$,

$$\begin{cases} k_0 \in K_M \\ s_0^\dagger \in S_0^\dagger . \end{cases}$$

Next, write

$$\gamma = k_\gamma^0 m_\gamma^0 a_\gamma^0 n_\gamma^0 \qquad (G = K \cdot P_0 (P_0 = M_0 \cdot A_0 \cdot N_0)).$$

Then

$$\begin{cases} a_\gamma^0 = a_\gamma^\dagger a_\gamma & (A_0 = A_0^\dagger \cdot A) \\ n_\gamma^0 = n_\gamma^\dagger n_\gamma & (N_0 = N_0^\dagger \cdot N), \end{cases}$$

hence

$$\begin{aligned} \gamma &= k_\gamma^0 m_\gamma^0 a_\gamma^0 n_\gamma^0 \\ &= k_\gamma^0 m_\gamma^0 a_\gamma^\dagger a_\gamma n_\gamma^\dagger n_\gamma \\ &= k_\gamma^0 \big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big) a_\gamma n_\gamma \in K \cdot M \cdot A \cdot N, \end{aligned}$$

implying that we can take

$$m_\gamma = m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger .$$

To recapitulate, we now have

$$\begin{aligned} k_\gamma^{-1} m_x k_\gamma m_\gamma &= k_0 a_0^\dagger s_0^\dagger \delta_0 m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \\ &= k_0 a_0^\dagger s_0^\dagger m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big)^{-1} \delta_0 \big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big) \\ &= k_0 \big(a_0^\dagger a_\gamma^\dagger \big) \big(a_\gamma^{-\dagger} s_0^\dagger m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big) \big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big)^{-1} \delta_0 \big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big). \end{aligned}$$

Since

$$\begin{cases} k_0 \in K_M \\ a_\gamma^{-\dagger} s_0^\dagger m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \in S_0^\dagger, \end{cases}$$

it remains only to show that

$$a_0^\dagger a_\gamma^\dagger \in A_0^\dagger(\mathbf{H} : \mathbf{H}_0)$$

and

$$\big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big)^{-1} \delta_0 \big(m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger \big) \in \Gamma_M .$$

The demonstration of the first point being quite analogous to that of the preceding lemma, pass to the second. With

$$m_\gamma = m_\gamma^0 a_\gamma^\dagger n_\gamma^\dagger ,$$

we shall establish that

$$m_\gamma^{-1} k_\gamma^{-1} \Gamma_{M_\gamma} k_\gamma m_\gamma \subset \Gamma_M$$

which will more than do it. Recalling that

$$\begin{cases} \Gamma_M = M \cap \Gamma \cdot N \\ \Gamma_{M_\gamma} = M_\gamma \cap \Gamma \cdot N_\gamma, \end{cases}$$

let $\delta^* \in \Gamma_{M_\gamma}$, say $\delta^* = \gamma^* n^*$ — then

$$\begin{aligned} &\delta^* \in M_\gamma \\ \Rightarrow\ & m_\gamma^{-1} k_\gamma^{-1} \delta^* k_\gamma m_\gamma \in M \\ \Rightarrow\ & m_\gamma^{-1} k_\gamma^{-1} \delta^* k_\gamma m_\gamma \\ &\quad = a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} \delta^* k_\gamma m_\gamma a_\gamma^{-1} \end{aligned}$$

or still

$$\begin{aligned} m_\gamma^{-1} k_\gamma^{-1} \delta^* k_\gamma m_\gamma & \\ &= \left(a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} \gamma^* k_\gamma m_\gamma a_\gamma\right)\left(a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} n^* k_\gamma m_\gamma a_\gamma\right) \\ &= n_\gamma\left(n_\gamma^{-1} a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} \gamma^* k_\gamma m_\gamma a_\gamma n_\gamma\right) n_\gamma^{-1} \\ &\quad \times \left(a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} n^* k_\gamma m_\gamma a_\gamma\right) \\ &= n_\gamma(\gamma^{-1}\gamma^*\gamma) n_\gamma^{-1}\left(a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} n^* k_\gamma m_\gamma a_\gamma\right). \end{aligned}$$

It is clear that

$$a_\gamma^{-1} m_\gamma^{-1} k_\gamma^{-1} n^* k_\gamma m_\gamma a_\gamma \in N.$$

In addition

$$\begin{aligned} &\gamma^* \in \Gamma \cap P_\gamma \\ \Rightarrow\ &\gamma^{-1}\gamma^*\gamma \in \Gamma \cap P. \end{aligned}$$

But

$$n_\gamma(\gamma^{-1}\gamma^*\gamma) n_\gamma^{-1} = (\gamma^{-1}\gamma^*\gamma)\left\{(\gamma^{-1}\gamma^*\gamma)^{-1} n_\gamma (\gamma^{-1}\gamma^*\gamma)\right\} n_\gamma^{-1} \in \Gamma \cdot N.$$

It therefore follows that

$$m_\gamma^{-1} k_\gamma^{-1} \delta^* k_\gamma m_\gamma \in M \cap \Gamma \cdot N = \Gamma_M,$$

completing the proof of the lemma. □

4. Estimates. The purpose of this section will be to formulate and prove a series of estimates which will then find application in the next section when we come to the truncation operator. Certain, more or less immediate, consequences of these estimates will, however, be given here, this being the place where they belong so to speak.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A — then, as before,

$$\begin{cases} \mathcal{C}_P(\mathfrak{a}) \\ \mathcal{D}_P(\mathfrak{a}) \end{cases}$$

denote, respectively, the positive chamber or positive cone of $(P, S; A)$,

$$\begin{cases} \mathcal{C}_P(\check{\mathfrak{a}}) \\ \mathcal{D}_P(\check{\mathfrak{a}}) \end{cases}$$

being the corresponding objects viewed in the dual $\check{\mathfrak{a}}$ of $\mathfrak{a}$. We shall agree to write

$$\begin{cases} \chi_{P,A:\mathcal{C}} \\ \chi_{P,A:\mathcal{D}} \end{cases} \qquad \begin{cases} \check{\chi}_{P,A:\mathcal{C}} \\ \check{\chi}_{P,A:\mathcal{D}} \end{cases}$$

for the associated characteristic functions.

Recall that one may attach to any Λ in $\check{\mathfrak{a}} + \sqrt{-1}\,\check{\mathfrak{a}}$ an Eisenstein series

$$E(P\,|\,A:1:\Lambda:x) = \sum_{\gamma\in\Gamma/\Gamma\cap P} a_{x\gamma}^{(\Lambda-\rho)},$$

ρ being as always. It is well-known that the series defining $E(P\,|\,A:1:\Lambda:x)$ is absolutely-uniformly convergent on compact subsets of the Cartesian product

$$\left(-(\rho + \mathcal{C}_P(\check{\mathfrak{a}})) + \sqrt{-1}\,\check{\mathfrak{a}}\right) \times G.$$

LEMMA 4.1. *Let $x \in G$ — then, for every $H \in \mathfrak{a}$,*

$$\#(\{\gamma \in \Gamma/\Gamma \cap P\colon H - H(x\gamma) \in \mathcal{D}_P(\mathfrak{a})\})$$

is majorized by

$$e^{3\rho(H)}\cdot E(P\,|\,A:1:-2\rho:x),$$

thus, in particular, is a finite set.

Proof. Suppose that $H - H(x\gamma) \in \mathcal{D}_P(\mathfrak{a})$ — then we can write

$$H(x\gamma) = H - H_{x\gamma}^{+}$$

where $H_{x\gamma}^{+} \in \mathcal{D}_P(\mathfrak{a})$. Consequently,

$$-3\rho(H(x\gamma)) = -3\rho(H) + 3\rho(H_{x\gamma}^{+}) \geq -3\rho(H),$$

so

$$e^{-3\rho(H(x\gamma))} \geq e^{-3\rho(H)}.$$

It therefore follows that

$$\begin{aligned} E(P \mid A : 1 : -2\rho : x) &= \sum_{\gamma \in \Gamma/\Gamma \cap P} e^{-3\rho(H(x\gamma))} \\ &\geq \sum_{\substack{\gamma \in \Gamma/\Gamma \cap P \\ H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a})}} e^{-3\rho(H)} \\ &= e^{-3\rho(H)} \cdot \#(\{\gamma \in \Gamma/\Gamma \cap P : H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a})\}) \end{aligned}$$

which is equivalent to our assertion. □

LEMMA 4.2. *Let $H \in \mathfrak{a}$ — then, for every compact set C in G,*

$$\#(\{\gamma \in \Gamma/\Gamma \cap P : H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a})(x \in C)\}) < +\infty.$$

Proof. Suppose not — then we can find infinitely many distinct elements γ_n in the set in question and elements x_n in C such that $H - H(x_n\gamma_n) \in \mathfrak{I}_P(\mathfrak{a})$ for all n. By passing to a subsequence if necessary, it can be assumed with no loss of generality that $x_n \to x$, say. Now fix an element $H_0 \in \mathfrak{I}_P(\mathfrak{a})$. Since

$$H(x_n\gamma_n) - H(x\gamma_n) \to 0,$$

there exists an index N with the property that if $n > N$, then

$$H(x_n\gamma_n) - H(x\gamma_n) \in \mathfrak{I}_P(\mathfrak{a}) - H_0$$

or still

$$H + H_0 - H(x\gamma_n) \in \mathfrak{I}_P(\mathfrak{a}).$$

As the number of γ_n for which this relation is true is infinite, we have contradicted the preceding lemma. □

Let f be a complex valued locally bounded (measurable) function on G/Γ — then, as usual, we write

$$f^P(x) = \int_{N/N \cap \Gamma} f(xn)\, d_N(n) \qquad (x \in G),$$

the compact quotient $N/N \cap \Gamma$ having total mass one. Given $H \in \mathfrak{a}$, put

$$T_{P|A}(H : f)(x) = \sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P,A:\mathfrak{I}}(H - H(x\gamma)) \cdot f^P(x\gamma) \qquad (x \in G).$$

For fixed x in G, Lemma 4.1 implies that the sum defining$T_{P|A}(H:f)(x)$ is actually finite. The assignment

$$T_{P|A}(H:f), \qquad x \mapsto T_{P|A}(H:f)(x),$$

thus defines a function (on G/Γ). As such it is locally bounded. Indeed, this is the case for f, so one need only quote Lemma 4.2.

PROPOSITION 4.3. *Suppose that f has compact support — then the same is true of $T_{P|A}(H:f)$.*

Proof. Choose a compact subset C of G such that $\mathrm{spt}(f) \subset C \cdot \Gamma/\Gamma$ — then, in view of Lemma 4.2, we can find a finite subset F_H of $\Gamma/\Gamma \cap$ P with the property that

$$H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a})(x \in C) \Rightarrow \gamma \in F_H.$$

Choose a finite subset F of Γ such that $F_H \subset F \cdot \Gamma \cap P/\Gamma \cap P$ and let $C(N)$ be a compact subset of N containing a fundamental domain for the action of $N \cap \Gamma$ on N — then $(C \cdot F \cdot C(N)) \cdot \Gamma/\Gamma$ is a compact subset of G/Γ and we claim that

$$\mathrm{spt}\big(T_{P|A}(H:f)\big) \subset (C \cdot F \cdot C(N)) \cdot \Gamma/\Gamma.$$

In fact, if

$$T_{P|A}(H:f)(x) \neq 0,$$

then there exists a $\gamma \in \Gamma$ such that

$$H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a}) \quad \text{and} \quad f^P(x\gamma) \neq 0,$$

thus an $n \in N$ such that

$$f(x\gamma n) \neq 0,$$

and finally a $\delta \in \Gamma$ such that

$$x\gamma n\delta \in C.$$

Put $y = x\gamma n\delta$ — then

$$\begin{aligned} H - H(y\delta^{-1}) &= H - H(x\gamma n) \\ &= H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a}) \end{aligned}$$

$$\Rightarrow$$

$$\delta^{-1} = \delta'\delta'' \qquad (\delta' \in F, \delta'' \in \Gamma \cap P).$$

Therefore

$$x\gamma = y\delta'\delta''n^{-1} = y\delta'(\delta''n^{-1}\delta''^{-1})\delta''.$$

Now write

$$\delta'' n^{-1} \delta''^{-1} = u\eta$$

where $u \in C(N)$ and $\eta \in \Gamma \cap P$ — then

$$x\gamma = y\delta' u \eta \delta''$$
$$\Rightarrow$$
$$x = y\delta' u(\eta\delta''\gamma^{-1}) \in C \cdot F \cdot C(N) \cdot \Gamma,$$

which settles the claim and, thereby, the proposition. □

If f is a cusp form, then

$$T_{P|A}(H:f) = 0 \qquad (P \neq G).$$

In general, some control can be gained by insisting that H be large and negative. More precisely:

PROPOSITION 4.4. *Let C be a compact subset of G. Supposing that $P \neq G$, let $H \in \mathfrak{a}$ be such that*

$$\rho(H) < -\frac{1}{3}\log\Big\{\sup_{x \in C} E(P \,|\, A : 1 : -2\rho : x)\Big\}.$$

Then

$$T_{P|A}(H:f) \,|\, C = 0.$$

Proof. The hypotheses at hand imply that

$$1 > e^{3\rho(H)} \cdot E(P \,|\, A : 1 : -2\rho : x) \qquad (\forall\, x \in C).$$

Therefore, thanks to the estimate provided by Lemma 4.1,

$$\{\gamma \in \Gamma/\Gamma \cap P \colon H - H(x\gamma) \in \mathcal{Q}_P(\mathfrak{a})\} = \varnothing \qquad (\forall\, x \in C).$$

Since an empty sum is conventionally null,

$$T_{P|A}(H:f) \,|\, C = 0,$$

as desired. □

The raison d'être for the introduction of $T_{P|A}(H:f)$ will become clear in the next section: There it will be seen that the truncation operator is an alternating linear combination of such entities, one for each element in a fixed set of representatives for the Γ-conjugacy classes of Γ-cuspidal split parabolic subgroups of G. The fact that the sum is alternating leads to certain analytical and combinatorial subtleties. By focusing on a generic term, these points will not arise in the present discussion.

There are two questions of equivariance which should be dealt with. Let us first consider the dependence of our definition on the split component A. Suppose that $n \in N$ — then it is immediate that

$$T_{P|nAn^{-1}}(n \cdot H:f) = T_{P|A}(H:f).$$

In other words, the definition of $T_{P|A}(H:f)$ is as independent of the choice of split component as can be expected. If now $\gamma \in \Gamma$, $x \in \gamma P$, then a direct application of the definition gives

$$T_{\gamma P\gamma^{-1}|xAx^{-1}}(x \cdot (H - H(\gamma)):f) = T_{P|A}(H:f).$$

Let $\gamma = kp$ ($k \in K$, $p \in P$) — then it follows in particular that

$$T_{\gamma P\gamma^{-1}|kAk^{-1}}(k \cdot (H - H(\gamma)):f) = T_{P|A}(H:f).$$

In this connection, note that were A the special split component of (P, S), then

$$k \cdot (H - H(\gamma)) = I_\Gamma(\gamma P\gamma^{-1} : P)(H),$$

the I_Γ-map being that from the preceding section. Accordingly, when we are working with special split components only, we shall write

$$T_P(H:f)$$

in place of

$$T_{P|A}(H:f).$$

So, for example, we have

$$T_{\gamma P\gamma^{-1}}(I_\Gamma(\gamma P\gamma^{-1} : P)(H):f) = T_P(H:f).$$

Before taking up our next result, we shall recall some definitions (cf. **[3.a]**) and a lemma of Langlands (cf. **[2.b]**). Let still (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A — then

$$\Xi_{P,A}: G \to \mathbf{R}$$

stands for the function defined by the rule

$$\Xi_{P,A}(x) = \inf_{\lambda \in \Sigma_P^0(\mathfrak{g}, \mathfrak{a})} a_x^\lambda \qquad (x \in G).$$

Since any two split components of (P, S) are N-conjugate, $\Xi_{P,A}$ is independent of the choice of A. It is thus permissible to write Ξ_P in place of $\Xi_{P,A}$.

Let f be a complex valued (measurable) function on G/Γ. Then:

(SI) f is said to be slowly increasing if there exists a real number r such that for every Siegel domain $\mathfrak{S}$ associated with a Γ-percuspidal parabolic subgroup P of G there is a positive constant C such that

$$|f(x)| \leq C \cdot \Xi_P(x)^r \qquad (x \in \mathfrak{S}).$$

(RD) f is said to be rapidly decreasing if for every real number r and for every Siegel domain $\mathfrak{S}$ associated with a Γ-percuspidal parabolic subgroup P of G there is a positive constant C such that

$$|f(x)| \leq C \cdot \Xi_P(x)^r \qquad (x \in \mathfrak{S}).$$

In either case, r is called an exponent of growth.

LEMMA 4.5. *Let $P_1, \ldots, P_{r_0}$ be Γ-percuspidal parabolic subgroups of G with associated Siegel domains $\mathfrak{S}_1, \ldots, \mathfrak{S}_{r_0}$ having the property that*

$$G = \bigcup_{i_0=1}^{r_0} \mathfrak{S}_{i_0} \cdot \Gamma.$$

Let f be a complex valued function on G/Γ — then f is slowly increasing (or rapidly decreasing) iff the requisite growth condition is met on the $\mathfrak{S}_{i_0}$ alone.

[The proof, while not difficult, is not entirely obvious either; see Langlands [**2.b**].]

The $T_{P|A}$-operation respects the slowly increasing functions on G/Γ in the following sense.

PROPOSITION 4.6. *Let f be a slowly increasing function on G/Γ — then*

$$T_{P|A}(H:f) \qquad (H \in \mathfrak{a})$$

is also a slowly increasing function on G/Γ.

We shall preface the proof with some comments of a general nature and a preliminary estimate. Let $S_r(G/\Gamma)$ be the set of slowly increasing functions f on G/Γ with exponent of growth r — then $S_r(G/\Gamma)$ is a Banach space under the norm

$$\|f\|_r = \max_{1 \leq i_0 \leq r_0} \sup_{x \in \mathfrak{S}_{i_0, \omega_0 \kappa_{i_0}}} \Xi_{P_{i_0}}(x)^{-r} |f(x)|.$$

Here the notation is as in the fundamental theorem of reduction (Theorem 3.1). If $S(G/\Gamma)$ is the set of all slowly increasing functions on G/Γ, then

$$S(G/\Gamma) = \bigcup_r S_r(G/\Gamma).$$

In passing, note that the union can be taken over all r less than some fixed r, e.g. -1. This said, we shall actually establish a somewhat more precise result, namely:

PROPOSITION 4.6. (bis) *For every $r < -1$ there exists an $r' < r$ such that*

$$T_{P|A}(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma).$$

Moreover, the (linear) operator

$$T_{P|A}\colon S_r(G/\Gamma) \to S_{r'}(G/\Gamma)$$

is bounded.

LEMMA 4.7. *Let f be a slowly increasing function on G/Γ with exponent of growth $r < -1$. Fix the index i_0 — then there exists a positive constant $C_{i_0}(f)$ such that*

$$|f(x)| \leq C_{i_0}(f)\cdot E\big(P_{i_0} | A_{i_0} : 1 : (2r+1)\rho_{i_0} : x\big)$$

for all $x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}$.

[Note: Observe that

$$r < -1 \Rightarrow (2r+1)\rho_{i_0} \in -\big(\rho_{i_0} + \mathcal{C}_{P_{i_0}}(\check{\mathfrak{a}}_{i_0})\big),$$

hence the Eisenstein series on the right-hand side of our estimate is convergent.]

Proof. Write $P_{i_0} = M_{i_0} \cdot A_{i_0} \cdot N_{i_0}$ so that

$$\mathfrak{S}_{t_0,\omega_0}\kappa_{i_0} = K \cdot A_{i_0}[t_0] \cdot \omega_{i_0}.$$

Decompose a given $x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}$ accordingly — then

$$\lambda(H_x) \leq \log t_0 \qquad \big(\forall\, \lambda \in \Sigma^0_{P_{i_0}}(\mathfrak{g}, \mathfrak{a}_{i_0})\big).$$

Put

$$2\rho_{i_0} = \sum_\lambda n_\lambda \lambda,$$

the n_λ being certain positive integers. Fix, for the moment, a λ_0. Since

$$n_{\lambda_0} r\lambda_0(H_x) \geq r\lambda_0(H_x) + (n_{\lambda_0} - 1) r \log t_0,$$

we have

$$2r\rho_{i_0}(H_x) \geq r\lambda_0(H_x) + \Big(\sum_\lambda n_\lambda - 1\Big) r \log t_0$$

from which it follows that

$$a_x^{2r\rho_{i_0}} \geq C_{i_0} \cdot a_x^{r\lambda_0}$$

where

$$\log C_{i_0} = \Big(\sum_\lambda n_\lambda - 1\Big) r \log t_0.$$

Taking inf's then gives

$$a_x^{2r\rho_{i_0}} \geq C_{i_0} \cdot \Xi_{P_{i_0}}(x)^r.$$

But now, from

$$|f(x)| \leq C \cdot \Xi_{P_{i_0}}(x)^r \qquad \big(x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}\big),$$

we may infer that

$$\begin{aligned} |f(x)| &\leq \big(C/C_{i_0}\big) \cdot a_x^{2r\rho_{i_0}} \\ &\leq \big(C/C_{i_0}\big) \cdot \sum_{\gamma \in \Gamma/\Gamma \cap P_{i_0}} a_{x\gamma}^{2r\rho_{i_0}} \\ &= \big(C/C_{i_0}\big) \cdot E\big(P_{i_0} \,|\, A_{i_0} : 1 : (2r+1)\rho_{i_0} : x\big) \end{aligned}$$

for all $x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}$. Taking $C_{i_0}(f) = C/C_{i_0}$ finishes the proof. □

With this preparation, we are in a position to broach the proof of Proposition 4.6 in the refined form indicated above (Proposition 4.6 (bis)). We can, of course, assume that f has exponent of growth $r < -1$. Bearing in mind that

$$G = \bigcup_{i_0=1}^{r_0} \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0} \cdot \Gamma,$$

on the basis of the preceding lemma, for any $x \in G$, we have

$$\begin{aligned} &|T_{P|A}(H:f)(x)| \\ &\leq \sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P,A:\mathfrak{s}}(H - H(x\gamma)) \cdot |f^P(x\gamma)| \\ &\leq \sum_{i_0=1}^{r_0} C_{i_0}(f) \\ &\quad \times \sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P,A:\mathfrak{s}}(H - H(x\gamma)) \cdot E^P\big(P_{i_0} \,|\, A_{i_0} : 1 : (2r+1)\rho_{i_0} : x\gamma\big). \end{aligned}$$

This makes it clear that we need only deal with

$$T_{P|A}\big(H : E\big(P_{i_0} \,|\, A_{\iota_0} : 1 : (2r+1)\rho_{i_0} : ?\big)\big)$$

for some fixed value of the index i_0. The function

$$E\big(P_{i_0} \,|\, A_{i_0} : 1 : (2r+1)\rho_{i_0} : ?\big)$$

is an automorphic form on G/Γ. Therefore, thanks to a well-known principle (cf. **[3.a]**), one has

$$E^P\big(P_{i_0} \,|\, A_{\iota_0} : 1 : (2r+1)\rho_{i_0} : k\,man\big) = \sum_i \Phi_\iota(k:m) p_i(H) e^{\Lambda_\iota(H)} \qquad (H = \log a)$$

where

$$\begin{cases} \Phi_i \text{ is an automorphic form on } K \times M/\{1\} \times \Gamma_M \\ p_i \text{ is a polynomial function on } \mathfrak{a} \\ \Lambda_i \text{ is a linear function on } \mathfrak{a}, \end{cases}$$

the summation being finite. [Note: $E^P(\ldots)$ is N-invariant.] Fix a set $\{(P_{\iota_0}, S_{\iota_0})\}$ of Γ-percuspidal split parabolic subgroups of G which are dominated predecessors of (P, S) and with the property that $\{(P^\dagger_{\iota_0}, S^\dagger_{\iota_0})\}$ is a set of representatives for the Γ_M-conjugacy classes of Γ_M-percuspidal split parabolic subgroups of M. Specifically:

$$(P, S; A) \succcurlyeq (P_{\iota_0}, S_{\iota_0}; A_{\iota_0}).$$

Because there exist Siegel domains $\mathfrak{S}^\dagger_{\iota_0}$ per $(P^\dagger_{\iota_0}, S^\dagger_{\iota_0}; A^\dagger_{\iota_0})$ with the property that

$$M = \bigcup_{\iota_0} \mathfrak{S}^\dagger_{\iota_0} \cdot \Gamma_M,$$

Lemma 4.7 (applied to the pair (M, Γ_M)) implies that

$$|\,\Phi_\iota(k:m)\,| \le \sum_{\iota_0} C_{\iota_0}(\Phi_i) \cdot E\big(P^\dagger_{\iota_0} \,|\, A^\dagger_{\iota_0} : 1 : (2r^\dagger + 1)\rho^\dagger_{\iota_0} : m\big)$$

for appropriate positive constants $C_{\iota_0}(\Phi_i)$ if only $r^\dagger \ll -1$. Here we had perhaps remind ourselves that as Φ_i is an automorphic form it is necessarily a slowly increasing function on $K \times M/\{1\} \times \Gamma_M$ or, equivalently, on M/Γ_M. All told, then, for the purpose of ascertaining the slow growth of

$$T_{P|A}(H : f),$$

it suffices to prove that $\forall \Lambda \in \check{\mathfrak{a}}$, $\exists\, r_\Lambda < -1$ such that for all indices ι_0 and all $r^\dagger < r_\Lambda$, the function

$$\sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P,A:\mathfrak{I}}(H - H(x\gamma)) \cdot a_{x\gamma}^{\Lambda} E\big(P_{\iota_0}^\dagger \,|\, A_{\iota_0}^\dagger : 1 : (2r^\dagger + 1)\rho_{\iota_0}^\dagger : m_{x\gamma}\big)$$

is slowly increasing on G/Γ. [Needless to say, $m_{x\gamma}$ is the M-component of $x\gamma$.] To this end, choose $r_\Lambda < -1$ in such a way as to force

$$\Lambda - 2r_\Lambda \rho \in \mathcal{C}_P(\check{\mathfrak{a}}).$$

Suppose that $r^\dagger < r_\Lambda$ — then

$$\Lambda - 2r^\dagger\rho = (\Lambda - 2r_\Lambda\rho) + 2\big(r_\Lambda - r^\dagger\big)\rho \in \mathcal{C}_P(\check{\mathfrak{a}}).$$

Thus

$$\begin{aligned} &H - H(x\gamma) \in \mathfrak{I}_P(\mathfrak{a}) \\ \Rightarrow\quad &0 \le \big\langle H - H(x\gamma), \Lambda - 2r^\dagger\rho \big\rangle \\ \Rightarrow\quad &\Lambda(H(x\gamma)) \le (\Lambda - 2r^\dagger\rho)(H) + 2r^\dagger\rho(H(x\gamma)) \\ \Rightarrow\quad &a_{x\gamma}^{\Lambda} \le e^{(\Lambda - 2r^\dagger\rho)(H)} a_{x\gamma}^{2r^\dagger\rho}. \end{aligned}$$

Our function is thereby seen to admit the majorization

$$e^{(\Lambda - 2r^\dagger\rho)(H)}$$

times

$$\sum_{\gamma \in \Gamma/\Gamma \cap P} a_{x\gamma}^{2r^\dagger\rho} E\big(P_{\iota_0}^\dagger \,|\, A_{\iota_0}^\dagger : 1 : (2r^\dagger + 1)\rho_{\iota_0}^\dagger : m_{x\gamma}\big).$$

But the last expression is, by the lemma of descent for Eisenstein series (cf. **[3.a]**), precisely

$$E\big(P_{\iota_0} \,|\, A_{\iota_0} : 1 : (2r^\dagger + 1)\rho_{\iota_0} : x\big).$$

The slow increase of

$$T_{P|A}(H : f)$$

is now apparent. Furthermore, our estimates make it plain that for every $r < -1$ there exists an $r' < r$ such that

$$T_{P|A}(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma).$$

Indeed, a quick perusal of the discussion supra leads at once to the conclusion that

$$
\begin{gathered}
|f(x)| \leq C \cdot \Xi_{P_{i_0}}(x)^r \\
\Rightarrow \\
|T_{P|A}(H:f)(x)| \leq CC' \cdot \Xi_{P_{i_0}}(x)^{r'} \quad \left(x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}\right),
\end{gathered}
$$

C' a positive constant depending on r but not on f (or i_0). This remark proves that

$$T_{P|A}(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma)$$

and, at the same time, exhibits the boundedness of the operation.

The next step in our investigation hinges upon an elementary formal computation.

PROPOSITION 4.8. *Let f, g be bounded measurable compactly supported functions on G/Γ — then*

$$\left(T_{P|A}(H:f), g\right) = \left(f, T_{P|A}(H:g)\right).$$

Proof. The left-hand side of the putative equality, i.e.

$$\int_{G/\Gamma} T_{P|A}(H:f)(x)\overline{g(x)}d_G(x),$$

is equal to

$$\int_{G/\Gamma}\Big(\sum_{\gamma\in\Gamma/\Gamma\cap P} \chi_{P,A:\mathfrak{S}}(H - H(x\gamma))\cdot f^P(x\gamma)\Big)\overline{g(x)}d_G(x)$$

or still

$$\int_{G/\Gamma\cap P} \chi_{P,A:\mathfrak{S}}(H - H(x))\cdot f^P(x)\overline{g(x)}d_G(x)$$

or still

$$\int_{G/\Gamma\cap P}\int_{N/N\cap\Gamma} \chi_{P,A:\mathfrak{S}}(H - H(x))f(xn)\overline{g(x)}d_N(n)\, d_G(x)$$

or still

$$\int_{G/\Gamma\cap P}\int_{N/N\cap\Gamma}\int_{N/N\cap\Gamma}$$

of

$$\chi_{P,A:\mathfrak{S}}(H - H(xn_1))f(xn_1n_2)\overline{g(xn_1)}$$

or still

$$\int_{G/\Gamma\cap P}\int_{N/N\cap\Gamma}\int_{N/N\cap\Gamma}$$

of

$$\chi_{P,A:\mathfrak{I}}(H - H(x))f(xn_2)\overline{g(xn_1)}.$$

Now switch the order of integration in n_1 and n_2 (legitimate because of our hypotheses) — then f and $\bar{g}$ are interchanged, so, by symmetry, we recover

$$\big(f, T_{P|A}(H:g)\big),$$

serving, therefore, to establish the desired equality. □

Inspection of the foregoing argument enables one to assert the validity of its conclusion under a weaker set of hypotheses, viz: f, g locally bounded measurable functions on G/Γ, one of which alone with compact support. For then either

$$\big(T_{P|A}(H:|f|), |g|\big) \quad \text{or} \quad \big(|f|, T_{P|A}(H:|g|)\big)$$

is finite and this allows the Fubini-type manipulations.

Looking back over what has been obtained so far, we see that the $T_{P|A}$-operation possesses a number of characteristic attributes. Let

$$Q = \sum r_i T_{P_i|A_i}(H_i:?) \qquad (r_i \in \mathbf{R})$$

be a real finite linear combination of such entities. Then

$$Q\colon S(G/\Gamma) \to S(G/\Gamma)$$

is a linear map having the property that for every $r < -1$ there exists an $r' < r$ such that

$$Q(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma),$$

$Q \mid S_r(G/\Gamma)$ being continuous. Moreover, for all bounded measurable compactly supported functions f, g on G/Γ,

$$(Qf, g) = (f, Qg),$$

this relation actually holding under the less stringent conditions indicated above.

PROPOSITION 4.9. *Assume that*

$$Q \circ Q = Q.$$

Then

$$Q(S(G/\Gamma) \cap L^2(G/\Gamma)) \subset L^2(G/\Gamma)$$

and the closure of

$$Q \mid S(G/\Gamma) \cap L^2(G/\Gamma)$$

is an orthogonal projection on $L^2(G/\Gamma)$.

Of course, the key new point is the hypothesis of idempotence: $Q \circ Q = Q$. $T_{P|A}(H:?)$ will generally not have this property but what is remarkable and, as it turns out, of crucial importance, certain real finite linear combinations Q of such entities will. This question will in fact be a central topic of the next section.

As for the proposition, the proof is easy enough. Suppose to begin with that f is a bounded measurable compactly supported function on G/Γ — then we have

$$\begin{aligned}(Qf, f) &= (Q \circ Qf, f)\\ &= (Qf, Qf)\end{aligned}$$

$$\Rightarrow$$

$$\|Qf\| \leq \|f\|.$$

Consequently, Q, restricted to the bounded measurable compactly supported functions on G/Γ, extends to a bounded self-adjoint idempotent operator on $L^2(G/\Gamma)$, that is, to an orthogonal projection on $L^2(G/\Gamma)$. Call this extension $\bar{Q}$. To complete our proof, we need only show that Q and $\bar{Q}$ agree on

$$S(G/\Gamma) \cap L^2(G/\Gamma).$$

Take a function f in this set. Let C be any compact subset of G/Γ, χ_C its characteristic function —then

$$\begin{aligned}(\bar{Q}f, \chi_C) &= (f, \bar{Q}\chi_C)\\ &= (f, Q\chi_C) = (Qf, \chi_C)\end{aligned}$$

$$\Rightarrow$$

$$\int_C (\bar{Q}f - Qf) = 0,$$

so, by inner regularity, $\bar{Q}f = Qf$ a.e. on G/Γ.

It will eventually be necessary to employ some estimates of a character quite different from those encountered supra. What we have in mind here are variants on well-known themes of Harish-Chandra and Langlands. But what they have is not exactly what we need so it will be safer to proceed from first principles.

Let $S_r^\infty(G/\Gamma)$ be the space of slowly increasing differentiable functions f on G/Γ with exponent of growth r such that for every right invariant differential operator D on G, Df is also slowly increasing with exponent of growth r — then the semi-norms

$$|f|_{r,D} = \max_{1 \leq i_0 \leq r_0} \sup_{x \in \mathfrak{S}_{i_0,\omega_0}\kappa_{i_0}} \Xi_{P_{i_0}}(x)^{-r} |Df(x)|$$

serve to equip $S_r^\infty(G/\Gamma)$ with the structure of a Fréchet space. The discussion in the remainder of this section will center on the estimation theory of $S_r^\infty(G/\Gamma)$.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with special split component A; assume, in addition, that $P \neq G$. Let F, F' be subsets of $\Sigma_P^0(\mathfrak{g}, \mathfrak{a})$; assume, in addition, that $F \neq \emptyset$, $F' \subset F$.

LEMMA 4.10. *There exist normal subgroups*

$$\{N_\mu : 1 \leq \mu \leq d + 1\}$$

of $N_{F'}$ *such that*

(1) $N_{F'} = N_1 \supset N_2 \supset \ldots \supset N_d \supset N_{d+1} = N_F$,
(2) $\dim(N_\mu/N_{\mu+1}) = 1$,
(3) N_μ *is* A*-stable,*
(4) $\Gamma \cap N_\mu$ *is a lattice in* N_μ.

Proof. Fix a Γ-percuspidal split parabolic subgroup P_0 of G with special split component A_0 such that

$$(P, S; A) \succcurlyeq (P_0, S_0; A_0).$$

The roots λ in $\Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ can be arranged in a lexicographic order so as to guarantee that if

$$\mathfrak{n}_{\Lambda_0} = \sum_{\lambda \geq \Lambda_0} \mathfrak{g}_\lambda, \qquad N_{\Lambda_0} = \exp(\mathfrak{n}_{\Lambda_0}),$$

then $\Gamma \cap N_{\Lambda_0}$ is a lattice in N_{Λ_0} (cf. [**3.a**]). There is no loss of generality in supposing that

$$\begin{cases} N_F = N_\Lambda \\ N_{F'} = N_{\Lambda'} \end{cases}$$

where $\Lambda' < \Lambda$. Then

$$\mathfrak{n}_{F'} = \mathfrak{n}_F \oplus \sum_{\{\lambda:\ \Lambda' \leq \lambda < \Lambda\}} \mathfrak{g}_\lambda.$$

List the elements of $\{\lambda: \Lambda' \leq \lambda < \Lambda\}$ in increasing order: $\Lambda' = \lambda_1 < \lambda_2 \ldots$ Fix, as is possible, a subgroup $\Gamma_{F'}$ of $\Gamma \cap N_{F'}$ with the property that

$$\log(\Gamma_{F'})$$

is a lattice in $\mathfrak{n}_{F'}$. Put

$$\Gamma_{F'}(1) = \mathfrak{g}_{\lambda_1} \cap \Big(\log(\Gamma_{F'}) + \sum_{\lambda > \lambda_1} \mathfrak{g}_\lambda\Big)$$

$$\Gamma_{F'}(2) = \mathfrak{g}_{\lambda_2} \cap \Big(\log(\Gamma_{F'}) + \sum_{\lambda > \lambda_2} \mathfrak{g}_\lambda\Big)$$

$$\ldots$$

Choose a basis $\{X_1, \ldots, X_d\}$ of

$$\sum_{\{\lambda:\ \Lambda' \leq \lambda < \Lambda\}} \mathfrak{g}_\lambda$$

such that

the first $\dim(\mathfrak{g}_{\lambda_1})$ come from $\Gamma_{F'}(1)$

the second $\dim(\mathfrak{g}_{\lambda_2})$ come from $\Gamma_{F'}(2)$

$$\ldots$$

Finally, set

$$\mathfrak{n}_\mu = \mathfrak{n}_F \oplus \operatorname{span}\{X_\mu, \ldots, X_d\}.$$

Then the

$$N_\mu = \exp(\mathfrak{n}_\mu)$$

satisfy all the requirements of our lemma. □

Keeping to the preceding notations and assumptions, let

$$\phi_{P,F} = \sum_{\{F':\ F' \subset F\}} (-1)^{\operatorname{rank}(P_{F'})} \cdot f^{P_{F'}},$$

f a complex valued locally bounded (measurable) function on G/Γ. It is the estimation of $\phi_{P,F}$ which is now our primary concern. Of course, functions of this type arise in the theory of Eisenstein series so it should not be unexpected that they will also play a role here.

Let us agree to write

$$\pi_F(f)$$

for f^{P_F} — then it is clear that

$$\pi_{F'} \circ \pi_{F''} = \pi_{F' \cap F''}.$$

Accordingly,

$$\begin{aligned}\phi_{P,F} &= \sum_{\{F' :\, F' \subset F\}} (-1)^{\#(F')} \pi_{F'}(f) \\ &= \prod_{\{F' :\, F' \subset F,\, \#(F - F') = 1\}} (\pi_F - \pi_{F'})(f).\end{aligned}$$

On the face of it, therefore, one might reasonably attempt to estimate $\phi_{P,F}$ by first estimating

$$(\pi_F - \pi_{F'})(f)$$

in a uniform manner and then taking products. This is indeed sufficient for many applications but, as it turns out, our situation is more delicate, so we shall have to proceed somewhat differently.

Upon writing

$$(P_F, S_F; A_F) \succcurlyeq (P_{F'}, S_{F'}; A_{F'}),$$

we determine a Γ_{M_F}-cuspidal split parabolic subgroup $(P^\dagger_{F'}, S^\dagger_{F'})$ of M_F with special split component $A^\dagger_{F'}$. One has

$$N_{F'} = N^\dagger_{F'} \cdot N_F$$

or still

$$N^\dagger_{F'} = N_{F'}/N_F,$$

hence

$$N^\dagger_{F'} \cap \Gamma_{M_F} = (N_{F'} \cap \Gamma) \cdot N_F/N_F.$$

This said, it then follows that

$$\begin{aligned}\int_{N_{F'}/N_{F'} \cap \Gamma} &= \int_{N_{F'}/(N_{F'} \cap \Gamma) \cdot N_F} \int_{(N_{F'} \cap \Gamma) \cdot N_F/N_{F'} \cap \Gamma} \\ &= \int_{N^\dagger_{F'}/N^\dagger_{F'} \cap \Gamma_{M_F}} \int_{N_F/N_F \cap \Gamma}\end{aligned}$$

or still

$$\pi_{F'} = \pi^\dagger_{F'} \circ \pi_F.$$

We can thus rewrite $\phi_{P,F}$, namely

$$\phi_{P,F} = \prod_{\{F' \colon F' \subset F, \#(F-F')=1\}} (1_F - \pi_{F'}^{\dagger})(\pi_F(f)).$$

The thrust of this remark lies in the observation that any partial product

$$\prod_{F'} (1_F - \pi_{F'})(\pi_F(f)),$$

qua a function on G, is invariant to the right under $(N \cap \Gamma) \cdot N_F$.

The next thing to do is to set the stage for an application of Lemma 4.10. As there, we have normal subgroups N_μ, $N_F \subset N_\mu \subset N_{F'}$, with the properties (1)–(4). Put

$$\pi_\mu(f) = \int_{N_\mu/N_\mu \cap \Gamma} f(?n_\mu)\, d_{N_\mu}(n_\mu).$$

Then

$$\pi_F - \pi_{F'} = \sum_{\mu=1}^{d} (\pi_{\mu+1} - \pi_\mu).$$

On the other hand, if we write

$$N_\mu^{\dagger} \quad \text{for} \quad N_\mu/N_F,$$

then an integral manipulation entirely analogous to the one carried out above gives

$$\pi_\mu = \pi_\mu^{\dagger} \circ \pi_F.$$

Consequently,

$$\left(1_F - \pi_{F'}^{\dagger}\right) \circ \pi_F = \sum_{\mu=1}^{d} \left(\pi_{\mu+1}^{\dagger} - \pi_\mu^{\dagger}\right) \circ \pi_F.$$

The quotient

$$N_\mu^{\dagger}/N_{\mu+1}^{\dagger} = N_\mu/N_{\mu+1}$$

is one dimensional. Pick an element $X_\mu \in \mathfrak{n}_\mu/\mathfrak{n}_F$ such that $\exp(X_\mu) \cdot N_{\mu+1}^{\dagger}$ generates

$$\left(N_\mu^{\dagger} \cap \Gamma_{M_F}\right) \cdot N_{\mu+1}^{\dagger}/N_{\mu+1}^{\dagger} = N_\mu^{\dagger} \cap \Gamma_{M_F}/N_{\mu+1}^{\dagger} \cap \Gamma_{M_F}.$$

If, in a general way, for $t \in \mathbf{R}$,

$$\Phi_\mu(t) = \int_{N_{\mu+1}^{\dagger}/N_{\mu+1}^{\dagger} \cap \Gamma_{M_F}} \Phi\left(\exp(tX_\mu) n_{\mu+1}\right) d_{N_{\mu+1}}(n_{\mu+1}),$$

then the difference

$$\Phi_\mu(t) - \int_0^1 \Phi_\mu(s)\, ds$$

computes

$$\left(\pi^\dagger_{\mu+1} - \pi^\dagger_\mu\right)(\Phi)\,|_t.$$

LEMMA 4.11. *Let $f \in C^\infty(\mathbf{R}/\mathbf{Z})$. Put*

$$\hat{f}(0) = \int_0^1 f(x)\, dx.$$

Then, for every non-negative integer k,

$$\| f - \hat{f}(0) \|_\infty \leq 2^{-k} \cdot \| f^{(k)} \|_\infty.$$

Proof. We shall give two proofs.

Method 1. Write

$$f(x) - \hat{f}(0) = \sum_{n \neq 0} \hat{f}(n) e^{-2\pi\sqrt{-1}\, nx}$$

where

$$\hat{f}(n) = \int_0^1 f(x) e^{2\pi\sqrt{-1}\, nx}\, dx.$$

Then

$$f(x) - \hat{f}(0) = \sum_{n \neq 0} \left(\frac{-1}{2\pi\sqrt{-1}\, n} \right)^k \cdot \hat{f}^{(k)}(n) e^{-2\pi\sqrt{-1}\, nx}$$

implying, therefore, that

$$\begin{aligned}
| f(x) - \hat{f}(0) | &\leq \sum_{n \neq 0} \left(\frac{1}{2\pi n} \right)^k \cdot | \hat{f}^{(k)}(n) | \\
&\leq \left(\sum_{n \neq 0} \left| \frac{1}{2\pi n} \right|^{2k} \right)^{1/2} \cdot \left(\sum_{n \neq 0} | \hat{f}^{(k)}(n) |^2 \right)^{1/2} \\
&\leq 2 \cdot \left(\frac{1}{2\pi} \right)^k \cdot \left(\sum_{n=1}^\infty \frac{1}{n^{2k}} \right)^{1/2} \cdot \| f^{(k)} \|_2 \\
&\leq 2^{-k} \cdot \left(\int_0^1 | f^{(k)}(x) |^2\, dx \right)^{1/2} \\
&\leq 2^{-k} \cdot \| f^{(k)} \|_\infty,
\end{aligned}$$

or still

$$\|f-\hat{f}(0)\|_\infty \leq 2^{-k}\cdot\|f^{(k)}\|_\infty,$$

as was to be shown.

Method 2. Write

$$\begin{aligned} f(x)-\hat{f}(0) &= f(x)-\hat{f}(0)-\int_{x-1}^{x}(f(y)-\hat{f}(0))\,dy \\ &= \int_{x-1}^{x}\big((f(x)-\hat{f}(0))-(f(y)-\hat{f}(0))\big)\,dy \\ &= \int_{x-1}^{x}(f(x)-f(y))\,dy \\ &= \int_{x-1}^{x}\left(\int_{y}^{x}f'(t)\,dt\right)dy \\ &= \int_{x-1}^{x}\left(\int_{x-1}^{t}f'(t)\,dy\right)dt \\ &= \int_{x-1}^{x}(t-(x-1))f'(t)\,dt. \end{aligned}$$

Consequently,

$$|f(x)-\hat{f}(0)|\leq\left(\int_0^1 t\,dt\right)\cdot\|f'\|_\infty = 2^{-1}\|f'\|_\infty.$$

Because f is periodic with period 1,

$$\int_0^1 f^{(k)}(x)\,dx = 0 \qquad (k\geq 1).$$

It thus follows by induction that

$$|f(x)-\hat{f}(0)|\leq 2^{-k}\cdot\|f^{(k)}\|_\infty$$

or still

$$\|f-\hat{f}(0)\|_\infty \leq 2^{-k}\cdot\|f^{(k)}\|_\infty,$$

as was to be shown.

Hence the lemma. □

To be able to apply estimates of the foregoing type, we need to impose conditions of differentiability on f. Since there is nothing to be gained by striving for maximum generality, we shall simply assume that f is C^∞ — then

$$f^{P_F}=\pi_F(f)\in C^\infty(G/(N\cap\Gamma)\cdot N_F).$$

Let $\{\lambda_i\}$ be an enumeration of the elements of F. Put $F_i = F - \{\lambda_i\}$ — then

$$\{F' : \#(F - F') = 1\} = \{F_i\}.$$

In these notations, with $p = \#(F)$,

$$\phi_{P,F} = \prod_{i=1}^{p} (\pi_F - \pi_{F_i})(f)$$

or still

$$\phi_{P,F} = \prod_{i=1}^{p} (1_F - \pi_i^\dagger)(\pi_F(f))$$

where, for simplicity, $\pi_i^\dagger = \pi_{F_i}^\dagger$.

Given a subset $\mathbb{S}$ of $\{1,\ldots,p\}$, put

$$\Phi(P:F:\mathbb{S}:f) = \prod_{i\in\mathbb{S}} (1_F - \pi_i^\dagger)(\pi_F(f)).$$

Then it is clear that

$$\Phi(P:F:\mathbb{S}:f) \in C^\infty(G/(N\cap\Gamma)\cdot N_F)$$

with

$$D\Phi(P:F:\mathbb{S}:f) = \Phi(P:F:\mathbb{S}:Df)$$

for any right invariant differential operator D on G.

We can now describe the basic idea behind the estimation of $\phi_{P,F}$. For any i between 1 and p, let

$$\mathbb{S}_i = \{1,\ldots,i\}.$$

Write

$$\phi_{P,F} = (1_F - \pi_p^\dagger)\big(\Phi(P:F:\mathbb{S}_{p-1}:f)\big).$$

It will then we shown that $\phi_{P,F}$ can be estimated in terms of certain derivatives D of

$$\Phi(P:F:\mathbb{S}_{p-1}:f).$$

Since

$$\Phi(P:F:\mathbb{S}_{p-1}:Df) = (1_F - \pi_{p-1}^\dagger)\big(\Phi(P:F:\mathbb{S}_{p-2}:Df)\big),$$

the argument proceeds via iteration on a step-by-step basis.

Before taking up the details, we had best establish a convention or two.

Let

$$\Phi \in C^\infty(G/(N \cap \Gamma)\cdot N_F).$$

In what follows, it will sometimes be necessary to view Φ as a function on $G \times N$:

$$\Phi(x:n) = \Phi(xn).$$

When this is done, we then employ without comment the usual tensor product formalism for differential operators on product spaces.

Given $F' \subset F$, $\#(F - F') = 1$, let $\omega_{F'}^\dagger$ be a compact neighborhood of 1 in $N_{F'}^\dagger$ with the property that

$$\omega_{F'}^\dagger \cdot \left(N_{F'}^\dagger \cap \Gamma_{M_F}\right) = N_{F'}^\dagger.$$

Write

$$\|?\|_{F'}^\dagger$$

for the sup norm calculated on $\omega_{F'}^\dagger$. If Φ is per supra, then

$$\|\Phi\|_{F'}^\dagger = \sup_{N_{F'}^\dagger} |\Phi|,$$

Φ being, in particular, right invariant under $N_{F'}^\dagger \cap \Gamma_{M_F}$.

LEMMA 4.12. *Let $F' \subset F$, $\#(F - F') = 1$ — then, for every non-negative integer k, and any*

$$\Phi \in C^\infty(G/(N \cap \Gamma)\cdot N_F),$$

$\forall\, x \in G$,

$$|(1_F - \pi_{F'}^\dagger)(\pi_F(\Phi))(x)| \leq 2^{-k}\cdot d \cdot \max_{1 \leq \mu \leq d} \|(1 \otimes X_\mu^k)\Phi(x:?)\|_{F'}^\dagger.$$

Proof. Write

$$\left(1_F - \pi_{F'}^\dagger\right) \circ \pi_F = \sum_{\mu=1}^{d} \left(\pi_{\mu+1}^\dagger - \pi_\mu^\dagger\right) \circ \pi_F.$$

Because

$$\pi_F(\Phi) = \Phi,$$

we have only to estimate

$$|(1_F - \pi_{F'}^\dagger)(\Phi)(x)|$$

or still, the individual

$$|\left(\pi_{\mu+1}^\dagger - \pi_\mu^\dagger\right)(\Phi)(x)|.$$

In turn, thanks to Lemma 4.11 and the remarks prefacing its formulation,

$$|(\pi^\dagger_{\mu+1} - \pi^\dagger_\mu)(\Phi)(x)|$$

can be estimated vis-à-vis

$$\int_{N^\dagger_{\mu+1}/N^\dagger_{\mu+1}\cap\Gamma_{M_F}} \Phi(x\exp(tX_\mu)n_{\mu+1})\, d_{N_{\mu+1}}(n_{\mu+1})$$

and the corresponding 'constant term', i.e., the associated integral from 0 to 1. In this way, we find that

$$|(\pi^\dagger_{\mu+1} - \pi^\dagger_\mu)(\Phi)(x)|$$

is majorized by 2^{-k} times

$$\sup\left|\frac{d^k}{dt^k}[?]\right|,$$

[?] being the t-dependent integral above. As the latter cannot exceed

$$\|(1\otimes X^k_\mu)\Phi(x:?)\|^\dagger_{F'},$$

an application of the triangle inequality completes the proof. □

In passing, let us observe that

$$(1\otimes X^k_\mu)\Phi(x:n) = (\mathrm{Ad}(x)X^k_\mu\cdot\Phi)(xn).$$

To set up the statement of the main result in this circle of ideas, make the following replacements in the data:

$$\begin{cases} F' \to F_i \\ \omega^\dagger_{F_i} \to \omega^\dagger_i \\ d \to d_i \\ X_\mu \to X_{\mu_i}. \end{cases}$$

PROPOSITION 4.13. *Let $f\in C^\infty(G/\Gamma)$ — then for every p-tuple $\mathbf{k} = (k_1,\ldots,k_p)$ of non-negative integers k_i there exists a positive constant $C_{\mathbf{k}}$ such that*

$$|\phi_{P,F}(x)|$$

is majorized by $C_{\mathbf{k}}$ times the maximum over all

$$\begin{cases} 1\le\mu_1\le d_1 \\ \vdots \\ 1\le\mu_p\le d_p \end{cases}$$

of the supremum over all

$$\begin{cases} n_1^\dagger \in \omega_1^\dagger \\ \vdots \\ n_p^\dagger \in \omega_p^\dagger \end{cases}$$

of the absolute value of

$$\operatorname{Ad}(xn_p^\dagger \ldots n_2^\dagger) X_{\mu_1}^{k_1} \cdot \operatorname{Ad}(xn_p^\dagger \ldots n_3^\dagger) X_{\mu_2}^{k_2}$$
$$\ldots \operatorname{Ad}(xn_p) X_{\mu_{p-1}}^{k_{p-1}} \cdot \operatorname{Ad}(x) X_{\mu_p}^{k_p} \cdot \pi_F(f)(xn_p^\dagger \ldots n_1^\dagger).$$

The importance (and therefore the significance) of this estimate will become clear in due course. At first glance, one might think that it would be awkward to use in actual practice. But this is not the case at all. For in the applications, x, which is a priori arbitrary, will be restricted in a certain way. Since

$$\omega_p^\dagger \ldots \omega_1^\dagger$$

is compact, something specific can then be said.

For instance, suppose that $f \in S_r^\infty(G/\Gamma)$. There is a strictly positive function E_r on G, a linear combination of Eisenstein series, such that $\forall D$

$$|Df(x)| \leq C(f, D) \cdot |E_r(x)| \qquad (x \in G),$$

$C(f, D)$ a positive constant. [Note: The existence of E_r is ensured by Lemma 4.7; E_r does not, of course, depend on f.] Now suppose that we confine x to a compact subset Ω of G — then the differential operators figuring in our proposition stay within a compact subset of all the right invariant differential operators on G (equipped with the usual LF-topology), so, ignoring positive constants,

$$\sup_{x \in \Omega} |\phi_{P,F}(x)|$$

is no more than

$$\sup_{x \in \Omega} \sup_{n^\dagger \in \omega_p^\dagger \ldots \omega_1^\dagger} |\pi_F(E_r)(xn^\dagger)|,$$

an inequality which is indeed fundamental.

Proof of Proposition 4.13. In view of the preparation which has been already undertaken, the proof itself is virtually obvious. One simply writes (cf. supra)

$$\phi_{P,F} = (1_F - \pi_p^\dagger)(\Phi(P : F : \mathbb{S}_{p-1} : f))$$

and then, to be completely formal about it, utilizes downward induction. □

We shall close this section with some remarks which stand by themselves although they will not be fully exploited until subsequent papers in this series.

Put

$$E_r(P_{i_0}:?) = E(P_{i_0}\,|\,A_{i_0}:1:(2r+1)\rho_{i_0}:?).$$

Then (cf. Lemma 4.7)

$$E_r = \sum_{i_0=1}^{r_0} E_r(P_{i_0}:?).$$

The role of the E_r on G/Γ is that of providing universal majorants for slowly increasing functions, a point of obvious technical value. It is then only natural to ask: Can one find analogues of the E_r for rapidly decreasing functions? We shall now take up this question.

Let q be a real parameter. Introduce

$$\zeta_q = \sum_{i_0=1}^{r_0} \zeta_q(P_{i_0}:?)$$

where, by definition,

$$\zeta_q(P_{i_0}:x) = \sum_{\gamma\in\Gamma/\Gamma\cap P_{i_0}} \exp\left(-q\cdot\|H_{P_{i_0}|A_{i_0}}(x\gamma)\|\right) \qquad (x\in G).$$

Convergence can be secured by assuming that, e.g., $q > 2\|\rho_{i_0}\|$, in which case the corresponding function is slowly increasing.

LEMMA 4.14. (i) $\forall\, c$, $\exists\, q_c$, Q_c *such that*

$$\zeta_{q_c} \le Q_c\cdot \Xi^c_{P_{i_0}} \quad on \quad \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}.$$

(ii) $\forall\, q$, $\exists\, c_q$, C_q *such that*

$$\Xi^{c_q}_{P_{i_0}} \le C_q\cdot\zeta_q \quad on \quad \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}.$$

This result carries with it the immediate consequence that the ζ_q are universal majorants for rapidly decreasing functions on G/Γ. Indeed, any such f has the property that $\forall\, q(> 2\|\rho_{i_0}\|)$ there exists a positive constant $C_f(q)$ such that

$$|f| \le C_f(q)\cdot\zeta_q$$

and conversely.

To prove Lemma 4.14 we shall need an estimate on the E_r which itself depends on still another estimate, the proof of which will be given later on.

LEMMA 4.15. *Let* $1 \leq i_0, j_0 \leq r_0$ — *then* $\exists\, C_r > 0$ *such that*

$$\forall x \in G$$
$$E_r(P_{j_0} : x) \leq C_r \cdot \exp\left(-2rC_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right).$$

[Note: C_0 is a positive constant which does not depend on r.]

Proof. Take, in the notations of Sublemma 4 (§7) infra,

$$\begin{cases} C' = K \\ C'' = \omega_{i_0}. \end{cases}$$

Given $x \in G$, write

$$x = k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x \delta_x$$

per

$$G = K \cdot A_{i_0} \cdot \omega_{i_0} \cdot \left(S_{i_0} \cap \Gamma\right).$$

Then that result implies that for every $\gamma \in \Gamma$,

$$\left\langle H_{P_{j_0}|A_{j_0}}\left(k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x \gamma\right), \rho_{j_0}\right\rangle$$
$$\geq \left\langle H_{P_{j_0}|A_{j_0}}(k_x \gamma), \rho_{j_0}\right\rangle - C_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \| - C_{00}$$

for certain positive constants C_0, C_{00}. It therefore follows that

$$\begin{aligned} E_r(P_{j_0} : x) &= E_r\left(P_{j_0} : k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x \delta_x\right) \\ &= E_r\left(P_{j_0} : k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x\right) \\ &\leq E_r\left(P_{j_0} : k_x\right) \\ &\qquad \times \exp\left(-2r\left(C_{00} + C_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right)\right) \\ &\leq C_r \cdot \exp\left(-2rC_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right) \end{aligned}$$

where

$$C_r = E_r\left(P_{j_0} : 1\right) \cdot \exp(-2rC_{00}).$$

Hence the lemma. □

Proof of Lemma 4.14(i). There is no loss of generality in supposing that $c > 1$. Fix j_0, $1 \le j_0 \le r_0$ — then it will be enough to show that, up to a positive constant,

$$\zeta_{q_c}(P_{i_0} : ?)$$

is majorized on $\mathfrak{S}_{t_0,\omega_0}\kappa_{j_0}$ by $\Xi^c_{P_{j_0}}$ for some $q_c \gg 0$. We have (cf. supra)

$$1 \le C_r \cdot E_r(P_{j_0} : x)^{-1} \exp(-2rC_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|).$$

Therefore, for any q,

$$\zeta_q(P_{i_0} : x) \le C_r \cdot E_r(P_{j_0} : x)^{-1} \zeta_{q(r)}(P_{i_0} : x)$$

where

$$q(r) = q + 2rC_0.$$

Let now

$$x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{j_0}.$$

There is a constant $c_\zeta < -1$ such that

$$\zeta_{3\|\rho_{i_0}\|}(P_{i_0} : x) \le C_\zeta \cdot \Xi_{P_{j_0}}(x)^{c_\zeta}.$$

On the other hand, as can be seen from the proof of Lemma 4.7, there is a constant $r < -1$ with the property that

$$E_r(P_{j_0} : x) \ge C_{j_0} \cdot \Xi_{P_{j_0}}(x)^{c_\zeta - c}.$$

Put

$$q_c = 3\|\rho_{i_0}\| - 2rC_0.$$

Then, on $\mathfrak{S}_{t_0,\omega_0}\kappa_{j_0}$, we have

$$\begin{aligned} &\zeta_{q_c}(P_{i_0} : x) \\ &\quad \le C_r \cdot E_r(P_{j_0} : x)^{-1} \zeta_{q_c(r)}(P_{i_0} : x) \\ &\quad \le C_r \cdot E_r(P_{j_0} : x)^{-1} \zeta_{3\|\rho_{i_0}\|}(P_{i_0} : x) \\ &\quad \le C_r C_\zeta C_{j_0}^{-1} \cdot \Xi_{P_{j_0}}(x)^c, \end{aligned}$$

from which our assertion follows. □

Proof of Lemma 4.14(ii). Given q, set

$$c_q = q / \max_\lambda \|\lambda\|,$$

λ running through $\Sigma^0_{P_{i_0}}(\mathfrak{g}, \mathfrak{a}_{i_0})$. Using definitions only, we then find that on $\mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}$,

$$\Xi_{P_{i_0}}(x)^{c_q} \leq C_q \cdot \exp\left(-q \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right),$$

C_q a positive constant which need not be explicated. Since

$$\exp\left(-q \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right) \leq \zeta_q(P_{i_0} : x) \leq \zeta_q(x),$$

we are done. □

5. The truncation operator. The purpose of this section is to define and study the truncation operator. The idea behind its introduction can be traced to the works of Langlands (especially [**2.a**]) who, however, only proceeded on an ad hoc basis in certain special cases. It was Arthur [**1.b**] who gave, in the adelic setting, a general definition and, in that situation, established its essential properties. We considered in [**3.b**] the case of Γ-rank one lattices. Here we shall deal with the general case. If it were only a question of one cusp, then the present discussion could be modeled, to some extent at least, after that of Arthur. But, of course, Γ will ordinarily possess more than one cusp, a point which causes a number of complications thereby necessitating a treatment which differs radically from Arthur's. The definition itself will be easy enough. From then on, though, there will be a host of difficulties to overcome. For this reason, we shall content ourselves initially with precise statements only, deferring the proofs to subsequent sections.

We begin by recasting the definition of

$$T_{P|A}(H : f)$$

from the preceding section. So, as there, let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A which we take now to be special. Recall that in this situation we write

$$T_P(H : f)$$

in place of

$$T_{P|A}(H : f).$$

Given $\mathbf{H} \in \mathfrak{a}$, define

$$T_P(\mathbf{H} : f)$$

by

$$T_P(I_P(\mathbf{H}) : f).$$

It is then the case that

$$T_{\gamma P\gamma^{-1}}(\mathbf{H}:f) = T_P(\mathbf{H}:f)$$

for all $\gamma \in \Gamma$.

Fix $\gamma \in \Gamma$; put $P_\gamma = \gamma P \gamma^{-1}$ — then the K-component of γ per the decomposition $G = K \cdot P$ takes the special split component A of P to the special split component A_γ of P_γ. Noting that

$$H_{P|A}(x\gamma) = I_\Gamma(P:P_\gamma)\big(H_{P_\gamma|A_\gamma}(x)\big),$$

the definitions then imply that

$$\begin{aligned}
&\chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\\
&\quad= \chi_{P,A:\mathfrak{H}}\Big(I_\Gamma(P:P_\gamma)\big(I_{P_\gamma}(\mathbf{H})\big) - I_\Gamma(P:P_\gamma)\big(H_{P_\gamma|A_\gamma}(x)\big)\Big)\\
&\quad= \chi_{P,A:\mathfrak{H}}\Big(I(P\,|\,A:P_\gamma\,|\,A_\gamma)\big[I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big]\Big)\\
&\quad= \chi_{P_\gamma,A_\gamma:\mathfrak{H}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big).
\end{aligned}$$

Furthermore,

$$f^{P_\gamma}(x) = f^P(x\gamma).$$

Let $\mathcal{C}_\Gamma(P)$ be the Γ-conjugacy class of P — then it follows that

$$\begin{aligned}
T_P(\mathbf{H}:f)(x) &= \sum_{\gamma\in\Gamma/\Gamma\cap P} \chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\cdot f^P(x\gamma)\\
&= \sum_{\gamma\in\Gamma/\Gamma\cap P} \chi_{P_\gamma,A_\gamma:\mathfrak{H}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big)\cdot f^{P_\gamma}(x)\\
&= \sum_{P_\gamma\in\mathcal{C}_\Gamma(P)} \chi_{P_\gamma,A_\gamma:\mathfrak{H}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big)\cdot f^{P_\gamma}(x),
\end{aligned}$$

an expression which turns out to be of considerable utility.

As before, let

$$\{(P_i, S_i): 1 \le i \le r\}$$

be a set of representatives for the Γ-conjugacy classes of Γ-cuspidal split parabolic subgroups of G. Given $\mathbf{H} \in \mathfrak{a}$, put, for any complex valued locally bounded (measurable) function f on G/Γ,

$$Q^{\mathbf{H}}f = \sum_{i=1}^{r} (-1)^{\operatorname{rank}(P_i)} T_{P_i}(\mathbf{H}:f),$$

$Q^{\mathbf{H}}$ then being the so-called truncation operator with which we shall be occupied for the remainder of this section.

There are a number of elementary observations which should be made immediately. In the first place, it is clear that the definition of $Q^{\mathbf{H}}$ is independent of the choice of the representatives P_i. Next, $Q^{\mathbf{H}}f$ is a locally bounded function on G/Γ which is even slowly increasing provided that f is so (cf. Proposition 4.6). If f has compact support, then $Q^{\mathbf{H}}f$ does too (cf. Proposition 4.3). On cusp forms, $Q^{\mathbf{H}}$ is the identity. Finally, while $Q^{\mathbf{H}}$ will not ordinarily respect the continuity or differentiability of a function, it is nevertheless always true that

$$\lim_{\mathbf{H}\to-\infty} Q^{\mathbf{H}}f = f$$

uniformly on compacta, as can be seen from Proposition 4.4.

It is a point of some importance that $Q^{\mathbf{H}}$ can also be written in terms of all the Γ-cuspidals. Thus let $\mathcal{C}_\Gamma$ be the set of all Γ-cuspidal split parabolic subgroups of G (cf. §3) — then, taking into account what was said above, we have

$$Q^{\mathbf{H}}f(x) = \sum_{P\in\mathcal{C}_\Gamma} (-1)^{\operatorname{rank}(P)} \chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(x)\big)\cdot f^P(x)$$

or still

$$f(x) + \sum_{\substack{P\in\mathcal{C}_\Gamma \\ P\neq G}} (-1)^{\operatorname{rank}(P)} \chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(x)\big)\cdot f^P(x).$$

We shall see that this alternative representation of $Q^{\mathbf{H}}$ is, from a technical point of view, decisive.

Our objective now will be to show that, under certain conditions, $Q^{\mathbf{H}}$ can be regarded as an orthogonal projection on $L^2(G/\Gamma)$. Owing to Proposition 4.9 (and supporting discussion), it all comes down to a question of idempotence. Ideally, one would like to say: If $\mathbf{H}$ is sufficiently regular, then $Q^{\mathbf{H}}$ (or rather its closure $\overline{Q^{\mathbf{H}}}$) is an orthogonal projection on $L^2(G/\Gamma)$. Unfortunately, due to the presence of several cusps, things are not quite so simple as this. Instead, our statements will have to be phrased in terms of a new ordering on $\mathfrak{a}$, an unexpected development.

Given $\mathbf{H}_1, \mathbf{H}_2$ in $\mathfrak{a}$, write

$$\mathbf{H}_1 \lessdot \mathbf{H}_2$$

if there exists an $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)$ such that

$$I(P_0\,|\,A_0 \colon P_{i_0}\,|\,A_{i_0})\big(I_{P_{i_0}}(\mathbf{H}_2) - I_{P_{i_0}}(\mathbf{H}_1)\big) = H_0$$

for all $i_0 = 1,\dots,r_0$. This relation partially orders $\mathfrak{a}$. Obviously (cf. Lemma 3.2):

$$\mathbf{H}_1 \lessdot \mathbf{H}_2 \Rightarrow \mathbf{H}_1 < \mathbf{H}_2.$$

Moreover, the two relations coincide if Γ possesses a single cusp but, as can be seen by example, this is not true in general.

THEOREM 5.1. *Fix* $\mathbf{H}_0$ *in* $\mathfrak{a}$ — *then there exists* $\mathbf{H}_{00} < \mathbf{H}_0$ *such that for all* $\mathbf{H} \lessdot \mathbf{H}_{00}$

$$Q^{\mathbf{H}} \circ Q^{\mathbf{H}} = Q^{\mathbf{H}}.$$

Consequently, under the hypotheses at hand, the closure $\overline{Q}^{\mathbf{H}}$ of

$$Q^{\mathbf{H}} \mid S(G/\Gamma) \cap L^2(G/\Gamma)$$

is an orthogonal projection on $L^2(G/\Gamma)$. Notationally, it will usually be unnecessary to distinguish between $Q^{\mathbf{H}}$ and $\overline{Q}^{\mathbf{H}}$.

The proof of Theorem 5.1 is by no means a simple exercise. Let us isolate the main issue. Fix a Γ-cuspidal split parabolic subgroup (P, S) of G with special split component $A(P \neq G)$. Consider

$$\chi_{P,A:\mathfrak{I}}\big(I_P(\mathbf{H}) - H_{P|A}(x)\big) \cdot \big(Q^{\mathbf{H}} f\big)^P(x).$$

Then idempotence would be established if it could be shown that, independently of P, for all $\mathbf{H}$ per supra

$$I_P(\mathbf{H}) - H_{P|A}(x) \in \mathfrak{I}_P(\mathfrak{a})$$

$$\Rightarrow$$

$$\big(Q^{\mathbf{H}} f\big)^P(x) = 0.$$

In reality, we shall actually prove somewhat more than this. Call, as usual,

$$\mathfrak{I}_P(\mathfrak{a})^-$$

the closure of $\mathfrak{I}_P(\mathfrak{a})$ — then

$$-\mathfrak{I}_P(\mathfrak{a})^- \cap \mathfrak{I}_P(\mathfrak{a}) = \varnothing.$$

LEMMA. *Let* $\mathbf{H}$ *be as above — then, independently of* P,

$$I_P(\mathbf{H}) - H_{P|A}(x) \notin -\mathfrak{I}_P(\mathfrak{a})^-$$

$$\Rightarrow$$

$$\big(Q^{\mathbf{H}} f\big)^P(x) = 0.$$

This result will be established in the next section. Here is a corollary. Take $\mathbf{H}', \mathbf{H}'' \in \mathfrak{a}$ per supra with $\mathbf{H}'' \lessdot \mathbf{H}'$ — then $\mathbf{H}'' < \mathbf{H}'$, thus, by definition,

$$I_P(\mathbf{H}') \in I_P(\mathbf{H}'') + \mathfrak{C}_P(\mathfrak{a}).$$

Suppose now that

$$I_P(\mathbf{H}'') - H_{P|A}(x) \in \mathfrak{I}_P(\mathfrak{a}),$$

so

$$I_P(\mathbf{H}') - H_{P|A}(x) \in \mathcal{D}_P(\mathfrak{a}) + \mathcal{C}_P(\mathfrak{a}) \subset \mathcal{D}_P(\mathfrak{a}).$$

In view of the lemma, we then have

$$(Q^{\mathbf{H}'}f)^P(x) = 0.$$

It therefore follows that

$$Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}.$$

REMARK. There is a small item of detail present. We have

$$\bar{Q}^{\mathbf{H}''} \circ \bar{Q}^{\mathbf{H}'} = \bar{Q}^{\mathbf{H}'},$$

so, upon taking adjoints,

$$\bar{Q}^{\mathbf{H}'} \circ \bar{Q}^{\mathbf{H}''} = \bar{Q}^{\mathbf{H}'}.$$

The point to be made now is that one cannot assert that necessarily

$$Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''} = Q^{\mathbf{H}'}.$$

Fortunately, this is not really serious. Claim: Let f be a complex valued locally bounded (measurable) function on G/Γ — then

$$Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f = Q^{\mathbf{H}'}f \quad \text{a.e.} \quad (\text{on } G/\Gamma).$$

Indeed, if C be any compact subset of G/Γ, χ_C its characteristic function, then

$$\begin{aligned}(Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f, \chi_C) &= (Q^{\mathbf{H}''}f, Q^{\mathbf{H}'}\chi_C) \\ &= (f, Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'}\chi_C) \\ &= (f, Q^{\mathbf{H}'}\chi_C) \\ &= (Q^{\mathbf{H}'}f, \chi_C)\end{aligned}$$

$\Rightarrow$

$$\int_C (Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f - Q^{\mathbf{H}'}f) = 0,$$

so, by inner regularity,

$$Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f = Q^{\mathbf{H}'}f \quad \text{a.e.} \quad (\text{on } G/\Gamma).$$

It is worth observing that the formula

$$Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}$$

retains its validity under cicumstances less restrictive than those above. To this end, let $\mathfrak{a}_I$ be the set of $\mathbf{H} \in \mathfrak{a}$ such that, independently of P,

$$I_P(\mathbf{H}) - H_{P|A}(x) \notin \mathcal{D}_P(\mathfrak{a})^-$$
$$\Rightarrow$$
$$(Q^{\mathbf{H}}f)^P(x) = 0.$$

The thrust of the main lemma, then, lies in describing conditions suffcient to ensure that $\mathbf{H} \in \mathfrak{a}_I$. Plainly,

$$\mathbf{H} \in \mathfrak{a}_I \Rightarrow Q^{\mathbf{H}} \circ Q^{\mathbf{H}} = Q^{\mathbf{H}}.$$

Accordingly, take $\mathbf{H}', \mathbf{H}'' \in \mathfrak{a}$ subject to the following requirements: (1) $\mathbf{H}'' < \mathbf{H}'$; (2) $\mathbf{H}' \in \mathfrak{a}_I$. As can be seen from the preceding argument, this is all that is needed to ensure that

$$Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}.$$

In passing, let us note that

$$\lim_{\mathbf{H} \to -\infty} Q^{\mathbf{H}} = \mathrm{ID}$$

in the strong operator topology, the approach to $-\infty$ being through $\mathfrak{a}_I$ vis-à-vis $<$. For purposes of calculation, we remark that one may associate with each pair $(\mathbf{H}_0, \mathbf{H}_{00})$ per Theorem 5.1 a cofinal subset of $(\mathfrak{a}_I, <)$, namely $\{\mathbf{H} \in \mathfrak{a} \colon \mathbf{H} \lessdot \mathbf{H}_{00}\}$, $<$ and $\lessdot$ agreeing there.

We mentioned earlier that for any $\mathbf{H} \in \mathfrak{a}$,

$$Q^{\mathbf{H}}(S(G/\Gamma)) \subset S(G/\Gamma).$$

Now fix anew an element $\mathbf{H}_0 \in \mathfrak{a}$ — then it follows from the proof of Proposition 4.6 (bis) that for every $r < -1$ there exists an $r' < r$ such that

$$Q^{\mathbf{H}}(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma) \qquad (\forall\, \mathbf{H} < \mathbf{H}_0).$$

Moreover,

$$Q^{\mathbf{H}} \colon S_r(G/\Gamma) \to S_{r'}(G/\Gamma)$$

is not only continuous but

$$\{Q^{\mathbf{H}} \colon \mathbf{H} < \mathbf{H}_0\}$$

is equicontinuous.

Suppose that we replace $S(G/\Gamma)$ by

$$R(G/\Gamma),$$

the space of rapidly decreasing functions on G/Γ. Is it true that

$$Q^{\mathbf{H}}(R(G/\Gamma)) \subset R(G/\Gamma)?$$

The answer is 'yes' provided the parameter $\mathbf{H}$ is suitably restricted (cf. infra). Although this fact is certainly of some independent interest, it turns out that in the actual applications a result of a rather different nature is the proper object of focus.

In what follows, let $\mathbf{H}_0, \mathbf{H}_{00} \in \mathfrak{a}$, $\mathbf{H}_{00} < \mathbf{H}_0$, be parameters such that $\mathbf{H} < \mathbf{H}_{00}$ forces the conclusion of Proposition 3.10.

THEOREM 5.2. *Fix* $\mathbf{H}_0$ *in* $\mathfrak{a}$. *Let* $f \in S_r^\infty(G/\Gamma)$ — *then, for all* $\mathbf{H} < \mathbf{H}_{00}$, $Q^{\mathbf{H}}f$ *is rapidly decreasing.*

The proof of Theorem 5.2 is far from obvious; it depends in an essential way on a suitable specialization of the estimate from Proposition 4.13. We shall defer the details until two sections hence.

In conclusion, we emphasize that the theorems formulated in this section capture the crucial properties of the truncation operator. On the other hand, it may come as a bit of a surprise that their proofs are quite different in both concept and execution. Additional comments may be found in §§8, 10 infra.

6. Idempotence of $Q^{\mathbf{H}}$. The purpose of this section will be to prove that the truncation operator $Q^{\mathbf{H}}$ is idempotent, as formulated in Theorem 5.1. In those notations, recall that, with $\mathbf{H}$ as there, the question is to show, independently of P, that

$$I_P(\mathbf{H}) - H_{P|A}(x) \notin -\mathfrak{I}_P(\mathfrak{a})^- \quad \Rightarrow \quad (Q^{\mathbf{H}}f)^P(x) = 0.$$

We shall start off with some structural preliminaries. Let $(P, S), (P^*, S^*)$ be Γ-cuspidal split parabolic subgroups of G with unipotent radicals N, N^*. It will be supposed throughout that $P^* \neq G$.

PROPOSITION 6.1. *There exists one and only one* Γ-*cuspidal parabolic subgroup* $\tau^*(P)$ *of* G *such that*:

(i) $\tau^*(P) \preccurlyeq P$;

(ii) $R_u(\tau^*(P)) = (P \cap N^*) \cdot N$.

[Note: R_u stands for unipotent radical.]

Since a parabolic subgroup is the normalizer of its unipotent radical, it is the existence of $\tau^*(P)$ with which we shall be concerned. Of course, it will have to turn out that

$$\tau^*(P) = N_G((P \cap N^*) \cdot N),$$

a recipe not depending on the various choices which will be made in the actual construction of $\tau^*(P)$.

Choose Γ-percuspidals P_0, P_0^* such that

$$\begin{cases} P \succcurlyeq P_0 \\ P^* \succcurlyeq P_0^*. \end{cases}$$

We can and will suppose that P_0, P_0^* have split components A_0, A_0^* in common. To justify this, simply remark that P_0, P_0^* are G-conjugate by some element from

$$\bigcup_{w \in W(A_0)} P_0 w P_0,$$

as was shown in [3.a]. Observe that $A_0 = A_0^*$ need not be θ-stable, thus may very well be non-special. Select split components A, A^* of $(P, S), (P^*, S^*)$ with the property that $A \subset A_0$, $A^* \subset A_0^*$ — then

$$\begin{cases} (P, S; A) \succcurlyeq (P_0, S_0; A_0) \\ (P^*, S^*; A^*) \succcurlyeq (P_0^*, S_0^*; A_0^*). \end{cases}$$

The argument now falls into a number of steps.

Step 1. We claim that

$$P \cap N^* = (M \cap N^*) \cdot (N \cap N^*).$$

It suffices to prove that the left-hand side is contained in the right-hand side, the opposite inclusion being obvious. For this purpose, note first that $P \cap N^*$ consists entirely of unipotent elements, so the determinant of its action on $\mathfrak{n}$ is $+1$. Accordingly,

$$P \cap N^* \subset S = M \cdot N.$$

Let $p = mn (m \in M, n \in N)$ be an arbitrary element of $P \cap N^*$ — then it must be shown that $m \in M \cap N^*$, $n \in N \cap N^*$. Fix a sequence $\{a_k\} \subset A$ such that $a_k \to -\infty$ relative to P. Since $a_k \in A$, $a_k p a_k^{-1} \in P (\forall\, k)$. On the other hand,

$$a_k \in A \subset A_0 = A_0^* \subset P_0^* \subset P^*,$$

so $a_k p a_k^{-1} \in N^* (\forall\, k)$. Therefore

$$a_k p a_k^{-1} \in P \cap N^*$$

for all k. Because $P \cap N^*$ is closed, we have

$$\begin{aligned} m &= m \cdot \lim_{k\to\infty} a_k n a_k^{-1} \\ &= \lim_{k\to\infty} m a_k n a_k^{-1} \\ &= \lim_{k\to\infty} a_k m n a_k^{-1} \\ &= \lim_{k\to\infty} a_k p a_k^{-1} \in P \cap N^* \\ &\Rightarrow \\ &m \in M \cap N^*. \end{aligned}$$

But then $n \in N \cap N^*$, completing the discussion.

Let us consider $M \cap N^*$, the centralizer of A in N^*. It is more or less direct that $M \cap N^*$ is connected with Lie algebra a sum of root spaces with respect to $\mathfrak{a}_0 = \mathfrak{a}_0^*$, the relevant roots being those whose restriction to $\mathfrak{a}$ is null. This suggests that $M \cap N^*$ may very well be the unipotent radical of a parabolic subgroup of M. We will in fact confirm this in the lines below. There would then remain the problem of Γ_M-cuspidality.

Step 2. We claim that

$$M \cap N^* \cap \Gamma_M$$

is a uniform lattice in $M \cap N^*$. On general grounds, that $\Gamma \cap S$ is a lattice in S and $\Gamma \cap N^*$ is a uniform lattice in N^* both combine to imply that

$$\Gamma \cap S \cap N^*$$

is a uniform lattice in $S \cap N^* (= P \cap N^*)$. This said, let $\{x_n\}$ be a sequence in $M \cap N^*$ — then the uniformity of

$$M \cap N^* \cap \Gamma_M$$

in $M \cap N^*$ will follow provided that it can be shown that $\{x_n\}$ contains a subsequence convergent mod

$$M \cap N^* \cap \Gamma_M.$$

But it is certainly true that $\{x_n\}$ contains a subsequence convergent mod

$$\Gamma \cap S \cap N^*$$

so the desired conclusion results by projection.

The following criterion was established in [**3.a**]: Let P be a parabolic subgroup of G such that $N \cap \Gamma$ is a lattice in N — then there exists S (necessarily unique) such that the split parabolic subgroup (P, S) is Γ-cuspidal.

Admitting still the fact that $M \cap N^*$ really is the unipotent radical of a parabolic subgroup of M, the aforementioned criterion (applied to the pair (M, Γ_M)), in conjunction with what has been said above, imply that the putative parabolic is Γ_M-cuspidal with unipotent radical $M \cap N^*$. Noting that

$$(P \cap N^*)\cdot N = (M \cap N^*)\cdot N,$$

the proof of our proposition is then finished via production of $\tau^*(P)$ by undaggering.

We have yet to exhibit a parabolic subgroup of M whose unipotent radical is $M \cap N^*$. Because

$$(P, S; A) \succcurlyeq (P_0, S_0; A_0),$$

we determine, in the usual way, a Γ_M-cuspidal split parabolic subgroup $(P_0^\dagger, S_0^\dagger)$ of M with split component $A_0^\dagger$. Furthermore, $P_0^\dagger$ is Γ_M-percuspidal.

Step 3. Fix $H^* \in \mathcal{C}_{P^*}(\mathfrak{a}^*)$. Let

$$a_t^* = \exp(tH^*).$$

Then

$$\mathfrak{n}^* = \left\{X \in \mathfrak{g}\colon \lim_{t\to-\infty} \operatorname{Ad}(a_t^*)X = 0\right\},$$

hence

$$\mathfrak{m} \cap \mathfrak{n}^* = \left\{X \in \mathfrak{m}\colon \lim_{t\to-\infty} \operatorname{Ad}(a_t^*)X = 0\right\}.$$

Relative to the orthogonal decomposition

$$\mathfrak{a}_0 = \mathfrak{a}_0^\dagger \oplus \mathfrak{a},$$

let $H^\dagger$ be the projection of H^* onto $\mathfrak{a}_0^\dagger$. Put

$$a_t^\dagger = \exp(tH^\dagger).$$

Taking into account the fact that $\mathfrak{m}$ and $\mathfrak{a}$ commute, we have still

$$\mathfrak{m} \cap \mathfrak{n}^* = \left\{X \in \mathfrak{m}\colon \lim_{t\to-\infty} \operatorname{Ad}(a_t^\dagger)X = 0\right\}.$$

These considerations serve to reduce our problem to an essentially familiar fact from the theory of parabolic subgroups. Working with (G, Γ) instead of (M, Γ_M), let A_0 be a split component of a Γ-percuspidal split parabolic subgroup of G. Let $\mathcal{P}(A_0)$ be the set of all split parabolic subgroups of G with A_0 as split component. [Note: Not every element of

$\mathcal{P}(A_0)$ need be Γ-percuspidal.] If by $\mathcal{C}(A_0)$ we understand the set of chambers of $\mathfrak{a}_0$, then, as is well-known (see, e.g., [**3.a**]), the map

$$\begin{cases} \mathcal{P}(A_0) \to \mathcal{C}(A_0) \\ (P_0, S_0; A_0) \mapsto \mathcal{C}_{P_0}(\mathfrak{a}_0) \end{cases}$$

sets up a bijection between $\mathcal{P}(A_0)$ and $\mathcal{C}(A_0)$.

SUBLEMMA. *Let $H_0 \in \mathfrak{a}_0$. Set*

$$\mathfrak{n} = \left\{X \in \mathfrak{g}: \lim_{t \to -\infty} \mathrm{Ad}(\exp(tH_0))X = 0\right\}.$$

Then $\mathfrak{n}$ is the Lie algebra of the unipotent radical of a parabolic subgroup P of G which is a dominant successor of any $P_0 \in \mathcal{P}(A_0)$ such that $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)^-$.

[The proof is, of course, canonical. Write

$$\mathfrak{a}_0 = \bigcup_{P_0} \mathcal{C}_{P_0}(\mathfrak{a}_0)^-.$$

Take any $P_0 \in \mathcal{P}(A_0)$ with $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)^-$. Enumerate the elements λ_i of $\Sigma^0_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ by requiring

$$\begin{cases} \lambda_i(H_0) = 0 & (1 \le i \le p) \\ \lambda_i(H_0) > 0 & (p < i \le l_0). \end{cases}$$

Then

$$\mathfrak{a} = \bigcap_{i=1}^{p} \mathrm{Ker}(\lambda_i)$$

determines a dominant successor of P_0 associated with $\mathfrak{n}$.]

We may view τ^* as a map

$$\tau^*: \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$$

characterized by conditions (i) and (ii) of the proposition supra. It is clear that

$$\tau^*(\gamma P \gamma^{-1}) = \gamma \tau^*(P) \gamma^{-1} \qquad (\gamma \in \Gamma \cap N^*).$$

Slightly less obvious is:

LEMMA 6.2. *Suppose that*

$$\tau^*(P) \preccurlyeq P' \preccurlyeq P.$$

Then

$$\tau^*(P) = \tau^*(P').$$

Proof. The a priori containments

$$\begin{aligned}((P \cap N^*)\cdot N) \cap N^* & \\ &\supset P \cap N^* \\ &\supset P' \cap N^* \\ &\supset \tau^*(P) \cap N^* \\ &\supset R_u(\tau^*(P)) \cap N^* \\ &\supset ((P \cap N^*)\cdot N) \cap N^*\end{aligned}$$

are actually equalities, hence, in particular

$$P \cap N^* = P' \cap N^*.$$

But then

$$\begin{aligned}R_u(\tau^*(P)) &= (P \cap N^*)\cdot N \\ &\subset (P \cap N^*)\cdot N' \\ &= (P' \cap N^*)\cdot N' \\ &= R_u(\tau^*(P')).\end{aligned}$$

However,

$$\begin{aligned}\tau^*(P) \preccurlyeq P' &\Rightarrow R_u(\tau^*(P)) \supset N' \\ &\Rightarrow R_u(\tau^*(P)) \supset (P \cap N^*)\cdot N' \\ &= R_u(\tau^*(P')).\end{aligned}$$

So, altogether,

$$R_u(\tau^*(P)) = R_u(\tau^*(P')),$$

implying, therefore, that

$$\tau^*(P) = \tau^*(P'),$$

as was to be shown. □

A corollary to this lemma is the fact that τ^*, viewed as a map $\mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$, is idempotent, i.e.

$$\tau^* \circ \tau^* = \tau^*.$$

Let

$$\mathcal{C}_\Gamma^* = \mathrm{Ran}(\tau^*).$$

Then a given $P \in \mathcal{C}_\Gamma$ belongs to $\mathcal{C}_\Gamma^*$ iff $\tau^*(P) = P$ or, equivalently, iff $P \cap N^* \subset N$. This being so, our next task will be to investigate the fiber

$(\tau^*)^{-1}(P)$ over a given $P \in \mathcal{C}_\Gamma^*$. There is an immediate global characterization, viz.

$$P' \in (\tau^*)^{-1}(P) \Leftrightarrow (P' \cap N^*)\cdot N' = N$$

or, in infinitesimal terms,

$$P' \in (\tau^*)^{-1}(P) \Leftrightarrow (\mathfrak{p}' \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n}.$$

The point we wish to make now is that $\mathfrak{p}'$ can be replaced by $\mathfrak{p}$ here, that is,

$$P' \in (\tau^*)^{-1}(P) \Leftrightarrow (\mathfrak{p} \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n}.$$

Indeed, if $P' \in (\tau^*)^{-1}(P)$, then necessarily

$$\begin{aligned} & P' \cap N^* = P \cap N^* \\ \Rightarrow\ & \\ & \mathfrak{p}' \cap \mathfrak{n}^* = \mathfrak{p} \cap \mathfrak{n}^* \\ \Rightarrow\ & \\ & (\mathfrak{p} \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n}. \end{aligned}$$

On the other hand, the equality and the containment

$$\begin{cases} (\mathfrak{p} \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n} \\ \mathfrak{p}' \supset \mathfrak{p} \end{cases}$$

$$\begin{aligned} \Rightarrow\ & \\ & (\mathfrak{p}' \cap \mathfrak{n}^*) + \mathfrak{n}' \supset \mathfrak{n} \\ \Rightarrow\ & \\ & R_u(\tau^*(P')) \supset R_u(P) \\ \Rightarrow\ & \\ & \tau^*(P') \subset P \subset P' \\ \Rightarrow\ & \\ & \tau^*(P') \preccurlyeq P \preccurlyeq P' \\ \Rightarrow\ & \\ & \tau^*(P') = \tau^*(P) = P \end{aligned}$$

by Lemma 6.2. [We explicitly observe that we have used the fact that containment is equivalent to domination on the set $\mathcal{C}_\Gamma$ (cf. **[3.a]**).] In root-theoretic terms, it can then be said that

$$\begin{aligned} & P' \in (\tau^*)^{-1}(P) \\ \Leftrightarrow\ & \\ & \forall\, \lambda \in \Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0) \text{st} \begin{cases} \mathfrak{g}_\lambda \subset \mathfrak{n} \\ \\ \mathfrak{g}_\lambda \not\subset \mathfrak{p} \cap \mathfrak{n}^* \end{cases} \Rightarrow \mathfrak{g}_\lambda \subset \mathfrak{n}'. \end{aligned}$$

To exploit this remark, fix $H^* \in \mathcal{C}_{P^*}(\mathfrak{a}^*)$ — then

$$\forall\, \lambda \in \Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$$

$$\mathfrak{g}_\lambda \subset \mathfrak{n}^* \Leftrightarrow \lambda(H^*) > 0.$$

It can be supposed that the elements λ_i of $\Sigma^0_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ have been so arranged that

$$\mathfrak{a} = \bigcap_{i=1}^{p} \operatorname{Ker}(\lambda_i).$$

Neither $\mathfrak{g}_{\lambda_i}$ nor $\mathfrak{g}_{-\lambda_i}$ is contained in $\mathfrak{n}^*$, so

$$\lambda_i(H^*) = 0 \qquad (1 \leq i \leq p).$$

There is no loss of generality in assuming that

$$\begin{cases} \lambda_i(H^*) > 0 & (p < i \leq p^{-*}) \\ \lambda_i(H^*) \leq 0 & (p^{-*} < i \leq l_0). \end{cases}$$

Call P^{-*} the dominant successor of P_0 corresponding to

$$\mathfrak{a}^{-*} = \bigcap_{i=1}^{p^{-*}} \operatorname{Ker}(\lambda_i).$$

LEMMA 6.3. *Let* $P \in \mathcal{C}^*_\Gamma$ — *then*

$$(\tau^*)^{-1}(P) = \{P' \in \mathcal{C}_\Gamma \colon P \preccurlyeq P' \preccurlyeq P^{-*}\}.$$

Proof. If $P' \in (\tau^*)^{-1}(P)$, then, as has been seen above, $P \preccurlyeq P'$. To establish the opposite domination, simply note that in the representation of $\mathfrak{a}'$ as the intersection of certain $\operatorname{Ker}(\lambda_i)$ any such index i must, of necessity, lie between 1 and p^{-*} implying, therefore, that $P' \preccurlyeq P^{-*}$. So, to complete our proof, we have only to show that $\tau^*(P^{-*}) = P$, i.e., that P^{-*} is on the fiber over P. For this purpose, it will be convenient to utilize the root-theoretic criterion set forth supra. Thus take a $\lambda \in \Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ such that

$$\begin{cases} \mathfrak{g}_\lambda \subset \mathfrak{n} \\ \mathfrak{g}_\lambda \not\subset \mathfrak{p} \cap \mathfrak{n}^*. \end{cases}$$

The claim then is that $\mathfrak{g}_\lambda \subset \mathfrak{n}^{-*}$. Write $\lambda = \Sigma c_i \lambda_i$. Since $\mathfrak{g}_\lambda \subset \mathfrak{n}$, there exists an $i_0 > p$ such that $c_{i_0} > 0$. If, additionally, $i_0 > p^{-*}$, then, of course, $\mathfrak{g}_\lambda \subset \mathfrak{n}^{-*}$ and we are done. Otherwise, $p < i_0 \leq p^{-*}$, hence $\lambda_{i_0}(H^*) > 0$. But

$$\mathfrak{g}_\lambda \not\subset \mathfrak{p} \cap \mathfrak{n}^* \Rightarrow \lambda(H^*) = \sum_{i=p+1}^{l_0} c_i \lambda_i(H^*) \leq 0.$$

Because

$$c_{i_0}\lambda_{i_0}(H^*)$$

is positive, there must exist another index j_0 such that $c_{j_0} > 0$ and $\lambda_{j_0}(H^*) < 0$. Such a j_0 must be $> p^{-*}$, hence once again $\mathfrak{g}_\lambda \subset \mathfrak{n}^{-*}$. □

Now where are we? Starting with the Γ-cuspidal parabolic subgroup $P^* \neq G$, we produce a map $\tau^*: \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$ and the associated set $\mathcal{C}_\Gamma^*$. We shall consistently write $P^* \succcurlyeq P_0^*$, P_0^* some Γ-percuspidal which has been and will remain fixed. The reader must realize, however, that the ambient split components can vary, the choice being dictated by the context. There is undoubtedly some potential for confusion here so we shall make every effort to be completely precise in order to minimize it.

In terms of canonical data associated with $(P_0^*, S_0^*; A_0^*)$, A_0^* any split component of (P_0^*, S_0^*), the set $\mathcal{C}_\Gamma^*$ admits a decomposition, the description of which may be formulated in the following way. Fix a finite subset F_0^* of G with the property that

$$\{x_0 P_0^* x_0^{-1}: x_0 \in F_0^*\}$$

is a set of representatives for the Γ-percuspidal parabolic subgroups of G. Given

$$\begin{cases} w_0 \in W(A_0^*) \\ x_0 \in F_0^*, \end{cases}$$

put

$$\Delta(w_0 : x_0) = \Gamma x_0 \cap P_0^* w_0 P_0^*.$$

Then

$$\mathcal{C}_\Gamma^* = \bigcup_{w_0 \in W(A_0^*)} \; \bigcup_{x_0 \in F_0^*} \; \bigcup_{P \succcurlyeq P_0^*} \; \bigcup_{\delta_0 \in \Delta(w_0 : x_0)} \{\delta_0 P \delta_0^{-1}\} \cap \mathcal{C}_\Gamma^*.$$

A natural question thus suggests itself. If $P \succcurlyeq P_0^*$, when is it true that $\delta_0 P \delta_0^{-1} \in \mathcal{C}_\Gamma^*$? Naturally, this is entirely equivalent to determining when

$$\tau^*(\delta_0 P \delta_0^{-1}) = \delta_0 P \delta_0^{-1}.$$

Since we wish to discuss $\tau^*(\delta_0 P \delta_0^{-1})$, $P \succcurlyeq P_0^*$, the situation changes slightly in that P is not the generic Γ-cuspidal (as it was in Proposition 6.1) but rather this time $\delta_0 P \delta_0^{-1}$ is. Using the fact that

$$\delta_0 \in \Delta(w_0 : x_0) \subset P_0^* w_0 P_0^*,$$

write

$$\delta_0 = n(\delta_0) w_0 p(\delta_0) \qquad (n(\delta_0) \in N_0^*, \, p(\delta_0) \in P_0^*).$$

Then in the picture

$$\begin{cases} \delta_0 P \delta_0^{-1} \geqslant \delta_0 P_0^* \delta_0^{-1} \\ P^* \geqslant P_0^*, \end{cases}$$

the split component shared by $\delta_0 P_0^* \delta_0^{-1}$ and P_0^* is

$$n(\delta_0) A_0^* n(\delta_0)^{-1}.$$

Supposing that

$$(P, S; A) \geqslant (P_0^*, S_0^*; A_0^*),$$

we have, accordingly,

$$\tau^*(\delta_0 P \delta_0^{-1}) = \delta_0 P \delta_0^{-1}$$

$$\Leftrightarrow$$

$$\delta_0 P \delta_0^{-1} \cap N^* \subset \delta_0 N \delta_0^{-1}$$

$$\Leftrightarrow$$

$$\mathrm{Ad}(\delta_0)\mathfrak{p} \cap \mathfrak{n}^* \subset \mathrm{Ad}(\delta_0)\mathfrak{n}$$

$$\Leftrightarrow$$

$$\begin{aligned} &\mathrm{Ad}(n(\delta_0)) w_0 \mathrm{Ad}(n(\delta_0)^{-1}) \cdot \mathrm{Ad}(n(\delta_0))\mathfrak{p} \cap \mathfrak{n}^* \\ &\subset \mathrm{Ad}(n(\delta_0)) w_0 \mathrm{Ad}(n(\delta_0)^{-1}) \cdot \mathrm{Ad}(n(\delta_0))\mathfrak{n} \end{aligned}$$

$$\Leftrightarrow$$

$$\mathrm{Ad}(n(\delta_0)) w_0 \mathrm{Ad}(n(\delta_0)^{-1}) \cdot \mathrm{Ad}(n(\delta_0))\mathfrak{m} \cap \mathfrak{n}^* = \{0\}$$

$$\Leftrightarrow$$

$$\mathrm{Ad}(n(\delta_0)) w_0 \mathfrak{m} \cap \mathfrak{n}^* = \{0\}$$

$$\Leftrightarrow$$

$$\mathrm{Ad}(n(\delta_0))(w_0 \mathfrak{m} \cap \mathfrak{n}^*) = \{0\}$$

$$\Leftrightarrow$$

$$w_0 \mathfrak{m} \cap \mathfrak{n}^* = \{0\}.$$

Observe that this Lie algebra-theoretic condition involves δ_0 only through w_0, the shifted data entering in the verification but not in the final conclusion. Write

$$(P^*, S^*; A^*) \geqslant (P_0^*, S_0^*; A_0^*).$$

We remark that the split component A^* is the same as the one figuring in the earlier constructions vis-à-vis

$$\begin{cases} P \geqslant P_0^* \\ P^* \geqslant P_0^* \end{cases} \qquad : A,\ A^* \subset A_0^*.$$

Because $w_0\mathfrak{m} \cap \mathfrak{n}^*$ is a sum of root spaces with respect to $\mathfrak{a}_0^*$, a given root occurring only if its negative appears simultaneously,

$$\begin{aligned} & w_0\mathfrak{m} \cap \mathfrak{n}^* = \{0\} \\ \Leftrightarrow\ & \forall \lambda \text{ st } \mathfrak{g}_\lambda \subset \mathfrak{m}: w_0\lambda \,|\, \mathfrak{a}^* = 0. \end{aligned}$$

Represent P per P_0^*, i.e. write

$$P = (P_0^*)_F.$$

Then

$$\begin{aligned} & w_0\mathfrak{m} \cap \mathfrak{n}^* = \{0\} \\ \Leftrightarrow\ & \forall \lambda_i \in F, w_0\lambda_i \,|\, \mathfrak{a}^* = 0. \end{aligned}$$

In recapitulation, therefore,

$$\begin{aligned} & \tau^*(\delta_0 P \delta_0^{-1}) = \delta_0 P \delta_0^{-1} \\ \Leftrightarrow\ & \forall \lambda_i \in F, w_0\lambda_i \,|\, \mathfrak{a}^* = 0. \end{aligned}$$

Let us assume now that $\delta_0 P \delta_0^{-1} \in \mathcal{C}_\Gamma^*$, Lemma 6.3 then providing a characterization of the fiber over $\delta_0 P \delta_0^{-1}$. Thanks to what has just been learned, it is not difficult to describe

$$(\delta_0 P \delta_0^{-1})^{-*}.$$

Indeed, if

$$F_{w_0}^{-*} = F \cup \{\lambda_i: w_0\lambda_i > 0 \text{ and } w_0\lambda_i \,|\, \mathfrak{a}^* \neq 0\},$$

then

$$(\delta_0 P \delta_0^{-1})^{-*} = \delta_0 (P_0^*)_{F_{w_0}^{-*}} \delta_0^{-1},$$

as can be readily seen by transporting the question to $w_0 P_0^* w_0^{-1}$ and using the definitions.

The preceding structural facts will all play a role in due course. Setting them aside for the time being, we are at last ready to come to grips with the purported idempotence of $Q^{\mathbf{H}}$. It will be best to restate our objective.

MAIN LEMMA. *Fix* $\mathbf{H}_0$ *in* $\mathfrak{a}$ — *then there exists* $\mathbf{H}_{00} \lessdot \mathbf{H}_0$ *such that for all* $\mathbf{H} \lessdot \mathbf{H}_{00}$, *independently of* $P^* \neq G$,

$$\begin{aligned} & I_{P^*}(\mathbf{H}) - H_{P^*|A^*}(x) \notin -\mathcal{D}_{P^*}(\mathfrak{a}^*)^- \\ \Rightarrow\ & (Q^{\mathbf{H}} f)^{P^*}(x) = 0. \end{aligned}$$

We hasten to stress that here, of course, A^* is the special split component of (P^*, S^*).

By way of explanation, recall that the definition of $Q^{\mathbf{H}}f$ was initially given in terms of the P_i, that is,

$$Q^{\mathbf{H}}f = \sum_{i=1}^{r} (-1)^{\operatorname{rank}(P_i)} T_{P_i}(\mathbf{H}:f),$$

it then being observed that still

$$Q^{\mathbf{H}}f = \sum_{P \in \mathcal{C}_\Gamma} (-1)^{\operatorname{rank}(P)} \chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(?)\big)\cdot f^P,$$

the latter formulation making it clear that the role of P is that of a running variable. [Note: Again, all split components are special.]

Our immediate intention is to discuss

$$(Q^{\mathbf{H}}f)^{P^*}.$$

Because the terms in the sum defining $Q^{\mathbf{H}}f$ are not $\Gamma \cap N^*$-invariant, it will first be necessary to split $\mathcal{C}_\Gamma$ into $\Gamma \cap N^*$ conjugacy classes. Denoting by $\mathcal{C}_\Gamma(N^*)$ a set of representatives for these, the diagram

$$\begin{array}{ccc} \mathcal{C}_\Gamma & \xrightarrow{\tau^*} & \mathcal{C}_\Gamma \\ \downarrow & & \downarrow \\ \mathcal{C}_\Gamma(N^*) & \cdots\cdots\succ & \mathcal{C}_\Gamma(N^*) \end{array}$$

can be rendered commutative provided the dotted arrow is defined according to the relation

$$\tau^*(\gamma P \gamma^{-1}) = \gamma \tau^*(P) \gamma^{-1} \qquad (\gamma \in \Gamma \cap N^*).$$

We then have

$$Q^{\mathbf{H}}f(x) = \sum_{P \in \mathcal{C}_\Gamma(N^*)} \sum_{\gamma \in \Gamma \cap N^*/\Gamma \cap N^* \cap P} \times (-1)^{\operatorname{rank}(P)} \chi_{P_\gamma, A_\gamma:\mathfrak{H}}\Big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\Big)\cdot f^{P_\gamma}(x)$$

or still

$$\sum_{P \in \mathcal{C}_\Gamma(N^*)} \sum_{\gamma \in \Gamma \cap N^*/\Gamma \cap N^* \cap P} \times (-1)^{\operatorname{rank}(P)} \chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\cdot f^P(x\gamma).$$

PROPOSITION 4.9. *Assume that*

$$Q \circ Q = Q.$$

Then

$$Q(S(G/\Gamma) \cap L^2(G/\Gamma)) \subset L^2(G/\Gamma)$$

and the closure of

$$Q \mid S(G/\Gamma) \cap L^2(G/\Gamma)$$

is an orthogonal projection on $L^2(G/\Gamma)$.

Of course, the key new point is the hypothesis of idempotence: $Q \circ Q = Q$. $T_{P|A}(H:?)$ will generally not have this property but what is remarkable and, as it turns out, of crucial importance, certain real finite linear combinations Q of such entities will. This question will in fact be a central topic of the next section.

As for the proposition, the proof is easy enough. Suppose to begin with that f is a bounded measurable compactly supported function on G/Γ — then we have

$$\begin{aligned} (Qf, f) &= (Q \circ Qf, f) \\ &= (Qf, Qf) \\ \Rightarrow & \\ \|Qf\| &\leq \|f\|. \end{aligned}$$

Consequently, Q, restricted to the bounded measurable compactly supported functions on G/Γ, extends to a bounded self-adjoint idempotent operator on $L^2(G/\Gamma)$, that is, to an orthogonal projection on $L^2(G/\Gamma)$. Call this extension $\bar{Q}$. To complete our proof, we need only show that Q and $\bar{Q}$ agree on

$$S(G/\Gamma) \cap L^2(G/\Gamma).$$

Take a function f in this set. Let C be any compact subset of G/Γ, χ_C its characteristic function —then

$$\begin{aligned} (\bar{Q}f, \chi_C) &= (f, \bar{Q}\chi_C) \\ &= (f, Q\chi_C) = (Qf, \chi_C) \\ \Rightarrow & \\ \int_C (\bar{Q}f - Qf) &= 0, \end{aligned}$$

so, by inner regularity, $\bar{Q}f = Qf$ a.e. on G/Γ.

It will eventually be necessary to employ some estimates of a character quite different from those encountered supra. What we have in mind here are variants on well-known themes of Harish-Chandra and Langlands. But what they have is not exactly what we need so it will be safer to proceed from first principles.

Let $S_r^\infty(G/\Gamma)$ be the space of slowly increasing differentiable functions f on G/Γ with exponent of growth r such that for every right invariant differential operator D on G, Df is also slowly increasing with exponent of growth r — then the semi-norms

$$|f|_{r,D} = \max_{1 \leq i_0 \leq r_0} \sup_{x \in \mathfrak{S}_{\iota_0, \omega_0} \kappa_{\iota_0}} \Xi_{P_{\iota_0}}(x)^{-r} |Df(x)|$$

serve to equip $S_r^\infty(G/\Gamma)$ with the structure of a Fréchet space. The discussion in the remainder of this section will center on the estimation theory of $S_r^\infty(G/\Gamma)$.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with special split component A; assume, in addition, that $P \neq G$. Let F, F' be subsets of $\Sigma_P^0(\mathfrak{g}, \mathfrak{a})$; assume, in addition, that $F \neq \varnothing$, $F' \subset F$.

LEMMA 4.10. *There exist normal subgroups*

$$\{N_\mu \colon 1 \leq \mu \leq d+1\}$$

of $N_{F'}$ *such that*

(1) $N_{F'} = N_1 \supset N_2 \supset \ldots \supset N_d \supset N_{d+1} = N_F$,
(2) $\dim(N_\mu / N_{\mu+1}) = 1$,
(3) N_μ *is* A*-stable*,
(4) $\Gamma \cap N_\mu$ *is a lattice in* N_μ.

Proof. Fix a Γ-percuspidal split parabolic subgroup P_0 of G with special split component A_0 such that

$$(P, S; A) \succcurlyeq (P_0, S_0; A_0).$$

The roots λ in $\Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ can be arranged in a lexicographic order so as to guarantee that if

$$\mathfrak{n}_{\Lambda_0} = \sum_{\lambda \geq \Lambda_0} \mathfrak{g}_\lambda, \qquad N_{\Lambda_0} = \exp(\mathfrak{n}_{\Lambda_0}),$$

then $\Gamma \cap N_{\Lambda_0}$ is a lattice in N_{Λ_0} (cf. [**3.a**]). There is no loss of generality in supposing that

$$\begin{cases} N_F = N_\Lambda \\ N_{F'} = N_{\Lambda'} \end{cases}$$

where $\Lambda' < \Lambda$. Then

$$\mathfrak{n}_{F'} = \mathfrak{n}_F \oplus \sum_{\{\lambda:\ \Lambda' \leq \lambda < \Lambda\}} \mathfrak{g}_\lambda .$$

List the elements of $\{\lambda: \Lambda' \leq \lambda < \Lambda\}$ in increasing order: $\Lambda' = \lambda_1 < \lambda_2 \ldots$ Fix, as is possible, a subgroup $\Gamma_{F'}$ of $\Gamma \cap N_{F'}$ with the property that

$$\log(\Gamma_{F'})$$

is a lattice in $\mathfrak{n}_{F'}$. Put

$$\Gamma_{F'}(1) = \mathfrak{g}_{\lambda_1} \cap \left(\log(\Gamma_{F'}) + \sum_{\lambda > \lambda_1} \mathfrak{g}_\lambda\right)$$

$$\Gamma_{F'}(2) = \mathfrak{g}_{\lambda_2} \cap \left(\log(\Gamma_{F'}) + \sum_{\lambda > \lambda_2} \mathfrak{g}_\lambda\right)$$

$$\cdots$$

Choose a basis $\{X_1, \ldots, X_d\}$ of

$$\sum_{\{\lambda:\ \Lambda' \leq \lambda < \Lambda\}} \mathfrak{g}_\lambda$$

such that

the first $\dim(\mathfrak{g}_{\lambda_1})$ come from $\Gamma_{F'}(1)$

the second $\dim(\mathfrak{g}_{\lambda_2})$ come from $\Gamma_{F'}(2)$

$$\cdots$$

Finally, set

$$\mathfrak{n}_\mu = \mathfrak{n}_F \oplus \operatorname{span}\{X_\mu, \ldots, X_d\}.$$

Then the

$$N_\mu = \exp(\mathfrak{n}_\mu)$$

satisfy all the requirements of our lemma. □

Keeping to the preceding notations and assumptions, let

$$\phi_{P,F} = \sum_{\{F':\ F' \subset F\}} (-1)^{\operatorname{rank}(P_{F'})} \cdot f^{P_{F'}},$$

f a complex valued locally bounded (measurable) function on G/Γ. It is the estimation of $\phi_{P,F}$ which is now our primary concern. Of course, functions of this type arise in the theory of Eisenstein series so it should not be unexpected that they will also play a role here.

Let us agree to write

$$\pi_F(f)$$

for f^{P_F} — then it is clear that

$$\pi_{F'} \circ \pi_{F''} = \pi_{F' \cap F''}.$$

Accordingly,

$$\begin{aligned}\phi_{P,F} &= \sum_{\{F' :\, F' \subset F\}} (-1)^{\#(F')} \pi_{F'}(f) \\ &= \prod_{\{F' :\, F' \subset F,\, \#(F - F') = 1\}} (\pi_F - \pi_{F'})(f).\end{aligned}$$

On the face of it, therefore, one might reasonably attempt to estimate $\phi_{P,F}$ by first estimating

$$(\pi_F - \pi_{F'})(f)$$

in a uniform manner and then taking products. This is indeed sufficient for many applications but, as it turns out, our situation is more delicate, so we shall have to proceed somewhat differently.

Upon writing

$$(P_F, S_F;\, A_F) \geqslant (P_{F'}, S_{F'};\, A_{F'}),$$

we determine a Γ_{M_F}-cuspidal split parabolic subgroup $(P_{F'}^{\dagger}, S_{F'}^{\dagger})$ of M_F with special split component $A_{F'}^{\dagger}$. One has

$$N_{F'} = N_{F'}^{\dagger} \cdot N_F$$

or still

$$N_{F'}^{\dagger} = N_{F'}/N_F,$$

hence

$$N_{F'}^{\dagger} \cap \Gamma_{M_F} = (N_{F'} \cap \Gamma) \cdot N_F/N_F.$$

This said, it then follows that

$$\begin{aligned}\int_{N_{F'}/N_{F'} \cap \Gamma} &= \int_{N_{F'}/(N_{F'} \cap \Gamma) \cdot N_F} \int_{(N_{F'} \cap \Gamma) \cdot N_F/N_{F'} \cap \Gamma} \\ &= \int_{N_{F'}^{\dagger}/N_{F'}^{\dagger} \cap \Gamma_{M_F}} \int_{N_F/N_F \cap \Gamma}\end{aligned}$$

or still

$$\pi_{F'} = \pi_{F'}^{\dagger} \circ \pi_F.$$

We can thus rewrite $\phi_{P,F}$, namely

$$\phi_{P,F} = \prod_{\{F':\, F' \subset F,\, \#(F-F')=1\}} (1_F - \pi_{F'}^{\dagger})(\pi_F(f)).$$

The thrust of this remark lies in the observation that any partial product

$$\prod_{F'} (1_F - \pi_{F'})(\pi_F(f)),$$

qua a function on G, is invariant to the right under $(N \cap \Gamma) \cdot N_F$.

The next thing to do is to set the stage for an application of Lemma 4.10. As there, we have normal subgroups N_μ, $N_F \subset N_\mu \subset N_{F'}$, with the properties (1)–(4). Put

$$\pi_\mu(f) = \int_{N_\mu/N_\mu \cap \Gamma} f(?n_\mu)\, d_{N_\mu}(n_\mu).$$

Then

$$\pi_F - \pi_{F'} = \sum_{\mu=1}^{d} (\pi_{\mu+1} - \pi_\mu).$$

On the other hand, if we write

$$N_\mu^{\dagger} \quad \text{for} \quad N_\mu/N_F,$$

then an integral manipulation entirely analogous to the one carried out above gives

$$\pi_\mu = \pi_\mu^{\dagger} \circ \pi_F.$$

Consequently,

$$\left(1_F - \pi_{F'}^{\dagger}\right) \circ \pi_F = \sum_{\mu=1}^{d} \left(\pi_{\mu+1}^{\dagger} - \pi_\mu^{\dagger}\right) \circ \pi_F.$$

The quotient

$$N_\mu^{\dagger}/N_{\mu+1}^{\dagger} = N_\mu/N_{\mu+1}$$

is one dimensional. Pick an element $X_\mu \in \mathfrak{n}_\mu/\mathfrak{n}_F$ such that $\exp(X_\mu) \cdot N_{\mu+1}^{\dagger}$ generates

$$\left(N_\mu^{\dagger} \cap \Gamma_{M_F}\right) \cdot N_{\mu+1}^{\dagger}/N_{\mu+1}^{\dagger} = N_\mu^{\dagger} \cap \Gamma_{M_F}/N_{\mu+1}^{\dagger} \cap \Gamma_{M_F}.$$

If, in a general way, for $t \in \mathbf{R}$,

$$\Phi_\mu(t) = \int_{N_{\mu+1}^{\dagger}/N_{\mu+1}^{\dagger} \cap \Gamma_{M_F}} \Phi\left(\exp(tX_\mu) n_{\mu+1}\right) d_{N_{\mu+1}}(n_{\mu+1}),$$

then the difference

$$\Phi_\mu(t) - \int_0^1 \Phi_\mu(s)\, ds$$

computes

$$(\pi^\dagger_{\mu+1} - \pi^\dagger_\mu)(\Phi)\,|_t.$$

LEMMA 4.11. *Let $f \in C^\infty(\mathbf{R}/\mathbf{Z})$. Put*

$$\hat{f}(0) = \int_0^1 f(x)\, dx.$$

Then, for every non-negative integer k,

$$\| f - \hat{f}(0) \|_\infty \leq 2^{-k} \cdot \| f^{(k)} \|_\infty.$$

Proof. We shall give two proofs.

Method 1. Write

$$f(x) - \hat{f}(0) = \sum_{n \neq 0} \hat{f}(n) e^{-2\pi\sqrt{-1}\, nx}$$

where

$$\hat{f}(n) = \int_0^1 f(x) e^{2\pi\sqrt{-1}\, nx}\, dx.$$

Then

$$f(x) - \hat{f}(0) = \sum_{n \neq 0} \left(\frac{-1}{2\pi\sqrt{-1}\, n} \right)^k \cdot \hat{f}^{(k)}(n) e^{-2\pi\sqrt{-1}\, nx}$$

implying, therefore, that

$$\begin{aligned}
|f(x) - \hat{f}(0)| &\leq \sum_{n \neq 0} \left(\frac{1}{2\pi n} \right)^k \cdot |\hat{f}^{(k)}(n)| \\
&\leq \left(\sum_{n \neq 0} \left| \frac{1}{2\pi n} \right|^{2k} \right)^{1/2} \cdot \left(\sum_{n \neq 0} |\hat{f}^{(k)}(n)|^2 \right)^{1/2} \\
&\leq 2 \cdot \left(\frac{1}{2\pi} \right)^k \cdot \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right)^{1/2} \cdot \| f^{(k)} \|_2 \\
&\leq 2^{-k} \cdot \left(\int_0^1 |f^{(k)}(x)|^2\, dx \right)^{1/2} \\
&\leq 2^{-k} \cdot \| f^{(k)} \|_\infty,
\end{aligned}$$

or still

$$\| f - \hat{f}(0) \|_\infty \leq 2^{-k} \cdot \| f^{(k)} \|_\infty ,$$

as was to be shown.

Method 2. Write

$$\begin{aligned} f(x) - \hat{f}(0) &= f(x) - \hat{f}(0) - \int_{x-1}^{x} (f(y) - \hat{f}(0))\, dy \\ &= \int_{x-1}^{x} ((f(x) - \hat{f}(0)) - (f(y) - \hat{f}(0)))\, dy \\ &= \int_{x-1}^{x} (f(x) - f(y))\, dy \\ &= \int_{x-1}^{x} \left(\int_{y}^{x} f'(t)\, dt \right) dy \\ &= \int_{x-1}^{x} \left(\int_{x-1}^{t} f'(t)\, dy \right) dt \\ &= \int_{x-1}^{x} (t - (x-1)) f'(t)\, dt. \end{aligned}$$

Consequently,

$$| f(x) - \hat{f}(0) | \leq \left(\int_0^1 t\, dt \right) \cdot \| f' \|_\infty = 2^{-1} \| f' \|_\infty .$$

Because f is periodic with period 1,

$$\int_0^1 f^{(k)}(x)\, dx = 0 \qquad (k \geq 1).$$

It thus follows by induction that

$$| f(x) - \hat{f}(0) | \leq 2^{-k} \cdot \| f^{(k)} \|_\infty$$

or still

$$\| f - \hat{f}(0) \|_\infty \leq 2^{-k} \cdot \| f^{(k)} \|_\infty ,$$

as was to be shown.

Hence the lemma. □

To be able to apply estimates of the foregoing type, we need to impose conditions of differentiability on f. Since there is nothing to be gained by striving for maximum generality, we shall simply assume that f is C^∞ — then

$$f^{P_F} = \pi_F(f) \in C^\infty(G/(N \cap \Gamma) \cdot N_F).$$

Let $\{\lambda_i\}$ be an enumeration of the elements of F. Put $F_i = F - \{\lambda_i\}$ — then

$$\{F' \colon \#(F - F') = 1\} = \{F_i\}.$$

In these notations, with $p = \#(F)$,

$$\phi_{P,F} = \prod_{i=1}^{p} (\pi_F - \pi_{F_i})(f)$$

or still

$$\phi_{P,F} = \prod_{i=1}^{p} \left(1_F - \pi_i^\dagger\right)(\pi_F(f))$$

where, for simplicity, $\pi_i^\dagger = \pi_{F_i}^\dagger$.

Given a subset $\mathfrak{S}$ of $\{1,\ldots,p\}$, put

$$\Phi(P:F:\mathfrak{S}:f) = \prod_{i\in\mathfrak{S}} \left(1_F - \pi_i^\dagger\right)(\pi_F(f)).$$

Then it is clear that

$$\Phi(P:F:\mathfrak{S}:f) \in C^\infty(G/(N\cap\Gamma)\cdot N_F)$$

with

$$D\Phi(P:F:\mathfrak{S}:f) = \Phi(P:F:\mathfrak{S}:Df)$$

for any right invariant differential operator D on G.

We can now describe the basic idea behind the estimation of $\phi_{P,F}$. For any i between 1 and p, let

$$\mathfrak{S}_i = \{1,\ldots,i\}.$$

Write

$$\phi_{P,F} = \left(1_F - \pi_p^\dagger\right)\left(\Phi\left(P:F:\mathfrak{S}_{p-1}:f\right)\right).$$

It will then we shown that $\phi_{P,F}$ can be estimated in terms of certain derivatives D of

$$\Phi\left(P:F:\mathfrak{S}_{p-1}:f\right).$$

Since

$$\Phi\left(P:F:\mathfrak{S}_{p-1}:Df\right) = \left(1_F - \pi_{p-1}^\dagger\right)\left(\Phi\left(P:F:\mathfrak{S}_{p-2}:Df\right)\right),$$

the argument proceeds via iteration on a step-by-step basis.

Before taking up the details, we had best establish a convention or two.

Let

$$\Phi \in C^\infty(G/(N \cap \Gamma) \cdot N_F).$$

In what follows, it will sometimes be necessary to view Φ as a function on $G \times N$:

$$\Phi(x : n) = \Phi(xn).$$

When this is done, we then employ without comment the usual tensor product formalism for differential operators on product spaces.

Given $F' \subset F$, $\#(F - F') = 1$, let $\omega_{F'}^\dagger$ be a compact neighborhood of 1 in $N_{F'}^\dagger$ with the property that

$$\omega_{F'}^\dagger \cdot \left(N_{F'}^\dagger \cap \Gamma_{M_F}\right) = N_{F'}^\dagger.$$

Write

$$\|?\|_{F'}^\dagger$$

for the sup norm calculated on $\omega_{F'}^\dagger$. If Φ is per supra, then

$$\|\Phi\|_{F'}^\dagger = \sup_{N_{F'}^\dagger} |\Phi|,$$

Φ being, in particular, right invariant under $N_{F'}^\dagger \cap \Gamma_{M_F}$.

LEMMA 4.12. *Let $F' \subset F$, $\#(F - F') = 1$ — then, for every non-negative integer k, and any*

$$\Phi \in C^\infty(G/(N \cap \Gamma) \cdot N_F),$$

$\forall\, x \in G$,

$$|(1_F - \pi_{F'}^\dagger)(\pi_F(\Phi))(x)| \le 2^{-k} \cdot d \cdot \max_{1 \le \mu \le d} \|(1 \otimes X_\mu^k)\Phi(x : ?)\|_{F'}^\dagger.$$

Proof. Write

$$\left(1_F - \pi_{F'}^\dagger\right) \circ \pi_F = \sum_{\mu=1}^{d} \left(\pi_{\mu+1}^\dagger - \pi_\mu^\dagger\right) \circ \pi_F.$$

Because

$$\pi_F(\Phi) = \Phi,$$

we have only to estimate

$$|(1_F - \pi_{F'}^\dagger)(\Phi)(x)|$$

or still, the individual

$$|(\pi_{\mu+1}^\dagger - \pi_\mu^\dagger)(\Phi)(x)|.$$

In turn, thanks to Lemma 4.11 and the remarks prefacing its formulation,

$$|\left(\pi_{\mu+1}^{\dagger} - \pi_{\mu}^{\dagger}\right)(\Phi)(x)|$$

can be estimated vis-à-vis

$$\int_{N_{\mu+1}^{\dagger}/N_{\mu+1}^{\dagger}\cap\Gamma_{M_F}} \Phi\left(x\exp\left(tX_{\mu}\right)n_{\mu+1}\right)d_{N_{\mu+1}}(n_{\mu+1})$$

and the corresponding 'constant term', i.e., the associated integral from 0 to 1. In this way, we find that

$$|\left(\pi_{\mu+1}^{\dagger} - \pi_{\mu}^{\dagger}\right)(\Phi)(x)|$$

is majorized by 2^{-k} times

$$\sup|\frac{d^k}{dt^k}[?]|,$$

[?] being the t-dependent integral above. As the latter cannot exceed

$$\|\left(1 \otimes X_{\mu}^{k}\right)\Phi(x:?)\|_{F'}^{\dagger},$$

an application of the triangle inequality completes the proof. □

In passing, let us observe that

$$\left(1 \otimes X_{\mu}^{k}\right)\Phi(x:n) = \left(\mathrm{Ad}(x)X_{\mu}^{k}\cdot\Phi\right)(xn).$$

To set up the statement of the main result in this circle of ideas, make the following replacements in the data:

$$\begin{cases} F' \to F_{\iota} \\ \omega_{F_{\iota}}^{\dagger} \to \omega_{\iota}^{\dagger} \\ d \to d_{\iota} \\ X_{\mu} \to X_{\mu_{\iota}}. \end{cases}$$

PROPOSITION 4.13. *Let $f \in C^{\infty}(G/\Gamma)$ — then for every p-tuple $\mathbf{k} = (k_1,\ldots,k_p)$ of non-negative integers k_i there exists a positive constant $C_{\mathbf{k}}$ such that*

$$|\phi_{P,F}(x)|$$

is majorized by $C_{\mathbf{k}}$ times the maximum over all

$$\begin{cases} 1 \le \mu_1 \le d_1 \\ \vdots \\ 1 \le \mu_p \le d_p \end{cases}$$

of the supremum over all

$$\begin{cases} n_1^\dagger \in \omega_1^\dagger \\ \vdots \\ n_p^\dagger \in \omega_p^\dagger \end{cases}$$

of the absolute value of

$$\mathrm{Ad}\big(xn_p^\dagger\ldots n_2^\dagger\big)X_{\mu_1}^{k_1}\cdot\mathrm{Ad}\big(xn_p^\dagger\ldots n_3^\dagger\big)X_{\mu_2}^{k_2}$$
$$\ldots\mathrm{Ad}(xn_p)X_{\mu_{p-1}}^{k_{p-1}}\cdot\mathrm{Ad}(x)X_{\mu_p}^{k_p}\cdot\pi_F(f)\big(xn_p^\dagger\ldots n_1^\dagger\big).$$

The importance (and therefore the significance) of this estimate will become clear in due course. At first glance, one might think that it would be awkward to use in actual practice. But this is not the case at all. For in the applications, x, which is a priori arbitrary, will be restricted in a certain way. Since

$$\omega_p^\dagger\ldots\omega_1^\dagger$$

is compact, something specific can then be said.

For instance, suppose that $f \in S_r^\infty(G/\Gamma)$. There is a strictly positive function E_r on G, a linear combination of Eisenstein series, such that $\forall D$

$$|\,Df(x)\,| \le C(f, D)\cdot|\,E_r(x)\,| \qquad (x \in G),$$

$C(f, D)$ a positive constant. [Note: The existence of E_r is ensured by Lemma 4.7; E_r does not, of course, depend on f.] Now suppose that we confine x to a compact subset Ω of G — then the differential operators figuring in our proposition stay within a compact subset of all the right invariant differential operators on G (equipped with the usual LF-topology), so, ignoring positive constants,

$$\sup_{x\in\Omega}|\,\phi_{P,F}(x)\,|$$

is no more than

$$\sup_{x\in\Omega}\ \sup_{n^\dagger\in\omega_p^\dagger\ldots\omega_1^\dagger}|\,\pi_F(E_r)(xn^\dagger)\,|\,,$$

an inequality which is indeed fundamental.

Proof of Proposition 4.13. In view of the preparation which has been already undertaken, the proof itself is virtually obvious. One simply writes (cf. supra)

$$\phi_{P,F} = \big(1_F - \pi_p^\dagger\big)\big(\Phi\big(P:F:\mathbb{S}_{p-1}:f\big)\big)$$

and then, to be completely formal about it, utilizes downward induction. □

We shall close this section with some remarks which stand by themselves although they will not be fully exploited until subsequent papers in this series.

Put

$$E_r(P_{i_0}:?) = E(P_{i_0} \mid A_{i_0}:1:(2r+1)\rho_{i_0}:?).$$

Then (cf. Lemma 4.7)

$$E_r = \sum_{i_0=1}^{r_0} E_r(P_{i_0}:?).$$

The role of the E_r on G/Γ is that of providing universal majorants for slowly increasing functions, a point of obvious technical value. It is then only natural to ask: Can one find analogues of the E_r for rapidly decreasing functions? We shall now take up this question.

Let q be a real parameter. Introduce

$$\zeta_q = \sum_{i_0=1}^{r_0} \zeta_q(P_{i_0}:?)$$

where, by definition,

$$\zeta_q(P_{i_0}:x) = \sum_{\gamma \in \Gamma/\Gamma \cap P_{i_0}} \exp\left(-q \cdot \| H_{P_{i_0}|A_{i_0}}(x\gamma) \|\right) \qquad (x \in G).$$

Convergence can be secured by assuming that, e.g., $q > 2\|\rho_{i_0}\|$, in which case the corresponding function is slowly increasing.

LEMMA 4.14. (i) $\forall\, c$, $\exists\, q_c$, Q_c *such that*

$$\zeta_{q_c} \leq Q_c \cdot \Xi_{P_{i_0}}^c \quad on \quad \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}.$$

(ii) $\forall\, q$, $\exists\, c_q$, C_q *such that*

$$\Xi_{P_{i_0}}^{c_q} \leq C_q \cdot \zeta_q \quad on \quad \mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}.$$

This result carries with it the immediate consequence that the ζ_q are universal majorants for rapidly decreasing functions on G/Γ. Indeed, any such f has the property that $\forall\, q(> 2\|\rho_{i_0}\|)$ there exists a positive constant $C_f(q)$ such that

$$|f| \leq C_f(q) \cdot \zeta_q$$

and conversely.

To prove Lemma 4.14 we shall need an estimate on the E_r which itself depends on still another estimate, the proof of which will be given later on.

LEMMA 4.15. *Let* $1 \leq i_0, j_0 \leq r_0$ — *then* $\exists\, C_r > 0$ *such that*

$$\forall x \in G$$
$$E_r(P_{j_0} : x) \leq C_r \cdot \exp\left(-2rC_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right).$$

[Note: C_0 is a positive constant which does not depend on r.]

Proof. Take, in the notations of Sublemma 4 (§7) infra,

$$\begin{cases} C' = K \\ C'' = \omega_{i_0}. \end{cases}$$

Given $x \in G$, write

$$x = k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x \delta_x$$

per

$$G = K \cdot A_{i_0} \cdot \omega_{i_0} \cdot \left(S_{i_0} \cap \Gamma\right).$$

Then that result implies that for every $\gamma \in \Gamma$,

$$\left\langle H_{P_{j_0}|A_{j_0}}\left(k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x \gamma\right), \rho_{j_0} \right\rangle$$
$$\geq \left\langle H_{P_{j_0}|A_{j_0}}(k_x \gamma), \rho_{j_0} \right\rangle - C_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \| - C_{00}$$

for certain positive constants C_0, C_{00}. It therefore follows that

$$\begin{aligned} E_r(P_{j_0} : x) &= E_r\left(P_{j_0} : k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x \delta_x\right) \\ &= E_r\left(P_{j_0} : k_x \exp\left(H_{P_{i_0}|A_{i_0}}(x)\right) s_x\right) \\ &\leq E_r(P_{j_0} : k_x) \\ &\quad \times \exp\left(-2r\left(C_{00} + C_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right)\right) \\ &\leq C_r \cdot \exp\left(-2rC_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|\right) \end{aligned}$$

where

$$C_r = E_r(P_{j_0} : 1) \cdot \exp(-2rC_{00}).$$

Hence the lemma. □

Proof of Lemma 4.14(i). There is no loss of generality in supposing that $c > 1$. Fix j_0, $1 \leq j_0 \leq r_0$ — then it will be enough to show that, up to a positive constant,

$$\zeta_{q_c}(P_{i_0} : ?)$$

is majorized on $\mathfrak{S}_{t_0,\omega_0}\kappa_{j_0}$ by $\Xi^c_{P_{j_0}}$ for some $q_c \gg 0$. We have (cf. supra)

$$1 \leq C_r \cdot E_r(P_{j_0} : x)^{-1} \exp(-2rC_0 \cdot \| H_{P_{i_0}|A_{i_0}}(x) \|).$$

Therefore, for any q,

$$\zeta_q(P_{i_0} : x) \leq C_r \cdot E_r(P_{j_0} : x)^{-1} \zeta_{q(r)}(P_{i_0} : x)$$

where

$$q(r) = q + 2rC_0.$$

Let now

$$x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{j_0}.$$

There is a constant $c_\zeta < -1$ such that

$$\zeta_{3\|\rho_{i_0}\|}(P_{i_0} : x) \leq C_\zeta \cdot \Xi_{P_{j_0}}(x)^{c_\zeta}.$$

On the other hand, as can be seen from the proof of Lemma 4.7, there is a constant $r < -1$ with the property that

$$E_r(P_{j_0} : x) \geq C_{j_0} \cdot \Xi_{P_{j_0}}(x)^{c_\zeta - c}.$$

Put

$$q_c = 3\|\rho_{i_0}\| - 2rC_0.$$

Then, on $\mathfrak{S}_{t_0,\omega_0}\kappa_{j_0}$, we have

$$\begin{aligned} \zeta_{q_c}&(P_{i_0} : x) \\ &\leq C_r \cdot E_r(P_{j_0} : x)^{-1} \zeta_{q_c(r)}(P_{i_0} : x) \\ &\leq C_r \cdot E_r(P_{j_0} : x)^{-1} \zeta_{3\|\rho_{i_0}\|}(P_{i_0} : x) \\ &\leq C_r C_\zeta C_{j_0}^{-1} \cdot \Xi_{P_{j_0}}(x)^c, \end{aligned}$$

from which our assertion follows. □

Proof of Lemma 4.14(ii). Given q, set

$$c_q = q / \max_\lambda \|\lambda\|,$$

λ running through $\Sigma^0_{P_{i_0}}(\mathfrak{g}, \mathfrak{a}_{i_0})$. Using definitions only, we then find that on $\mathfrak{S}_{t_0,\omega_0}\kappa_{i_0}$,

$$\Xi_{P_{i_0}}(x)^{c_q} \leq C_q \cdot \exp\left(-q \cdot \| H_{P_{i_0}|A_{i_0}}(x)\|\right),$$

C_q a positive constant which need not be explicated. Since

$$\exp\left(-q \cdot \| H_{P_{i_0}|A_{i_0}}(x)\|\right) \leq \zeta_q(P_{i_0} : x) \leq \zeta_q(x),$$

we are done. □

5. The truncation operator. The purpose of this section is to define and study the truncation operator. The idea behind its introduction can be traced to the works of Langlands (especially [**2.a**]) who, however, only proceeded on an ad hoc basis in certain special cases. It was Arthur [**1.b**] who gave, in the adelic setting, a general definition and, in that situation, established its essential properties. We considered in [**3.b**] the case of Γ-rank one lattices. Here we shall deal with the general case. If it were only a question of one cusp, then the present discussion could be modeled, to some extent at least, after that of Arthur. But, of course, Γ will ordinarily possess more than one cusp, a point which causes a number of complications thereby necessitating a treatment which differs radically from Arthur's. The definition itself will be easy enough. From then on, though, there will be a host of difficulties to overcome. For this reason, we shall content ourselves initially with precise statements only, deferring the proofs to subsequent sections.

We begin by recasting the definition of

$$T_{P|A}(H : f)$$

from the preceding section. So, as there, let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A which we take now to be special. Recall that in this situation we write

$$T_P(H : f)$$

in place of

$$T_{P|A}(H : f).$$

Given $\mathbf{H} \in \mathfrak{a}$, define

$$T_P(\mathbf{H} : f)$$

by

$$T_P(I_P(\mathbf{H}) : f).$$

It is then the case that

$$T_{\gamma P \gamma^{-1}}(\mathbf{H}:f) = T_P(\mathbf{H}:f)$$

for all $\gamma \in \Gamma$.

Fix $\gamma \in \Gamma$; put $P_\gamma = \gamma P \gamma^{-1}$ — then the K-component of γ per the decomposition $G = K \cdot P$ takes the special split component A of P to the special split component A_γ of P_γ. Noting that

$$H_{P|A}(x\gamma) = I_\Gamma(P:P_\gamma)\big(H_{P_\gamma|A_\gamma}(x)\big),$$

the definitions then imply that

$$\begin{aligned}
\chi_{P,A:\mathfrak{H}}&\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\\
&= \chi_{P,A:\mathfrak{H}}\Big(I_\Gamma(P:P_\gamma)\big(I_{P_\gamma}(\mathbf{H})\big) - I_\Gamma(P:P_\gamma)\big(H_{P_\gamma|A_\gamma}(x)\big)\Big)\\
&= \chi_{P,A:\mathfrak{H}}\Big(I(P\,|\,A:P_\gamma\,|\,A_\gamma)\big[I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big]\Big)\\
&= \chi_{P_\gamma,A_\gamma:\mathfrak{H}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big).
\end{aligned}$$

Furthermore,

$$f^{P_\gamma}(x) = f^P(x\gamma).$$

Let $\mathcal{C}_\Gamma(P)$ be the Γ-conjugacy class of P — then it follows that

$$\begin{aligned}
T_P(\mathbf{H}:f)(x) &= \sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P,A:\mathfrak{H}}\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\cdot f^P(x\gamma)\\
&= \sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P_\gamma,A_\gamma:\mathfrak{H}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big)\cdot f^{P_\gamma}(x)\\
&= \sum_{P_\gamma \in \mathcal{C}_\Gamma(P)} \chi_{P_\gamma,A_\gamma:\mathfrak{H}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big)\cdot f^{P_\gamma}(x),
\end{aligned}$$

an expression which turns out to be of considerable utility.

As before, let

$$\{(P_i, S_i):\ 1 \le i \le r\}$$

be a set of representatives for the Γ-conjugacy classes of Γ-cuspidal split parabolic subgroups of G. Given $\mathbf{H} \in \mathfrak{a}$, put, for any complex valued locally bounded (measurable) function f on G/Γ,

$$Q^{\mathbf{H}} f = \sum_{i=1}^{r} (-1)^{\operatorname{rank}(P_i)} T_{P_i}(\mathbf{H}:f),$$

$Q^{\mathbf{H}}$ then being the so-called truncation operator with which we shall be occupied for the remainder of this section.

There are a number of elementary observations which should be made immediately. In the first place, it is clear that the definition of $Q^{\mathbf{H}}$ is independent of the choice of the representatives P_i. Next, $Q^{\mathbf{H}}f$ is a locally bounded function on G/Γ which is even slowly increasing provided that f is so (cf. Proposition 4.6). If f has compact support, then $Q^{\mathbf{H}}f$ does too (cf. Proposition 4.3). On cusp forms, $Q^{\mathbf{H}}$ is the identity. Finally, while $Q^{\mathbf{H}}$ will not ordinarily respect the continuity or differentiability of a function, it is nevertheless always true that

$$\lim_{\mathbf{H}\to -\infty} Q^{\mathbf{H}}f = f$$

uniformly on compacta, as can be seen from Proposition 4.4.

It is a point of some importance that $Q^{\mathbf{H}}$ can also be written in terms of all the Γ-cuspidals. Thus let $\mathcal{C}_\Gamma$ be the set of all Γ-cuspidal split parabolic subgroups of G (cf. §3) — then, taking into account what was said above, we have

$$Q^{\mathbf{H}}f(x) = \sum_{P\in\mathcal{C}_\Gamma} (-1)^{\mathrm{rank}(P)} \chi_{P,A:\mathfrak{D}}\big(I_P(\mathbf{H}) - H_{P|A}(x)\big)\cdot f^P(x)$$

or still

$$f(x) + \sum_{\substack{P\in\mathcal{C}_\Gamma \\ P\neq G}} (-1)^{\mathrm{rank}(P)} \chi_{P,A:\mathfrak{D}}\big(I_P(\mathbf{H}) - H_{P|A}(x)\big)\cdot f^P(x).$$

We shall see that this alternative representation of $Q^{\mathbf{H}}$ is, from a technical point of view, decisive.

Our objective now will be to show that, under certain conditions, $Q^{\mathbf{H}}$ can be regarded as an orthogonal projection on $L^2(G/\Gamma)$. Owing to Proposition 4.9 (and supporting discussion), it all comes down to a question of idempotence. Ideally, one would like to say: If $\mathbf{H}$ is sufficiently regular, then $Q^{\mathbf{H}}$ (or rather its closure $\overline{Q^{\mathbf{H}}}$) is an orthogonal projection on $L^2(G/\Gamma)$. Unfortunately, due to the presence of several cusps, things are not quite so simple as this. Instead, our statements will have to be phrased in terms of a new ordering on $\mathfrak{a}$, an unexpected development.

Given $\mathbf{H}_1, \mathbf{H}_2$ in $\mathfrak{a}$, write

$$\mathbf{H}_1 \lessdot \mathbf{H}_2$$

if there exists an $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)$ such that

$$I(P_0\,|\,A_0\colon P_{i_0}\,|\,A_{i_0})\big(I_{P_{i_0}}(\mathbf{H}_2) - I_{P_{i_0}}(\mathbf{H}_1)\big) = H_0$$

for all $i_0 = 1,\ldots,r_0$. This relation partially orders $\mathfrak{a}$. Obviously (cf. Lemma 3.2):

$$\mathbf{H}_1 \lessdot \mathbf{H}_2 \Rightarrow \mathbf{H}_1 < \mathbf{H}_2.$$

Moreover, the two relations coincide if Γ possesses a single cusp but, as can be seen by example, this is not true in general.

THEOREM 5.1. *Fix* $\mathbf{H}_0$ *in* $\mathfrak{a}$ — *then there exists* $\mathbf{H}_{00} < \mathbf{H}_0$ *such that for all* $\mathbf{H} \lessdot \mathbf{H}_{00}$

$$Q^{\mathbf{H}} \circ Q^{\mathbf{H}} = Q^{\mathbf{H}}.$$

Consequently, under the hypotheses at hand, the closure $\overline{Q^{\mathbf{H}}}$ of

$$Q^{\mathbf{H}} \mid S(G/\Gamma) \cap L^2(G/\Gamma)$$

is an orthogonal projection on $L^2(G/\Gamma)$. Notationally, it will usually be unnecessary to distinguish between $Q^{\mathbf{H}}$ and $\overline{Q^{\mathbf{H}}}$.

The proof of Theorem 5.1 is by no means a simple exercise. Let us isolate the main issue. Fix a Γ-cuspidal split parabolic subgroup (P, S) of G with special split component $A (P \neq G)$. Consider

$$\chi_{P,A:\mathfrak{D}}\big(I_P(\mathbf{H}) - H_{P|A}(x)\big)\cdot \big(Q^{\mathbf{H}}f\big)^P(x).$$

Then idempotence would be established if it could be shown that, independently of P, for all $\mathbf{H}$ per supra

$$I_P(\mathbf{H}) - H_{P|A}(x) \in \mathfrak{D}_P(\mathfrak{a})$$

$$\Rightarrow$$

$$\big(Q^{\mathbf{H}}f\big)^P(x) = 0.$$

In reality, we shall actually prove somewhat more than this. Call, as usual,

$$\mathfrak{D}_P(\mathfrak{a})^-$$

the closure of $\mathfrak{D}_P(\mathfrak{a})$ — then

$$-\mathfrak{D}_P(\mathfrak{a})^- \cap \mathfrak{D}_P(\mathfrak{a}) = \varnothing.$$

LEMMA. *Let* $\mathbf{H}$ *be as above — then, independently of* P,

$$I_P(\mathbf{H}) - H_{P|A}(x) \notin -\mathfrak{D}_P(\mathfrak{a})^-$$

$$\Rightarrow$$

$$\big(Q^{\mathbf{H}}f\big)^P(x) = 0.$$

This result will be established in the next section. Here is a corollary. Take $\mathbf{H}', \mathbf{H}'' \in \mathfrak{a}$ per supra with $\mathbf{H}'' \lessdot \mathbf{H}'$ — then $\mathbf{H}'' < \mathbf{H}'$, thus, by definition,

$$I_P(\mathbf{H}') \in I_P(\mathbf{H}'') + \mathfrak{C}_P(\mathfrak{a}).$$

Suppose now that

$$I_P(\mathbf{H}'') - H_{P|A}(x) \in \mathfrak{D}_P(\mathfrak{a}),$$

so

$$I_P(\mathbf{H}') - H_{P|A}(x) \in \mathfrak{D}_P(\mathfrak{a}) + \mathfrak{C}_P(\mathfrak{a}) \subset \mathfrak{D}_P(\mathfrak{a}).$$

In view of the lemma, we then have

$$(Q^{\mathbf{H}'}f)^P(x) = 0.$$

It therefore follows that

$$Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}.$$

REMARK. There is a small item of detail present. We have

$$\overline{Q}^{\mathbf{H}''} \circ \overline{Q}^{\mathbf{H}'} = \overline{Q}^{\mathbf{H}'},$$

so, upon taking adjoints,

$$\overline{Q}^{\mathbf{H}'} \circ \overline{Q}^{\mathbf{H}''} = \overline{Q}^{\mathbf{H}'}.$$

The point to be made now is that one cannot assert that necessarily

$$Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''} = Q^{\mathbf{H}'}.$$

Fortunately, this is not really serious. Claim: Let f be a complex valued locally bounded (measurable) function on G/Γ — then

$$Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f = Q^{\mathbf{H}'}f \quad \text{a.e.} \quad (\text{on } G/\Gamma).$$

Indeed, if C be any compact subset of G/Γ, χ_C its characteristic function, then

$$\begin{aligned}(Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f, \chi_C) &= (Q^{\mathbf{H}''}f, Q^{\mathbf{H}'}\chi_C) \\ &= (f, Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'}\chi_C) \\ &= (f, Q^{\mathbf{H}'}\chi_C) \\ &= (Q^{\mathbf{H}'}f, \chi_C)\end{aligned}$$

$\Rightarrow$

$$\int_C (Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f - Q^{\mathbf{H}'}f) = 0,$$

so, by inner regularity,

$$Q^{\mathbf{H}'} \circ Q^{\mathbf{H}''}f = Q^{\mathbf{H}'}f \quad \text{a.e.} \quad (\text{on } G/\Gamma).$$

It is worth observing that the formula

$$Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}$$

retains its validity under cicumstances less restrictive than those above. To this end, let $\mathfrak{a}_I$ be the set of $\mathbf{H} \in \mathfrak{a}$ such that, independently of P,

$$I_P(\mathbf{H}) - H_{P|A}(x) \notin \mathcal{C}_P(\mathfrak{a})^-$$

$$\Rightarrow$$

$$(Q^{\mathbf{H}}f)^P(x) = 0.$$

The thrust of the main lemma, then, lies in describing conditions suffcient to ensure that $\mathbf{H} \in \mathfrak{a}_I$. Plainly,

$$\mathbf{H} \in \mathfrak{a}_I \Rightarrow Q^{\mathbf{H}} \circ Q^{\mathbf{H}} = Q^{\mathbf{H}}.$$

Accordingly, take $\mathbf{H}', \mathbf{H}'' \in \mathfrak{a}$ subject to the following requirements: (1) $\mathbf{H}'' < \mathbf{H}'$; (2) $\mathbf{H}' \in \mathfrak{a}_I$. As can be seen from the preceding argument, this is all that is needed to ensure that

$$Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}.$$

In passing, let us note that

$$\lim_{\mathbf{H} \to -\infty} Q^{\mathbf{H}} = \mathrm{ID}$$

in the strong operator topology, the approach to $-\infty$ being through $\mathfrak{a}_I$ vis-à-vis $<$. For purposes of calculation, we remark that one may associate with each pair $(\mathbf{H}_0, \mathbf{H}_{00})$ per Theorem 5.1 a cofinal subset of $(\mathfrak{a}_I, <)$, namely $\{\mathbf{H} \in \mathfrak{a}: \mathbf{H} \lessdot \mathbf{H}_{00}\}$, $<$ and $\lessdot$ agreeing there.

We mentioned earlier that for any $\mathbf{H} \in \mathfrak{a}$,

$$Q^{\mathbf{H}}(S(G/\Gamma)) \subset S(G/\Gamma).$$

Now fix anew an element $\mathbf{H}_0 \in \mathfrak{a}$ — then it follows from the proof of Proposition 4.6 (bis) that for every $r < -1$ there exists an $r' < r$ such that

$$Q^{\mathbf{H}}(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma) \qquad (\forall\, \mathbf{H} < \mathbf{H}_0).$$

Moreover,

$$Q^{\mathbf{H}}\colon S_r(G/\Gamma) \to S_{r'}(G/\Gamma)$$

is not only continuous but

$$\{Q^{\mathbf{H}}\colon \mathbf{H} < \mathbf{H}_0\}$$

is equicontinuous.

Suppose that we replace $S(G/\Gamma)$ by

$$R(G/\Gamma),$$

the space of rapidly decreasing functions on G/Γ. Is it true that

$$Q^{\mathbf{H}}(R(G/\Gamma)) \subset R(G/\Gamma)?$$

The answer is 'yes' provided the parameter $\mathbf{H}$ is suitably restricted (cf. infra). Although this fact is certainly of some independent interest, it turns out that in the actual applications a result of a rather different nature is the proper object of focus.

In what follows, let $\mathbf{H}_0, \mathbf{H}_{00} \in \mathfrak{a}$, $\mathbf{H}_{00} < \mathbf{H}_0$, be parameters such that $\mathbf{H} < \mathbf{H}_{00}$ forces the conclusion of Proposition 3.10.

THEOREM 5.2. *Fix $\mathbf{H}_0$ in $\mathfrak{a}$. Let $f \in S_r^\infty(G/\Gamma)$ — then, for all $\mathbf{H} < \mathbf{H}_{00}$, $Q^{\mathbf{H}}f$ is rapidly decreasing.*

The proof of Theorem 5.2 is far from obvious; it depends in an essential way on a suitable specialization of the estimate from Proposition 4.13. We shall defer the details until two sections hence.

In conclusion, we emphasize that the theorems formulated in this section capture the crucial properties of the truncation operator. On the other hand, it may come as a bit of a surprise that their proofs are quite different in both concept and execution. Additional comments may be found in §§8, 10 infra.

6. Idempotence of $Q^{\mathbf{H}}$. The purpose of this section will be to prove that the truncation operator $Q^{\mathbf{H}}$ is idempotent, as formulated in Theorem 5.1. In those notations, recall that, with $\mathbf{H}$ as there, the question is to show, independently of P, that

$$I_P(\mathbf{H}) - H_{P|A}(x) \notin -\mathcal{D}_P(\mathfrak{a})^{-}$$

$$\Rightarrow$$

$$(Q^{\mathbf{H}}f)^P(x) = 0.$$

We shall start off with some structural preliminaries. Let $(P, S), (P^*, S^*)$ be Γ-cuspidal split parabolic subgroups of G with unipotent radicals N, N^*. It will be supposed throughout that $P^* \neq G$.

PROPOSITION 6.1. *There exists one and only one Γ-cuspidal parabolic subgroup $\tau^*(P)$ of G such that:*

(i) $\tau^*(P) \preccurlyeq P$;

(ii) $R_u(\tau^*(P)) = (P \cap N^*) \cdot N$.

[Note: R_u stands for unipotent radical.]

Since a parabolic subgroup is the normalizer of its unipotent radical, it is the existence of $\tau^*(P)$ with which we shall be concerned. Of course, it will have to turn out that

$$\tau^*(P) = N_G((P \cap N^*) \cdot N),$$

a recipe not depending on the various choices which will be made in the actual construction of $\tau^*(P)$.

Choose Γ-percuspidals P_0, P_0^* such that

$$\begin{cases} P \succcurlyeq P_0 \\ P^* \succcurlyeq P_0^*. \end{cases}$$

We can and will suppose that P_0, P_0^* have split components A_0, A_0^* in common. To justify this, simply remark that P_0, P_0^* are G-conjugate by some element from

$$\bigcup_{w \in W(A_0)} P_0 w P_0,$$

as was shown in [3.a]. Observe that $A_0 = A_0^*$ need not be θ-stable, thus may very well be non-special. Select split components A, A^* of $(P, S), (P^*, S^*)$ with the property that $A \subset A_0, A^* \subset A_0^*$ — then

$$\begin{cases} (P, S; A) \succcurlyeq (P_0, S_0; A_0) \\ (P^*, S^*; A^*) \succcurlyeq (P_0^*, S_0^*; A_0^*). \end{cases}$$

The argument now falls into a number of steps.

Step 1. We claim that

$$P \cap N^* = (M \cap N^*) \cdot (N \cap N^*).$$

It suffices to prove that the left-hand side is contained in the right-hand side, the opposite inclusion being obvious. For this purpose, note first that $P \cap N^*$ consists entirely of unipotent elements, so the determinant of its action on $\mathfrak{n}$ is $+1$. Accordingly,

$$P \cap N^* \subset S = M \cdot N.$$

Let $p = mn(m \in M, n \in N)$ be an arbitrary element of $P \cap N^*$ — then it must be shown that $m \in M \cap N^*$, $n \in N \cap N^*$. Fix a sequence $\{a_k\} \subset A$ such that $a_k \to -\infty$ relative to P. Since $a_k \in A$, $a_k p a_k^{-1} \in P (\forall\, k)$. On the other hand,

$$a_k \in A \subset A_0 = A_0^* \subset P_0^* \subset P^*,$$

so $a_k p a_k^{-1} \in N^* (\forall\, k)$. Therefore

$$a_k p a_k^{-1} \in P \cap N^*$$

for all k. Because $P \cap N^*$ is closed, we have

$$\begin{aligned} m &= m \cdot \lim_{k \to \infty} a_k n a_k^{-1} \\ &= \lim_{k \to \infty} m a_k n a_k^{-1} \\ &= \lim_{k \to \infty} a_k m n a_k^{-1} \\ &= \lim_{k \to \infty} a_k p a_k^{-1} \in P \cap N^* \\ &\Rightarrow \\ &m \in M \cap N^*. \end{aligned}$$

But then $n \in N \cap N^*$, completing the discussion.

Let us consider $M \cap N^*$, the centralizer of A in N^*. It is more or less direct that $M \cap N^*$ is connected with Lie algebra a sum of root spaces with respect to $\mathfrak{a}_0 = \mathfrak{a}_0^*$, the relevant roots being those whose restriction to $\mathfrak{a}$ is null. This suggests that $M \cap N^*$ may very well be the unipotent radical of a parabolic subgroup of M. We will in fact confirm this in the lines below. There would then remain the problem of Γ_M-cuspidality.

Step 2. We claim that

$$M \cap N^* \cap \Gamma_M$$

is a uniform lattice in $M \cap N^*$. On general grounds, that $\Gamma \cap S$ is a lattice in S and $\Gamma \cap N^*$ is a uniform lattice in N^* both combine to imply that

$$\Gamma \cap S \cap N^*$$

is a uniform lattice in $S \cap N^* (= P \cap N^*)$. This said, let $\{x_n\}$ be a sequence in $M \cap N^*$ — then the uniformity of

$$M \cap N^* \cap \Gamma_M$$

in $M \cap N^*$ will follow provided that it can be shown that $\{x_n\}$ contains a subsequence convergent mod

$$M \cap N^* \cap \Gamma_M.$$

But it is certainly true that $\{x_n\}$ contains a subsequence convergent mod

$$\Gamma \cap S \cap N^*$$

so the desired conclusion results by projection.

The following criterion was established in [**3.a**]: Let P be a parabolic subgroup of G such that $N \cap \Gamma$ is a lattice in N — then there exists S (necessarily unique) such that the split parabolic subgroup (P, S) is Γ-cuspidal.

Admitting still the fact that $M \cap N^*$ really is the unipotent radical of a parabolic subgroup of M, the aforementioned criterion (applied to the pair (M, Γ_M)), in conjunction with what has been said above, imply that the putative parabolic is Γ_M-cuspidal with unipotent radical $M \cap N^*$. Noting that

$$(P \cap N^*) \cdot N = (M \cap N^*) \cdot N,$$

the proof of our proposition is then finished via production of $\tau^*(P)$ by undaggering.

We have yet to exhibit a parabolic subgroup of M whose unipotent radical is $M \cap N^*$. Because

$$(P, S; A) \succcurlyeq (P_0, S_0; A_0),$$

we determine, in the usual way, a Γ_M-cuspidal split parabolic subgroup $(P_0^\dagger, S_0^\dagger)$ of M with split component $A_0^\dagger$. Furthermore, $P_0^\dagger$ is Γ_M-percuspidal.

Step 3. Fix $H^* \in \mathcal{C}_{P^*}(\mathfrak{a}^*)$. Let

$$a_t^* = \exp(tH^*).$$

Then

$$\mathfrak{n}^* = \left\{X \in \mathfrak{g}: \lim_{t \to -\infty} \operatorname{Ad}(a_t^*)X = 0\right\},$$

hence

$$\mathfrak{m} \cap \mathfrak{n}^* = \left\{X \in \mathfrak{m}: \lim_{t \to -\infty} \operatorname{Ad}(a_t^*)X = 0\right\}.$$

Relative to the orthogonal decomposition

$$\mathfrak{a}_0 = \mathfrak{a}_0^\dagger \oplus \mathfrak{a},$$

let $H^\dagger$ be the projection of H^* onto $\mathfrak{a}_0^\dagger$. Put

$$a_t^\dagger = \exp(tH^\dagger).$$

Taking into account the fact that $\mathfrak{m}$ and $\mathfrak{a}$ commute, we have still

$$\mathfrak{m} \cap \mathfrak{n}^* = \left\{X \in \mathfrak{m}: \lim_{t \to -\infty} \operatorname{Ad}(a_t^\dagger)X = 0\right\}.$$

These considerations serve to reduce our problem to an essentially familiar fact from the theory of parabolic subgroups. Working with (G, Γ) instead of (M, Γ_M), let A_0 be a split component of a Γ-percuspidal split parabolic subgroup of G. Let $\mathcal{P}(A_0)$ be the set of all split parabolic subgroups of G with A_0 as split component. [Note: Not every element of

$\mathcal{P}(A_0)$ need be Γ-percuspidal.] If by $\mathcal{C}(A_0)$ we understand the set of chambers of $\mathfrak{a}_0$, then, as is well-known (see, e.g., [3.a]), the map

$$\begin{cases} \mathcal{P}(A_0) \to \mathcal{C}(A_0) \\ (P_0, S_0; A_0) \mapsto \mathcal{C}_{P_0}(\mathfrak{a}_0) \end{cases}$$

sets up a bijection between $\mathcal{P}(A_0)$ and $\mathcal{C}(A_0)$.

SUBLEMMA. *Let $H_0 \in \mathfrak{a}_0$. Set*

$$\mathfrak{n} = \left\{ X \in \mathfrak{g} : \lim_{t \to -\infty} \mathrm{Ad}(\exp(tH_0))X = 0 \right\}.$$

Then $\mathfrak{n}$ is the Lie algebra of the unipotent radical of a parabolic subgroup P of G which is a dominant successor of any $P_0 \in \mathcal{P}(A_0)$ such that $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)^-$.

[The proof is, of course, canonical. Write

$$\mathfrak{a}_0 = \bigcup_{P_0} \mathcal{C}_{P_0}(\mathfrak{a}_0)^-.$$

Take any $P_0 \in \mathcal{P}(A_0)$ with $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)^-$. Enumerate the elements λ_i of $\Sigma^0_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ by requiring

$$\begin{cases} \lambda_i(H_0) = 0 & (1 \le i \le p) \\ \lambda_i(H_0) > 0 & (p < i \le l_0). \end{cases}$$

Then

$$\mathfrak{a} = \bigcap_{i=1}^{p} \mathrm{Ker}(\lambda_i)$$

determines a dominant successor of P_0 associated with $\mathfrak{n}$.]

We may view τ^* as a map

$$\tau^* : \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$$

characterized by conditions (i) and (ii) of the proposition supra. It is clear that

$$\tau^*(\gamma P \gamma^{-1}) = \gamma \tau^*(P) \gamma^{-1} \qquad (\gamma \in \Gamma \cap N^*).$$

Slightly less obvious is:

LEMMA 6.2. *Suppose that*

$$\tau^*(P) \preccurlyeq P' \preccurlyeq P.$$

Then

$$\tau^*(P) = \tau^*(P').$$

Proof. The a priori containments

$$\begin{aligned}((P \cap N^*)\cdot N) \cap N^* \\ &\supset P \cap N^* \\ &\supset P' \cap N^* \\ &\supset \tau^*(P) \cap N^* \\ &\supset R_u(\tau^*(P)) \cap N^* \\ &\supset ((P \cap N^*)\cdot N) \cap N^*\end{aligned}$$

are actually equalities, hence, in particular

$$P \cap N^* = P' \cap N^*.$$

But then

$$\begin{aligned}R_u(\tau^*(P)) &= (P \cap N^*)\cdot N \\ &\subset (P \cap N^*)\cdot N' \\ &= (P' \cap N^*)\cdot N' \\ &= R_u(\tau^*(P')).\end{aligned}$$

However,

$$\begin{aligned}\tau^*(P) \preccurlyeq P' &\Rightarrow R_u(\tau^*(P)) \supset N' \\ &\Rightarrow R_u(\tau^*(P)) \supset (P \cap N^*)\cdot N' \\ &= R_u(\tau^*(P')).\end{aligned}$$

So, altogether,

$$R_u(\tau^*(P)) = R_u(\tau^*(P')),$$

implying, therefore, that

$$\tau^*(P) = \tau^*(P'),$$

as was to be shown. □

A corollary to this lemma is the fact that τ^*, viewed as a map $\mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$, is idempotent, i.e.

$$\tau^* \circ \tau^* = \tau^*.$$

Let

$$\mathcal{C}_\Gamma^* = \mathrm{Ran}(\tau^*).$$

Then a given $P \in \mathcal{C}_\Gamma$ belongs to $\mathcal{C}_\Gamma^*$ iff $\tau^*(P) = P$ or, equivalently, iff $P \cap N^* \subset N$. This being so, our next task will be to investigate the fiber

$(\tau^*)^{-1}(P)$ over a given $P \in \mathcal{C}_\Gamma^*$. There is an immediate global characterization, viz.

$$P' \in (\tau^*)^{-1}(P) \Leftrightarrow (P' \cap N^*) \cdot N' = N$$

or, in infinitesimal terms,

$$P' \in (\tau^*)^{-1}(P) \Leftrightarrow (\mathfrak{p}' \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n}.$$

The point we wish to make now is that $\mathfrak{p}'$ can be replaced by $\mathfrak{p}$ here, that is,

$$P' \in (\tau^*)^{-1}(P) \Leftrightarrow (\mathfrak{p} \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n}.$$

Indeed, if $P' \in (\tau^*)^{-1}(P)$, then necessarily

$$\begin{aligned} & P' \cap N^* = P \cap N^* \\ \Rightarrow \quad & \\ & \mathfrak{p}' \cap \mathfrak{n}^* = \mathfrak{p} \cap \mathfrak{n}^* \\ \Rightarrow \quad & \\ & (\mathfrak{p} \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n}. \end{aligned}$$

On the other hand, the equality and the containment

$$\begin{cases} (\mathfrak{p} \cap \mathfrak{n}^*) + \mathfrak{n}' = \mathfrak{n} \\ \mathfrak{p}' \supset \mathfrak{p} \end{cases}$$

$$\begin{aligned} \Rightarrow \quad & \\ & (\mathfrak{p}' \cap \mathfrak{n}^*) + \mathfrak{n}' \supset \mathfrak{n} \\ \Rightarrow \quad & \\ & R_u(\tau^*(P')) \supset R_u(P) \\ \Rightarrow \quad & \\ & \tau^*(P') \subset P \subset P' \\ \Rightarrow \quad & \\ & \tau^*(P') \preccurlyeq P \preccurlyeq P' \\ \Rightarrow \quad & \\ & \tau^*(P') = \tau^*(P) = P \end{aligned}$$

by Lemma 6.2. [We explicitly observe that we have used the fact that containment is equivalent to domination on the set $\mathcal{C}_\Gamma$ (cf. **[3.a]**).] In root-theoretic terms, it can then be said that

$$\begin{aligned} & P' \in (\tau^*)^{-1}(P) \\ \Leftrightarrow \quad & \\ & \forall \lambda \in \Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0) \text{st} \begin{cases} \mathfrak{g}_\lambda \subset \mathfrak{n} \\ \\ \mathfrak{g}_\lambda \not\subset \mathfrak{p} \cap \mathfrak{n}^* \end{cases} \Rightarrow \mathfrak{g}_\lambda \subset \mathfrak{n}'. \end{aligned}$$

To exploit this remark, fix $H^* \in \mathcal{C}_{P^*}(\mathfrak{a}^*)$ — then

$$\forall\, \lambda \in \Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$$
$$\mathfrak{g}_\lambda \subset \mathfrak{n}^* \Leftrightarrow \lambda(H^*) > 0.$$

It can be supposed that the elements λ_i of $\Sigma^0_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ have been so arranged that

$$\mathfrak{a} = \bigcap_{i=1}^{p} \operatorname{Ker}(\lambda_i).$$

Neither $\mathfrak{g}_{\lambda_i}$ nor $\mathfrak{g}_{-\lambda_i}$ is contained in $\mathfrak{n}^*$, so

$$\lambda_i(H^*) = 0 \qquad (1 \le i \le p).$$

There is no loss of generality in assuming that

$$\begin{cases} \lambda_i(H^*) > 0 & (p < i \le p^{-*}) \\ \lambda_i(H^*) \le 0 & (p^{-*} < i \le l_0). \end{cases}$$

Call P^{-*} the dominant successor of P_0 corresponding to

$$\mathfrak{a}^{-*} = \bigcap_{i=1}^{p^{-*}} \operatorname{Ker}(\lambda_i).$$

LEMMA 6.3. *Let* $P \in \mathcal{C}^*_\Gamma$ — *then*

$$(\tau^*)^{-1}(P) = \{P' \in \mathcal{C}_\Gamma : P \preccurlyeq P' \preccurlyeq P^{-*}\}.$$

Proof. If $P' \in (\tau^*)^{-1}(P)$, then, as has been seen above, $P \preccurlyeq P'$. To establish the opposite domination, simply note that in the representation of $\mathfrak{a}'$ as the intersection of certain $\operatorname{Ker}(\lambda_i)$ any such index i must, of necessity, lie between 1 and p^{-*} implying, therefore, that $P' \preccurlyeq P^{-*}$. So, to complete our proof, we have only to show that $\tau^*(P^{-*}) = P$, i.e., that P^{-*} is on the fiber over P. For this purpose, it will be convenient to utilize the root-theoretic criterion set forth supra. Thus take a $\lambda \in \Sigma_{P_0}(\mathfrak{g}, \mathfrak{a}_0)$ such that

$$\begin{cases} \mathfrak{g}_\lambda \subset \mathfrak{n} \\ \mathfrak{g}_\lambda \not\subset \mathfrak{p} \cap \mathfrak{n}^*. \end{cases}$$

The claim then is that $\mathfrak{g}_\lambda \subset \mathfrak{n}^{-*}$. Write $\lambda = \sum c_i \lambda_i$. Since $\mathfrak{g}_\lambda \subset \mathfrak{n}$, there exists an $i_0 > p$ such that $c_{i_0} > 0$. If, additionally, $i_0 > p^{-*}$, then, of course, $\mathfrak{g}_\lambda \subset \mathfrak{n}^{-*}$ and we are done. Otherwise, $p < i_0 \le p^{-*}$, hence $\lambda_{i_0}(H^*) > 0$. But

$$\mathfrak{g}_\lambda \not\subset \mathfrak{p} \cap \mathfrak{n}^* \Rightarrow \lambda(H^*) = \sum_{i=p+1}^{l_0} c_i \lambda_i(H^*) \le 0.$$

Because

$$c_{i_0}\lambda_{i_0}(H^*)$$

is positive, there must exist another index j_0 such that $c_{j_0} > 0$ and $\lambda_{j_0}(H^*) < 0$. Such a j_0 must be $> p^{-*}$, hence once again $\mathfrak{g}_\lambda \subset \mathfrak{n}^{-*}$. □

Now where are we? Starting with the Γ-cuspidal parabolic subgroup $P^* \neq G$, we produce a map $\tau^*: \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$ and the associated set $\mathcal{C}_\Gamma^*$. We shall consistently write $P^* \succcurlyeq P_0^*$, P_0^* some Γ-percuspidal which has been and will remain fixed. The reader must realize, however, that the ambient split components can vary, the choice being dictated by the context. There is undoubtedly some potential for confusion here so we shall make every effort to be completely precise in order to minimize it.

In terms of canonical data associated with $(P_0^*, S_0^*; A_0^*)$, A_0^* any split component of (P_0^*, S_0^*), the set $\mathcal{C}_\Gamma^*$ admits a decomposition, the description of which may be formulated in the following way. Fix a finite subset F_0^* of G with the property that

$$\{x_0 P_0^* x_0^{-1} : x_0 \in F_0^*\}$$

is a set of representatives for the Γ-percuspidal parabolic subgroups of G. Given

$$\begin{cases} w_0 \in W(A_0^*) \\ x_0 \in F_0^*, \end{cases}$$

put

$$\Delta(w_0 : x_0) = \Gamma x_0 \cap P_0^* w_0 P_0^*.$$

Then

$$\mathcal{C}_\Gamma^* = \bigcup_{w_0 \in W(A_0^*)} \bigcup_{x_0 \in F_0^*} \bigcup_{P \succcurlyeq P_0^*} \bigcup_{\delta_0 \in \Delta(w_0 : x_0)} \{\delta_0 P \delta_0^{-1}\} \cap \mathcal{C}_\Gamma^*.$$

A natural question thus suggests itself. If $P \succcurlyeq P_0^*$, when is it true that $\delta_0 P \delta_0^{-1} \in \mathcal{C}_\Gamma^*$? Naturally, this is entirely equivalent to determining when

$$\tau^*(\delta_0 P \delta_0^{-1}) = \delta_0 P \delta_0^{-1}.$$

Since we wish to discuss $\tau^*(\delta_0 P \delta_0^{-1})$, $P \succcurlyeq P_0^*$, the situation changes slightly in that P is not the generic Γ-cuspidal (as it was in Proposition 6.1) but rather this time $\delta_0 P \delta_0^{-1}$ is. Using the fact that

$$\delta_0 \in \Delta(w_0 : x_0) \subset P_0^* w_0 P_0^*,$$

write

$$\delta_0 = n(\delta_0) w_0 p(\delta_0) \qquad (n(\delta_0) \in N_0^*,\ p(\delta_0) \in P_0^*).$$

Then in the picture

$$\begin{cases} \delta_0 P \delta_0^{-1} \geqslant \delta_0 P_0^* \delta_0^{-1} \\ P^* \geqslant P_0^*, \end{cases}$$

the split component shared by $\delta_0 P_0^* \delta_0^{-1}$ and P_0^* is

$$n(\delta_0) A_0^* n(\delta_0)^{-1}.$$

Supposing that

$$(P, S; A) \geqslant (P_0^*, S_0^*; A_0^*),$$

we have, accordingly,

$$\begin{aligned}
& \tau^*(\delta_0 P \delta_0^{-1}) = \delta_0 P \delta_0^{-1} \\
\Leftrightarrow \quad & \\
& \delta_0 P \delta_0^{-1} \cap N^* \subset \delta_0 N \delta_0^{-1} \\
\Leftrightarrow \quad & \\
& \mathrm{Ad}(\delta_0)\mathfrak{p} \cap \mathfrak{n}^* \subset \mathrm{Ad}(\delta_0)\mathfrak{n} \\
\Leftrightarrow \quad & \\
& \mathrm{Ad}(n(\delta_0)) w_0 \mathrm{Ad}(n(\delta_0)^{-1}) \cdot \mathrm{Ad}(n(\delta_0))\mathfrak{p} \cap \mathfrak{n}^* \\
& \subset \mathrm{Ad}(n(\delta_0)) w_0 \mathrm{Ad}(n(\delta_0)^{-1}) \cdot \mathrm{Ad}(n(\delta_0))\mathfrak{n}
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad & \\
& \mathrm{Ad}(n(\delta_0)) w_0 \mathrm{Ad}(n(\delta_0)^{-1}) \cdot \mathrm{Ad}(n(\delta_0))\mathfrak{m} \cap \mathfrak{n}^* = \{0\} \\
\Leftrightarrow \quad & \\
& \mathrm{Ad}(n(\delta_0)) w_0 \mathfrak{m} \cap \mathfrak{n}^* = \{0\} \\
\Leftrightarrow \quad & \\
& \mathrm{Ad}(n(\delta_0))(w_0 \mathfrak{m} \cap \mathfrak{n}^*) = \{0\} \\
\Leftrightarrow \quad & \\
& w_0 \mathfrak{m} \cap \mathfrak{n}^* = \{0\}.
\end{aligned}$$

Observe that this Lie algebra-theoretic condition involves δ_0 only through w_0, the shifted data entering in the verification but not in the final conclusion. Write

$$(P^*, S^*; A^*) \geqslant (P_0^*, S_0^*; A_0^*).$$

We remark that the split component A^* is the same as the one figuring in the earlier constructions vis-à-vis

$$\begin{cases} P \geqslant P_0^* \\ P^* \geqslant P_0^* \end{cases} \qquad : A,\ A^* \subset A_0^*.$$

Because $w_0\mathfrak{m} \cap \mathfrak{n}^*$ is a sum of root spaces with respect to $\mathfrak{a}_0^*$, a given root occurring only if its negative appears simultaneously,

$$w_0\mathfrak{m} \cap \mathfrak{n}^* = \{0\} \Leftrightarrow \forall \lambda \text{ st } \mathfrak{g}_\lambda \subset \mathfrak{m}: w_0\lambda \,|\, \mathfrak{a}^* = 0.$$

Represent P per P_0^*, i.e. write

$$P = (P_0^*)_F.$$

Then

$$w_0\mathfrak{m} \cap \mathfrak{n}^* = \{0\} \Leftrightarrow \forall \lambda_\iota \in F, w_0\lambda_\iota \,|\, \mathfrak{a}^* = 0.$$

In recapitulation, therefore,

$$\tau^*(\delta_0 P \delta_0^{-1}) = \delta_0 P \delta_0^{-1} \Leftrightarrow \forall \lambda_i \in F, w_0\lambda_\iota \,|\, \mathfrak{a}^* = 0.$$

Let us assume now that $\delta_0 P \delta_0^{-1} \in \mathcal{C}_\Gamma^*$, Lemma 6.3 then providing a characterization of the fiber over $\delta_0 P \delta_0^{-1}$. Thanks to what has just been learned, it is not difficult to describe

$$(\delta_0 P \delta_0^{-1})^{-*}.$$

Indeed, if

$$F_{w_0}^{-*} = F \cup \{\lambda_\iota: w_0\lambda_\iota > 0 \text{ and } w_0\lambda_\iota \,|\, \mathfrak{a}^* \neq 0\},$$

then

$$(\delta_0 P \delta_0^{-1})^{-*} = \delta_0 (P_0^*)_{F_{w_0}^{-*}} \delta_0^{-1},$$

as can be readily seen by transporting the question to $w_0 P_0^* w_0^{-1}$ and using the definitions.

The preceding structural facts will all play a role in due course. Setting them aside for the time being, we are at last ready to come to grips with the purported idempotence of $Q^{\mathbf{H}}$. It will be best to restate our objective.

MAIN LEMMA. *Fix* $\mathbf{H}_0$ *in* $\mathfrak{a}$ — *then there exists* $\mathbf{H}_{00} \lessdot \mathbf{H}_0$ *such that for all* $\mathbf{H} \lessdot \mathbf{H}_{00}$, *independently of* $P^* \neq G$,

$$I_{P^*}(\mathbf{H}) - H_{P^*|A^*}(x) \notin -\mathcal{D}_{P^*}(\mathfrak{a}^*)^- \Rightarrow (Q^{\mathbf{H}}f)^{P^*}(x) = 0.$$

We hasten to stress that here, of course, A^* is the special split component of (P^*, S^*).

By way of explanation, recall that the definition of $Q^{\mathbf{H}}f$ was initially given in terms of the P_i, that is,

$$Q^{\mathbf{H}}f = \sum_{i=1}^{r} (-1)^{\operatorname{rank}(P_i)} T_{P_i}(\mathbf{H}:f),$$

it then being observed that still

$$Q^{\mathbf{H}}f = \sum_{P \in \mathcal{C}_\Gamma} (-1)^{\operatorname{rank}(P)} \chi_{P,A:\mathfrak{I}}\big(I_P(\mathbf{H}) - H_{P|A}(?)\big)\cdot f^P,$$

the latter formulation making it clear that the role of P is that of a running variable. [Note: Again, all split components are special.]

Our immediate intention is to discuss

$$\left(Q^{\mathbf{H}}f\right)^{P^*}.$$

Because the terms in the sum defining $Q^{\mathbf{H}}f$ are not $\Gamma \cap N^*$-invariant, it will first be necessary to split $\mathcal{C}_\Gamma$ into $\Gamma \cap N^*$ conjugacy classes. Denoting by $\mathcal{C}_\Gamma(N^*)$ a set of representatives for these, the diagram

$$\begin{array}{ccc} \mathcal{C}_\Gamma & \xrightarrow{\tau^*} & \mathcal{C}_\Gamma \\ \downarrow & & \downarrow \\ \mathcal{C}_\Gamma(N^*) & \cdots\cdots\succ & \mathcal{C}_\Gamma(N^*) \end{array}$$

can be rendered commutative provided the dotted arrow is defined according to the relation

$$\tau^*(\gamma P \gamma^{-1}) = \gamma \tau^*(P) \gamma^{-1} \qquad (\gamma \in \Gamma \cap N^*).$$

We then have

$$Q^{\mathbf{H}}f(x) = \sum_{P \in \mathcal{C}_\Gamma(N^*)} \sum_{\gamma \in \Gamma \cap N^*/\Gamma \cap N^* \cap P} \times (-1)^{\operatorname{rank}(P)} \chi_{P_\gamma, A_\gamma:\mathfrak{I}}\big(I_{P_\gamma}(\mathbf{H}) - H_{P_\gamma|A_\gamma}(x)\big)\cdot f^{P_\gamma}(x)$$

or still

$$\sum_{P \in \mathcal{C}_\Gamma(N^*)} \sum_{\gamma \in \Gamma \cap N^*/\Gamma \cap N^* \cap P} \times (-1)^{\operatorname{rank}(P)} \chi_{P,A:\mathfrak{I}}\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\cdot f^P(x\gamma).$$

SUBLEMMA. *There exists a positive constant* K_0 *and a positive integer* k_0 *such that*

$$\forall\, H \in \mathfrak{a}$$

$$\sigma_*^F(H_0 - H) \neq 0$$

$$\Rightarrow$$

$$\|H(F)\| \leq K_0 - k_0 \cdot \sum_{i=1}^{p} \lambda_i^{F_i}(H).$$

Proof. We begin by reminding ourselves that

$$\mathfrak{a} = \mathfrak{a}(F) \oplus \mathfrak{a}_F.$$

This said, put

$$\|?\|_F = \sum_{i=1}^{p} |\langle ?, \lambda_i^{F_i} \rangle| .$$

Since

$$\lambda_i^{F_i} = c_i \lambda'_F \qquad (\exists\, c_i > 0),$$

$\|?\|_F$ is a norm on $\mathfrak{a}(F)$. The fact that

$$\sigma_*^F(H_0 - H) \neq 0$$

implies that

$$H_0(F) - H(F) \in \mathcal{C}_{P(F)}(\mathfrak{a}(F)).$$

Accordingly,

$$\begin{aligned} \|H(F)\|_F &\leq \|H_0(F)\|_F + \|H_0(F) - H(F)\|_F \\ &\leq \|H_0(F)\|_F + \sum_{i=1}^{p} \langle H_0(F) - H(F), \lambda_i^{F_i} \rangle \\ &= \|H_0(F)\|_F + \sum_{i=1}^{p} \langle H_0(F), \lambda_i^{F_i} \rangle - \sum_{i=1}^{p} \langle H(F), \lambda_i^{F_i} \rangle. \end{aligned}$$

By equivalence of norms, there exists a positive integer k_0 such that

$$\|?\| \leq k_0 \cdot \|?\|_F.$$

Because

$$\begin{cases} \langle H_0(F), \lambda_i^{F_i} \rangle = \lambda_i^{F_i}(H_0) \\ \langle H(F), \lambda_i^{F_i} \rangle = \lambda_i^{F_i}(H), \end{cases}$$

choosing

$$K_0 = k_0 \cdot \left(\| H_0(F) \|_F + \sum_{i=1}^{p} \lambda_i^{F_i}(H_0) \right)$$

serves to complete the proof. □

To deal with Lemma 7.2, we have only to take

$$H_0 = I_P(\mathbf{H})$$

in the preceding considerations. As for Lemma 7.3, it is clearly a consequence of the following more general statement which, in and of itself, will be needed in order to achieve our goal.

LEMMA 7.3. (bis) *There exists a positive constant k_r and a positive integer K_r such that*

$$\forall\, y \in G$$

$$F_P(\mathbf{H} : \mathbf{H}_0 : y) \cdot \sigma_*^F\big(I_P(\mathbf{H}) - H_{P|A}(y)\big) \neq 0$$

$$\Rightarrow$$

$$| E_r(y) | \leq k_r \cdot \exp\big(K_r \cdot \| H_{P|A}(y) \|\big).$$

We shall proceed via a series of sublemmas.

SUBLEMMA 1. *Let $T \in \mathfrak{gl}(n, \mathbf{C})$ — then*

$$\exp(\| T \|_{\mathrm{OP}}) \geq \| e^T \|_{\mathrm{OP}}.$$

Proof. Let v be a non-zero vector — then

$$\begin{aligned} \| e^T v \| &= \left\| \sum_{n=0}^{\infty} \frac{1}{n!} T^n v \right\| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \| T^n v \| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \| T^n \|_{\mathrm{OP}} \| v \| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \| T \|_{\mathrm{OP}}^n \| v \| \\ &= \exp(\| T \|_{\mathrm{OP}}) \| v \|, \end{aligned}$$

which is clearly equivalent to our assertion. □

Let us agree to denote by

$$\lambda_{i_0} - \lambda^{i_0}$$

generic (simple) roots — dual roots attached to $(P_{i_0}, S_{i_0}; A_{i_0})$.

SUBLEMMA 2. *There exists a positive constant* C_{i_0} *such that*

$$\inf_{X \in \mathfrak{g}} \left\{ C_{i_0} \cdot \|X\|_\theta + \left\langle H_{P_{i_0}|A_{i_0}}(\exp(X)), \lambda^{i_0} \right\rangle \right\} \geq 0.$$

[Note: $\|?\|_\theta$ is the Euclidean norm on $\mathfrak{g}$ canonically associated with the bilinear form B (cf. §3).]

Proof. Choose, as is possible, a finite dimensional representation π_{i_0} of G^0 (the identity component of G) on a complex Hilbert space E_{i_0}, a positive real number r_{i_0}, and a unit vector v_{i_0} such that

$$\|\pi_{i_0}(x)v_{i_0}\| = \exp\left(\left\langle H_{P_{i_0}|A_{i_0}}(x), r_{i_0}\lambda^{i_0} \right\rangle\right) \qquad (x \in G^0).$$

Let $X \in \mathfrak{g}$ — then

$$\|\pi_{i_0}(\exp(X))v_{i_0}\| \geq \|\pi_{i_0}(\exp(-X))\|_{\mathrm{OP}}^{-1}$$

$$\Rightarrow$$

$$\left\langle H_{P_{i_0}|A_{i_0}}(\exp(X)), \lambda^{i_0} \right\rangle \geq -\frac{1}{r_{i_0}} \cdot \log\left(\|\pi_{i_0}(\exp(-X))\|_{\mathrm{OP}}\right).$$

There is a positive constant C, not depending on X, with the property that

$$\|d\pi_{i_0}(-X))\|_{\mathrm{OP}} \leq C \cdot \|X\|_\theta.$$

Consequently, if

$$C_{i_0} = C/r_{i_0},$$

then

$$C_{i_0} \cdot \|X\|_\theta + \left\langle H_{P_{i_0}|A_{i_0}}(\exp(X)), \lambda^{i_0} \right\rangle \geq \frac{1}{r_{i_0}} \cdot \left[\|d\pi_{i_0}(-X))\|_{\mathrm{OP}} - \log\left(\|\pi_{i_0}(\exp(-X))\|_{\mathrm{OP}}\right)\right].$$

But

$$\pi_{i_0}(\exp(-X)) = \exp\left(d\pi_{i_0}(-X)\right).$$

Therefore, thanks to Sublemma 1, the quantity inside [...] is non-negative, so the inf over all $X \in \mathfrak{g}$ of our expression is bounded below by zero, as desired. □

SUBLEMMA 3. *Let C', C'' be compact subsets of G — then there exists a positive constant $C(i_0)$ such that*

$$\inf_{c' \in C', c'' \in C''; H \in \mathfrak{a}} \left\{ C(i_0) \cdot \|H\| + \left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c''), \lambda^{i_0} \right\rangle \right\} > -\infty.$$

Proof. Write

$$\left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c''), \lambda^{i_0} \right\rangle = \left\langle H_{P_{i_0}|A_{i_0}}\left(c'c'' \exp\left(\mathrm{Ad}(c'')^{-1} H\right)\right), \lambda^{i_0} \right\rangle$$

$$= \left\langle H_{P_{i_0}|A_{i_0}}(c'c''k), \lambda^{i_0} \right\rangle + \left\langle H_{P_{i_0}|A_{i_0}}\left(\exp\left(\mathrm{Ad}(c'')^{-1} H\right)\right), \lambda^{i_0} \right\rangle,$$

k the K-component of

$$\exp\left(\mathrm{Ad}(c'')^{-1} H\right)$$

per $G = K \cdot P_{i_0}$. Owing to Sublemma 2,

$$\left\langle H_{P_{i_0}|A_{i_0}}\left(\exp\left(\mathrm{Ad}(c'')^{-1} H\right)\right), \lambda^{i_0} \right\rangle \geq -C_{i_0} \cdot \|\mathrm{Ad}(c'')^{-1} H\|_\theta.$$

Since

$$\|\mathrm{Ad}(c'')^{-1} H\|_\theta \leq \sup_{C''} \|\mathrm{Ad}(?)^{-1}\|_{\mathrm{OP}} \cdot \|H\|,$$

it follows that with

$$C(i_0) = C_{i_0} \cdot \sup_{C''} \|\mathrm{Ad}(?)^{-1}\|_{\mathrm{OP}},$$

we have

$$C(i_0) \cdot \|H\| + \left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c''), \lambda^{i_0} \right\rangle$$

$$\geq \inf_{C' \cdot C'' \cdot K} \left\langle H_{P_{i_0}|A_{i_0}}(?), \lambda^{i_0} \right\rangle > -\infty,$$

thereby finishing the proof. □

Assign to the symbol ρ_{i_0} the customary interpretation.

SUBLEMMA 4. *Let C', C'' be compact subsets of G — then there exist positive constants C_0, C_{00} such that*

$$\forall x \in G, \quad \forall H \in \mathfrak{a}, \quad \forall c' \in C', \quad \forall c'' \in C'',$$

$$\left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c'' x), \rho_{i_0} \right\rangle$$

$$\geq \left\langle H_{P_{i_0}|A_{i_0}}(c'x), \rho_{i_0} \right\rangle - C_0 \cdot \|H\| - C_{00}.$$

Proof. Because ρ_{i_0} can be written as a positive linear combination of the λ^{i_0}, we need only produce C_0 and C_{00} with the property that

$$\left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c'' x), \lambda^{i_0} \right\rangle \geq \left\langle H_{P_{i_0}|A_{i_0}}(c' x), \lambda^{i_0} \right\rangle - C_0 \cdot \|H\| - C_{00}.$$

Now, on the one hand,

$$\left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c'' x), \lambda^{i_0} \right\rangle = \left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c'' k_x), \lambda^{i_0} \right\rangle + \left\langle H_{P_{i_0}|A_{i_0}}(x), \lambda^{i_0} \right\rangle,$$

while, on the other,

$$\left\langle H_{P_{i_0}|A_{i_0}}(c' x), \lambda^{i_0} \right\rangle = \left\langle H_{P_{i_0}|A_{i_0}}(c' k_x), \lambda^{i_0} \right\rangle + \left\langle H_{P_{i_0}|A_{i_0}}(x), \lambda^{i_0} \right\rangle.$$

This means that it is enough to establish the existence of C_0, C_{00} such that

$$\forall k \in K, \quad \forall H \in \mathfrak{a}, \quad \forall c' \in C', \quad \forall c'' \in C'',$$

$$\left\langle H_{P_{i_0}|A_{i_0}}(c' k), \lambda^{i_0} \right\rangle \leq C_{00} + C_0 \cdot \|H\| + \left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c'' k), \lambda^{i_0} \right\rangle.$$

Since the left-hand side stays bounded, Sublemma 3 guarantees the validity of our inequality. □

Proof of Lemma 7.3. (bis) By definition (cf. Lemma 4.7), E_r is a positive linear combination of the

$$E\left(P_{i_0} \mid A_{i_0} : 1 : (2r+1)\rho_{i_0} : ?\right).$$

Accordingly, we need only make our estimate for the latter. In turn, thanks to the usual invariances, there is no loss of generality in supposing that

$$y = c' \exp(H) c'' \qquad \left(H = H_{P|A}(y)\right)$$

where c', c'' are confined to certain compact sets C', C'', say. This being so, we have

$$\begin{aligned} E\left(P_{i_0} \mid A_{i_0} : 1 : (2r+1)\rho_{i_0} : y\right) &= E\left(P_{i_0} \mid A_{i_0} : 1 : (2r+1)\rho_{i_0} : c' \exp(H) c''\right) \\ &= \sum_{\gamma \in \Gamma / \Gamma \cap P_{i_0}} \exp\left(\left\langle H_{P_{i_0}|A_{i_0}}(c' \exp(H) c'' \gamma), 2r\rho_{i_0} \right\rangle\right) \\ &\leq \exp(-2r(C_0 \cdot \|H\| + C_{00})) \\ &\quad \times \sum_{\gamma \in \Gamma / \Gamma \cap P_{i_0}} \exp\left(\left\langle H_{P_{i_0}|A_{i_0}}(c' \gamma), 2r\rho_{i_0} \right\rangle\right). \end{aligned}$$

Here, of course, Sublemma 4 has been invoked. In this connection, let us also recall that r is negative (in fact < -1; cf. Lemma 4.7). Since

$$\sum_{\gamma\in\Gamma/\Gamma\cap P_{i_0}} \exp\left(\left\langle H_{P_{i_0}|A_{i_0}}(c'\gamma), 2r\rho_{i_0}\right\rangle\right) = E\left(P_{i_0}\,|\,A_{i_0}:1:\,(2r+1)\rho_{i_0}:c'\right),$$

the supremum

$$\sup_{c'\in C'} \sum_{\gamma\in\Gamma/\Gamma\cap P_{i_0}} \exp\left(\left\langle H_{P_{i_0}|A_{i_0}}(c'\gamma), 2r\rho_{i_0}\right\rangle\right)$$

is finite. Bearing in mind that $H = H_{P|A}(y)$, the existence of k_r and K_r is therefore clear. □

All the preparation which is needed to prove Theorem 5.2 has now been completed, i.e., at this point we are in a position to dispense with:

Proof of Objective. (bis) We can and will suppose that the purported exponent of growth c is > 1. It is then a question of establishing the existence of a positive constant C_c and a positive integer K_c such that

$$\forall\, x \in \mathfrak{S}_{t_0,\omega_0}\kappa_{t_0}, \quad \forall\, \gamma\in\Gamma$$

$$F_P(\mathbf{H}:\mathbf{H}_0:x\gamma)\cdot\sigma_*^F\left(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\right) \neq 0$$

$$\Rightarrow$$

$$\exp\left(-K_c\cdot\|H_{P|A}(x\gamma)\|\right) \leq C_c\cdot\Xi_{P_{i_0}}(x)^c.$$

Indeed, if this be done, we would have

$$\begin{aligned}\sum_{\gamma\in\Gamma/\Gamma\cap P} & F_P(\mathbf{H}:\mathbf{H}_0:x\gamma)\cdot\sigma_*^F\left(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\right) \\ &\times\exp\left(-K_c\cdot\|H_{P|A}(x\gamma)\|\right) \\ &\leq C_c\cdot\Xi_{P_{i_0}}(x)^c\cdot\sum_{\gamma\in\Gamma/\Gamma\cap P}\chi_{P,A:\mathfrak{H}}\left(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\right) \\ &= C_c\cdot\Xi_{P_{i_0}}(x)^c\cdot T_P(\mathbf{H}:1)(x),\end{aligned}$$

so it would only be necessary to absorb the exponent of growth of the slowly increasing function $T_P(\mathbf{H}:1)$ into c. Put $r = -c(< -1)$ — then, as can be seen from the proof of Lemma 4.7,

$$E\left(P_{i_0}\,|\,A_{i_0}:1:\,(2r+1)\rho_{i_0}:x\right) \geq C_{i_0}\cdot\Xi_{P_{i_0}}(x)^r,$$

C_{i_0} as there. But, from the proof of Lemma 7.3 (bis) supra,

$$E\big(P_{i_0}\,|\,A_{i_0}:1:(2r+1)\rho_{i_0}:x\big) = E\big(P_{i_0}\,|\,A_{i_0}:1:(2r+1)\rho_{i_0}:x\gamma\big)$$
$$\leq k_r\cdot\exp\big(K_r\cdot\|H_{P|A}(x\gamma)\|\big).$$

Take

$$\begin{cases} K_c = K_r \\ C_c = k_r/C_{i_0}. \end{cases}$$

Then it follows that

$$\exp\big(-K_c\cdot\|H_{P|A}(x\gamma)\|\big) \leq k_r\cdot E\big(P_{i_0}\,|\,A_{i_0}:1:(2r+1)\rho_{i_0}:x\big)^{-1}$$
$$\leq \big(k_r/C_{i_0}\big)\cdot\Xi_{P_{i_0}}(x)^{-r} = C_c\cdot\Xi_{P_{i_0}}(x)^{c},$$

the sought for conclusion. □

We shall close by justifying a comment made in § 5, namely that

$$Q^{\mathbf{H}}(R(G/\Gamma)) \subset R(G/\Gamma)$$

so long as the parameter $\mathbf{H}$ is subject to the assumptions set forth in Proposition 3.10. Naturally, the preceding argument is not immediately applicable: The elements of $R(G/\Gamma)$ need not even be differentiable. It is easy to see, however, what needs to be done. Thus fix $f \in R(G/\Gamma)$ — then, in view of what has gone before, to ascertain that

$$Q^{\mathbf{H}}f \in R(G/\Gamma)$$

we have only to prove that

$$\forall\, F \in \mathcal{P}_{\mathfrak{L}} \quad (F \neq \varnothing), \quad \forall\, K > 0, \quad \exists\, C_K > 0$$

such that

$$\forall\, y \in G$$
$$F_P(\mathbf{H}:\mathbf{H}_0:y)\cdot\sigma_*^F\big(I_P(\mathbf{H}) - H_{P|A}(y)\big) \neq 0$$
$$\Rightarrow$$
$$|\,\phi_{P,F}(y)\,| \leq C_K\cdot\exp\big(-K\cdot\|H_{P|A}(y)\|\big).$$

Remembering that

$$\phi_{P,F} = \sum_{\{F' :\, F' \subset F\}} (-1)^{\operatorname{rank}(P_{F'})}\cdot f^{P_{F'}},$$

it is clearly enough to prove that

$$\forall F \in \mathcal{P}_{\mathfrak{L}} \quad (F \neq \varnothing), \quad \forall K > 0, \quad \exists C_K > 0$$

such that

$$\forall y \in G$$
$$F_P(\mathbf{H} : \mathbf{H}_0 : y) \cdot \sigma_*^F\big(I_P(\mathbf{H}) - H_{P|A}(y)\big) \neq 0$$
$$\Rightarrow$$
$$|f(y)| \leq C_K \cdot \exp\big(-K \cdot \| H_{P|A}(y) \|\big).$$

LEMMA 7.4. *For every positive integer $K \gg 0$ there exists a positive constant C_K such that for all $x \in G$*

$$|f(x)| \leq C_K \cdot \exp\big(K \cdot \langle H_{P|A}(x), \rho \rangle\big).$$

Admit this result for the moment — then our verification can be completed as follows. To begin with, note that

$$\sigma_*^F\big(I_P(\mathbf{H}) - H_{P|A}(y)\big) \neq 0$$
$$\Rightarrow$$
$$I_P(\mathbf{H}) - H_{P|A}(y) \in \mathcal{D}_P(\mathfrak{a}).$$

This being the case, write

$$\rho = \sum_{i=1}^{l} c_\rho^i \lambda^i \qquad \big(c_\rho^i > 0\big).$$

Put

$$\|?\|_\rho = \sum_{i=1}^{l} c_\rho^i \cdot |\langle ?, \lambda^i \rangle| .$$

Then $\|?\|_\rho$ is a norm on $\mathfrak{a}$, thus, by equivalence of norms, there exists a positive constant C_ρ such that

$$\|?\| \leq C_\rho \cdot \|?\|_\rho .$$

We have now

$$\begin{aligned}
\langle H_{P|A}(y), \rho \rangle &= \langle I_P(\mathbf{H}), \rho \rangle - \langle I_P(\mathbf{H}) - H_{P|A}(y), \rho \rangle \\
&= \langle I_P(\mathbf{H}), \rho \rangle - \sum_{i=1}^{l} c_\rho^i \cdot \langle I_P(\mathbf{H}) - H_{P|A}(y), \lambda^i \rangle \\
&= \langle I_P(\mathbf{H}), \rho \rangle - \sum_{i=1}^{l} c_\rho^i \cdot |\langle I_P(\mathbf{H}) - H_{P|A}(y), \lambda^i \rangle| \\
&= \langle I_P(\mathbf{H}), \rho \rangle - \| I_P(\mathbf{H}) - H_{P|A}(y) \|_\rho ,
\end{aligned}$$

from which we derive the estimate

$$\begin{aligned}\|H_{P|A}(y)\| &\leq C_\rho \cdot \|H_{P|A}(y)\|_\rho \\ &\leq C_\rho \cdot \left(\|I_P(\mathbf{H}) - H_{P|A}(y)\|_\rho + \|I_P(\mathbf{H})\|_\rho\right) \\ &= C_\rho \cdot \left(\|I_P(\mathbf{H})\|_\rho + \langle I_P(\mathbf{H}), \rho\rangle - \langle H_{P|A}(y), \rho\rangle\right).\end{aligned}$$

Let K be a large positive integer — then

$$-K \cdot \|H_{P|A}(y)\| \geq -K_{\mathbf{H}} + C_\rho K \cdot \langle H_{P|A}(y), \rho\rangle$$

where

$$K_{\mathbf{H}} = KC_\rho \cdot \left(\|I_P(\mathbf{H})\|_\rho + \langle I_P(\mathbf{H}), \rho\rangle\right).$$

Hence

$$\begin{aligned}|f(y)| &\leq C_{[C_\rho K]} \cdot \exp\left([C_\rho K] \cdot \langle H_{P|A}(y), \rho\rangle\right) \\ &\leq C_{[C_\rho K]} \cdot \exp\left(C_\rho K \cdot \langle H_{P|A}(y), \rho\rangle\right) \leq C_K \cdot \exp\left(-K \cdot \|H_{P|A}(y)\|\right),\end{aligned}$$

C_K being, by definition,

$$C_{[C_\rho K]} \cdot \exp(K_{\mathbf{H}}).$$

The desired majorization of f is therefore established

There remains the proof of Lemma 7.4. It will be recalled that E_r is a positive linear combination of the

$$E\left(P_{i_0} | A_{i_0} : 1 : (2r + 1)\rho_{i_0} : ?\right) \qquad (r < -1).$$

Since f is rapidly decreasing, there exists a positive constant C_r such that

$$|f(x)| \leq C_r \cdot E_r(x)^{-1} \qquad (x \in G).$$

Let $K \gg 0$ — then

$$\begin{aligned}E(P | A : 1 : (-K + 1)\rho : x) &= \sum_{\gamma \in \Gamma/\Gamma \cap P} \exp\left(-K \cdot \langle H_{P|A}(x\gamma), \rho\rangle\right) \\ &\geq \exp\left(-K \cdot \langle H_{P|A}(x), \rho\rangle\right) \qquad (x \in G).\end{aligned}$$

But

$$E(P | A : 1 : (-K + 1)\rho : ?)$$

is a slowly increasing function so, in view of Lemma 4.7, there exists an $r < -1$ and a positive constant $C_{K,r}$ such that

$$E(P | A : 1 : (-K + 1)\rho : x) \leq C_{K,r} \cdot E_r(x)$$

for all x in G. Consequently,

$$|f(x)| \le C_r \cdot E_r(x)^{-1} \le (C_r C_{K,r}) \cdot E(P|A:1:(-K+1)\rho:x)^{-1}$$
$$\le (C_r C_{K,r}) \cdot \exp\left(K \cdot \left\langle H_{P|A}(x), \rho \right\rangle\right).$$

Setting

$$C_K = C_r C_{K,r}$$

completes the proof.

8. Additional properties of the truncation operator. The purpose of this section will be to carry the study of the truncation operator $Q^{\mathbf{H}}$ a little further. In contradistinction to Theorems 5.1 and 5.2 which, undoubtedly, are the main results, the properties appearing here lie less deep, being considerably more elementary and formal in character. Nevertheless, they all will play a role in due course.

We begin with some functorial remarks which are most easily expressed via certain commutative diagrams.

The parameter space $\mathfrak{a}$ depends on Γ. When this needs to be emphasized, we append a subscript: $\mathfrak{a}_\Gamma$.

Suppose that F is a finite subgroup of Γ, normal in G — then $\mathfrak{a}_\Gamma = \mathfrak{a}_{\Gamma/F}$ and, for any $\mathbf{H}$, the diagram

$$\begin{array}{ccccc} S(G/\Gamma) & \cup & L^2(G/\Gamma) = S((G/F)/(\Gamma/F)) & \cup & L^2((G/F)/(\Gamma/F)) \\ & Q^{\mathbf{H}}\downarrow & & \downarrow Q^{\mathbf{H}} & \\ S(G/\Gamma) & \cup & L^2(G/\Gamma) = S((G/F)/(\Gamma/F)) & \cup & L^2((G/F)/(\Gamma/F)) \end{array}$$

commutes.

Suppose that Γ' is a lattice in G, satisfying the usual assumption, such that $\Gamma' \supset \Gamma$ — then there is a canonical arrow of injection

$$\mathfrak{a}_{\Gamma'} \hookrightarrow \mathfrak{a}_\Gamma.$$

Given $\mathbf{H}' \in \mathfrak{a}_{\Gamma'}$, call $\mathbf{H}$ its image in $\mathfrak{a}_\Gamma$ — then the diagram

$$\begin{array}{ccccccc} S(G/\Gamma') & \cup & L^2(G/\Gamma') & \hookrightarrow & S(G/\Gamma) & \cup & L^2(G/\Gamma) \\ & Q^{\mathbf{H}'}\downarrow & & & & \downarrow Q^{\mathbf{H}} & \\ S(G/\Gamma') & \cup & L^2(G/\Gamma') & \hookrightarrow & S(G/\Gamma) & \cup & L^2(G/\Gamma) \end{array}$$

commutes.

Suppose that G' is an open subgroup of G, Γ' a lattice in G', satisfying the usual assumption. Put $\Gamma = \Gamma'$ — then $\mathfrak{a}_\Gamma = \mathfrak{a}_{\Gamma'}$. There is a canonical morphism of extension

$$\text{Ext}: Fnc(G'/\Gamma') \to Fnc(G/\Gamma)$$

and a canonical morphism of restriction

$$\mathrm{Res}\colon Fnc(G/\Gamma) \to Fnc(G'/\Gamma').$$

Let $\mathbf{H} = \mathbf{H}' \in \mathfrak{a}_\Gamma = \mathfrak{a}_{\Gamma'}$ — then there are commutative diagrams

$$\begin{array}{ccccccc} S(G'/\Gamma') & \cup & L^2(G'/\Gamma') & \overset{\mathrm{Ext}}{\to} & S(G/\Gamma) & \cup & L^2(G/\Gamma) \\ & Q^{\mathbf{H}'}\downarrow & & & & \downarrow Q^{\mathbf{H}} & \\ S(G'/\Gamma') & \cup & L^2(G'/\Gamma') & \underset{\mathrm{Ext}}{\to} & S(G/\Gamma) & \cup & L^2(G/\Gamma) \end{array}$$

$$\begin{array}{ccccccc} S(G/\Gamma) & \cup & L^2(G/\Gamma) & \overset{\mathrm{Res}}{\to} & S(G'/\Gamma') & \cup & L^2(G'/\Gamma') \\ & Q^{\mathbf{H}}\downarrow & & & & \downarrow Q^{\mathbf{H}'} & \\ S(G/\Gamma) & \cup & L^2(G/\Gamma) & \underset{\mathrm{Res}}{\to} & S(G'/\Gamma') & \cup & L^2(G'/\Gamma'). \end{array}$$

Suppose that

$$\begin{cases} G = G_1 \times G_2 \\ \Gamma = \Gamma_1 \times \Gamma_2, \end{cases}$$

so that

$$\mathfrak{a}_\Gamma = \mathfrak{a}_{\Gamma_1} \oplus \mathfrak{a}_{\Gamma_2}.$$

Write $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2)$ — then the diagram

$$\begin{array}{ccccc} \big(S(G_1/\Gamma_1) \cup L^2(G_1/\Gamma_1)\big) & \times & \big(S(G_2/\Gamma_2) \cup L^2(G_2/\Gamma_2)\big) & = & S(G/\Gamma) \cup L^2(G/\Gamma) \\ & Q^{\mathbf{H}_1} \times Q^{\mathbf{H}_2} \Big\downarrow & & & \downarrow Q^{(\mathbf{H}_1,\mathbf{H}_2)} \\ \big(S(G_1/\Gamma_1) \cup L^2(G_1/\Gamma_1)\big) & \times & \big(S(G_2/\Gamma_2) \cup L^2(G_2/\Gamma_2)\big) & = & S(G/\Gamma) \cup L^2(G/\Gamma) \end{array}$$

commutes.

One can view the discussion heretofore of the truncation operator as reflecting the 'G/Γ-picture'. In anticipation of the inductive arguments which will arise eventually, we shall also need the '$G/\Gamma \cap P$-picture', to which we now direct our attention.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G — then a complex valued (measurable) function f on $G/\Gamma \cap P$ is said to be slowly increasing provided the following condition is met. Given

$$\begin{cases} \text{a } \Gamma\text{-percuspidal } P_0 \text{ dominated by } P \\ \text{a Langlands decomposition } M \cdot A \cdot N \text{ of } P \\ \text{a Siegel domain } \mathfrak{S}_0^\dagger \text{ in } M \text{ per } P_0^\dagger, \end{cases}$$

there exist constants C, c, $r_0^\dagger$ such that

$$\forall\, k \in K, \quad \forall\, m \in \mathfrak{S}_0^\dagger, \quad \forall\, a \in A, \quad \forall\, n \in N,$$

$$|f(kman)| \leq C \cdot e^{c\|\log a\|} \cdot \Xi_{P_0^\dagger}(m)^{r_0^\dagger}.$$

We shall agree to write

$$S(G/\Gamma \cap P)$$

for the set of all such f.

LEMMA 8.1. *Let*

$$\mathfrak{F}_P\colon Fnc(G/\Gamma) \to Fnc(G/\Gamma \cap P)$$

be the canonical morphism — then

$$\mathfrak{F}_P(S(G/\Gamma)) \subset S(G/\Gamma \cap P).$$

Proof. The proof is very easy, modulo one remark. In view of Lemma 4.7, there is no loss of generality in replacing f by a suitable E_r (some $r < -1$). The remark, then, is this. Inspect the proof of Lemma 7.3 (bis) — then it will be seen that σ_*^F plays no real role at all. To put it a different way, upon choosing the compact sets C' and C'' of that argument so as to reflect the definition of $F_P(\mathbf{H}:\mathbf{H}_0:?)$, we find that

$$\forall\, y \in G$$

$$F_P(\mathbf{H}:\mathbf{H}_0:y) \neq 0$$

$$\Rightarrow$$

$$|E_r(y)| \leq k_r \cdot \exp\left(K_r \cdot \|H_{P|A}(y)\|\right).$$

Now specialize the choice of P to a Γ-percuspidal P_0, it being supposed that $P_0 \preccurlyeq P$, P as at the beginning — then

$$F_{P_0}(\mathbf{H}:\mathbf{H}_0:?) = 1_G$$

$$\Rightarrow$$

$$|E_r(x)| \leq k_r \cdot \exp\left(K_r \cdot \|H_{P_0|A_0}(x)\|\right)$$

for all x in G. Write $x = kman$ — then

$$H_{P_0|A_0}(x) = \log a + H_{P_0^\dagger|A_0^\dagger}(m),$$

implying that

$$\|H_{P_0|A_0}(x)\| \leq \|\log a\| + \|H_{P_0^\dagger|A_0^\dagger}(m)\|.$$

Assume in addition that m is confined to a Siegel domain $\mathfrak{S}_0^\dagger$ — then there exist constants $C_0^\dagger$, $c_0^\dagger$ such that

$$C_0^\dagger \cdot \exp\left(c_0^\dagger \|H_{P_0^\dagger|A_0^\dagger}(m)\|\right)$$

is bounded by a power of

$$\Xi_{P_0^\dagger}(m).$$

The contention of the lemma is therefore plain. □

Let

$$\{(P_i', S_i'): 1 \leq i \leq r_M\}$$

be a set of Γ-cuspidal split parabolic subgroups of G which are dominated predecessors of (P, S) and with the property that

$$\{('P_i, 'S_i): 1 \leq i \leq r_M\}$$

is a set of representatives for the Γ_M-conjugacy classes of Γ_M-cuspidal split parabolic subgroups of M. [Note: This notation is in accordance with that of Proposition 3.7.] Given $\mathbf{H} \in \mathfrak{a}$, put for any complex valued locally bounded (measurable) function f on $G/\Gamma \cap P$,

$$Q_P^{\mathbf{H}} f(x) = \sum_{i=1}^{r_M} (-1)^{\operatorname{rank}('P_i)}$$
$$\times \sum_{\gamma_i' \in \Gamma \cap P/\Gamma \cap P_i'} \chi_{'P_i,'A_i:\mathfrak{I}}\left(I_{'P_i}(\mathbf{H}) - H_{'P_i|'A_i}(m_{x\gamma_i'})\right) \cdot f^{P_i'}(x\gamma_i'),$$

$Q_P^{\mathbf{H}}$ then being the so-called partial truncation operator, the properties of which are more or less the same as those of $Q^{\mathbf{H}}$ itself; cf. infra. Eg: It is clear that $Q_P^{\mathbf{H}} f$ is again a locally bounded function on $G/\Gamma \cap P$.

We can view $Q_P^{\mathbf{H}}$ as a map

$$Q_P^{\mathbf{H}}: S(G/\Gamma \cap P) \to S(G/\Gamma \cap P).$$

As such, there is a commutative diagram

$$\begin{array}{ccc} S(G/\Gamma \cap P) & \overset{R_\gamma}{\to} & S(G/\Gamma \cap P_\gamma) \\ Q_P^{\mathbf{H}} \downarrow & & \downarrow Q_{P_\gamma}^{\mathbf{H}} \\ S(G/\Gamma \cap P) & \underset{R_\gamma}{\to} & S(G/\Gamma \cap P_\gamma), \end{array}$$

R_γ the right translation operator.

Let

$$S(G/(\Gamma \cap P)\cdot N)$$

stand for the functions in $S(G/\Gamma \cap P)$ which are invariant to the right under N. Assign to the symbol

$$S(K \times M/\Gamma_M \times A)$$

the obvious interpretation. Call

$$\mathfrak{F}_{P,N}\colon Fnc(G/(\Gamma \cap P)\cdot N) \to Fnc(K \times M/\Gamma_M \times A)$$

the canonical morphism — then, of course,

$$\mathfrak{F}_{P,N}(S(G/(\Gamma \cap P)\cdot N)) \subset S(K \times M/\Gamma_M \times A).$$

Furthermore, the diagram

$$\begin{array}{ccc} S(G/(\Gamma \cap P)\cdot N) & \overset{\mathfrak{F}_{P,N}}{\to} & S(K \times M/\Gamma_M \times A) \\ Q_P^{\mathbf{H}} \downarrow & & \downarrow 1 \times Q^{I_M(\mathbf{H})} \times 1 \\ S(G/(\Gamma \cap P)\cdot N) & \underset{\mathfrak{F}_{P,N}}{\to} & S(K \times M/\Gamma_M \times A) \end{array}$$

commutes. Bearing in mind Lemma 3.8, this fact leads at once to idempotence, in the sense of Theorem 5.1. It is clear that

$$S(G/\Gamma \cap P) \cap L^2(G/(\Gamma \cap P)\cdot A \cdot N)$$

is dense in

$$L^2(G/(\Gamma \cap P)\cdot A \cdot N).$$

Since self-adjointness is direct, we thereby obtain from $Q_P^{\mathbf{H}}$ an orthogonal projection on

$$L^2(G/(\Gamma \cap P)\cdot A \cdot N).$$

Let

$$S(G/(\Gamma \cap P)\cdot A \cdot N)$$

stand for the functions in $S(G/\Gamma \cap P)$ which are invariant to the right under $A \cdot N$. Then it is clear what one is to understand by

$$S_r^\infty(G/(\Gamma \cap P)\cdot A \cdot N).$$

On the other hand,

$$Fnc(K \times M/\{1\} \times \Gamma_M) \Leftrightarrow Fnc(G/(\Gamma \cap P)\cdot A \cdot N),$$

so there is no difficulty in defining

$$R(G/(\Gamma \cap P)\cdot A \cdot N).$$

It follows that, under the expected conditions,

$$Q_P^{\mathbf{H}}(S_r^\infty(G/(\Gamma \cap P)\cdot A \cdot N)) \subset R(G/(\Gamma \cap P)\cdot A \cdot N),$$

i.e. the analogue of Theorem 5.2 is in force for the partial truncation operator too.

Later on (see §10) we shall define a cofinal subset of $(\mathfrak{a}, <)$ having the property that for all $\mathbf{H}$ in this set everything that one wants to be true for the $Q_P^{\mathbf{H}}$ will be true *simultaneously* for all P.

Just as for $Q^{\mathbf{H}}f$, there are alternative ways to write $Q_P^{\mathbf{H}}f$. Thus

$$Q_P^{\mathbf{H}}f(x) = \sum_{P' \in \mathrm{Dom}_\Gamma(P)} (-1)^{\mathrm{rank}('P)} \times \chi_{'P,'A:\mathfrak{I}}(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x))\cdot f^{P'}(x)$$

or still

$$Q_P^{\mathbf{H}}f(x) = \sum_{'P \in \mathcal{C}_{\Gamma_M}} (-1)^{\mathrm{rank}('P)} \times \chi_{'P,'A:\mathfrak{I}}(I_{'P}(\mathbf{H}) - H_{'P|'A}(\dot{m}_x))\cdot (f^P)^{'P}.$$

PROPOSITION 8.2. *Fix P — then*

$$f^P(x) = \sum_{P' \in \mathrm{Dom}_\Gamma(P)} \chi_{'P,'A:\mathcal{C}}(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x))\cdot Q_{P'}^{\mathbf{H}}f(x).$$

Proof. It is a question of unraveling the right-hand side. Suppose that $P'' \preccurlyeq P'$ — then by $'P'$ we shall understand the entity obtained by daggering P'' into P'. This said, we have

$$\sum_{P' \in \mathrm{Dom}_\Gamma(P)} \chi_{'P,'A:\mathcal{C}}(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x))\cdot Q_{P'}^{\mathbf{H}}f(x)$$

$$= \sum_{P' \in \mathrm{Dom}_\Gamma(P)} \sum_{P'' \in \mathrm{Dom}_\Gamma(P')} (-1)^{\mathrm{rank}('P')}$$

$$\times \chi_{'P,'A:\mathcal{C}}(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x))\cdot \chi_{'P','A':\mathfrak{I}}(I_{'P'}(\mathbf{H}) - H_{'P'|'A'}(m_x'))$$

$$\times f^{P''}(x)$$

or still

$$\sum_{P''\in \mathrm{Dom}_\Gamma(P)} [\ldots]\cdot f^{P''}(x)$$

where $[\ldots]$ is equal to

$$\sum_{\{P'\in \mathcal{C}_\Gamma:\ P''\preccurlyeq P'\preccurlyeq P\}} (-1)^{\operatorname{rank}('P')}$$

$$\times \chi_{'P,'A:\mathcal{C}}\big(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x)\big)\cdot \chi_{'P','A':\mathfrak{H}}\big(I_{'P'}(\mathbf{H}) - H_{'P'|'A'}(m'_x)\big).$$

The latter sum can be viewed as being taken over the subsets of

$$\Sigma^0_{P''}(\mathfrak{m}, {''}\mathfrak{a}).$$

Since

$$''\mathfrak{a} = {'}\mathfrak{a}' \oplus {'}\mathfrak{a},$$

we have

$$\operatorname{rank}('P') = \operatorname{rank}(''P) - \operatorname{rank}('P).$$

Accordingly, our sum is a $\sigma^{\varnothing}_{\varnothing}$, hence is zero except when $P'' = P$, giving one in that case. In toto, therefore, the right-hand side of the initial expression yields precisely $f^P(x)$, as was to be shown $\square$.

It turns out that this proposition provides a characterization of the truncation operator. We shall, however, defer a precise discussion to §10.

9. An inner product formula. The purpose of this section is to obtain a formula for the (L^2) inner product of two truncated Eisenstein series associated with cusp forms. In the special case when Γ has just one cusp, a result of this type was advanced by Langlands [**2.a**]; he did not, however, give a proof. Here we shall treat the general case, supplying, of course, all the particulars of the argument. The significance of such a formula will become apparent only in subsequent papers in this series. Indeed, via the philosophy of Eisenstein systems, it will provide the springboard for an analogous formula vis-à-vis arbitrary Eisenstein series which, in turn, constitutes one of the main steps in the derivation of the Selberg trace formula.

We shall start off by establishing the notation and recalling certain basic facts which, for the most part, can be found in [**3.a**]. This done, our objective will then be indicated, albeit informally, it being best to defer a precise statement until later, taking up instead the steps needed for its realization in their natural order.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A which we take to be special. Given a K-type δ and an M-type $\mathcal{O}$, introduce, as usual, the finite dimensional Hilbert space $\mathcal{E}_{\text{cus}}(\delta, \mathcal{O})$. Attached to each $\Phi \in \mathcal{E}_{\text{cus}}(\delta, \mathcal{O})$ is the Eisenstein series

$$E(P \mid A : \Phi : \Lambda : x) = \sum_{\gamma \in \Gamma/\Gamma \cap P} a_{x\gamma}^{(\Lambda - \rho)} \cdot \Phi(x\gamma).$$

Put

$$\mathcal{T}_P(\check{\mathfrak{a}}) = -\left(\rho + \mathcal{C}_P(\check{\mathfrak{a}})\right).$$

Then it is known that the series defining $E(P \mid A : \Phi : \Lambda : x)$ is absolutely-uniformly convergent on compact subsets of the Cartesian product

$$\left(\mathcal{T}_P(\check{\mathfrak{a}}) + \sqrt{-1}\,\check{\mathfrak{a}}\right) \times G.$$

$E(P \mid A : \Phi : \Lambda : x)$ is a differentiable function of (Λ, x) and a holomorphic function of Λ. Moreover, $E(P \mid A : \Phi : \Lambda : x)$ can be meromorphically continued as a function of Λ from

$$\mathcal{T}_P(\check{\mathfrak{a}}) + \sqrt{-1}\,\check{\mathfrak{a}} \quad \text{to} \quad \check{\mathfrak{a}} + \sqrt{-1}\,\check{\mathfrak{a}}.$$

As such, the singularities lie along hyperplanes.

Let

$$\begin{cases} \langle \delta \rangle \text{ be a finite set of } K\text{-types} \\ \langle \mathcal{O} \rangle \text{ be a finite set of } M\text{-types.} \end{cases}$$

Set

$$\mathcal{E}_{\text{cus}}(\langle \delta \rangle, \langle \mathcal{O} \rangle) = \sum_{\langle \delta \rangle, \langle \mathcal{O} \rangle} \oplus\, \mathcal{E}_{\text{cus}}(\delta, \mathcal{O}).$$

If

$$\Phi = \sum_{\langle \delta \rangle, \langle \mathcal{O} \rangle} \Phi_{\delta, \mathcal{O}}$$

is in $\mathcal{E}_{\text{cus}}(\langle \delta \rangle, \langle \mathcal{O} \rangle)$, then we put

$$E(P \mid A : \Phi : \Lambda : x) = \sum_{\langle \delta \rangle, \langle \mathcal{O} \rangle} E(P \mid A : \Phi_{\delta, \mathcal{O}} : \Lambda : x).$$

Suppose that

$$(P, S; A) \succcurlyeq (P', S'; A').$$

Let δ be a K-type, $\mathcal{O}'$ an M'-type. The reduction of δ to K_M determines a finite set $\langle \delta_M \rangle$ of K_M-types. The tensor product $\delta \otimes \delta_M$ is a $K \times K_M$-type.

Maintaining the customary practice, introduce the finite dimensional Hilbert space $\mathcal{E}_{\text{cus}}(\langle \delta \otimes \delta_M \rangle, \delta \otimes \mathcal{O}')$. The assignment $\Phi' \mapsto {}'\Phi$,

$$'\Phi(k:m) = \Phi'(km),$$

defines an injection

$$\mathcal{E}_{\text{cus}}(\delta, \mathcal{O}') \to \mathcal{E}_{\text{cus}}(\langle \delta \otimes \delta_M \rangle, \delta \otimes \mathcal{O}')$$

with image the set of elements invariant under the action of K_M on $K \times M$ given by

$$k_M \cdot (k, m) = \left(kk_M^{-1}, k_M m\right).$$

The Eisenstein series

$$E\left(K \times {}'P \,|\, \{1\} \times {}'A : {}'\Phi : {}'\Lambda : (k, m)\right)$$

is a function on $K \times M$ invariant under the action of K_M, hence determines a function on G which we shall denote by

$$E({}'P \,|\, {}'A : {}'\Phi : {}'\Lambda : x).$$

The lemma of descent for Eisenstein series then says that

$$E(P' \,|\, A' : \Phi' : \Lambda' : x)$$

is equal to

$$\sum_{\gamma \in \Gamma / \Gamma \cap P} a_{x\gamma}^{(\Lambda - \rho)} \cdot E({}'P \,|\, {}'A : {}'\Phi : {}'\Lambda : x\gamma) \qquad (\Lambda' = {}'\Lambda + \Lambda).$$

Let $(P_1, S_1), (P_2, S_2)$ be Γ-cuspidal split parabolic subgroups of G with special split components A_1, A_2. Fix $\Phi_1 \in \mathcal{E}_{\text{cus}}(\delta, \mathcal{O}_1)$. Forming the Eisenstein series $E(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : x)$, let us consider

$$\begin{aligned} &E^{P_2}(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : x) \\ &\quad = \int_{N_2 / N_2 \cap \Gamma} E(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : x n_2) d_{N_2}(n_2). \end{aligned}$$

There are two possibilities.

Assume that (P_1, S_1) and (P_2, S_2) are not associate — then

$$E^{P_2}(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : x) \sim 0.$$

Assume that (P_1, S_1) and (P_2, P_2) are associate. Call $\mathcal{O}_2$ the M_2-type associated with $\mathcal{O}_1$ — then

$$\begin{aligned} &E^{P_2}(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : x) \\ &= \sum_{w_{21} \in W(A_2, A_1)} a_2(x)^{(w_{21}\Lambda_1 - \rho_2)} \cdot \left(c_{\text{cus}}(P_2 \,|\, A_2 : P_1 \,|\, A_1 : w_{21} : \Lambda_1)\Phi_1\right)(x), \end{aligned}$$

the c-function

$$c_{\text{cus}}(P_2 \mid A_2 : P_1 \mid A_1 : w_{21} : \Lambda_1)$$

being a linear transformation from $\mathcal{E}_{\text{cus}}(\delta, \mathcal{O}_1)$ to $\mathcal{E}_{\text{cus}}(\delta, \mathcal{O}_2)$. It is known that

$$c_{\text{cus}}(P_2 \mid A_2 : P_1 \mid A_1 : w_{21} : \Lambda_1)$$

is a holomorphic function in $\mathfrak{T}_{P_1}(\check{\mathfrak{a}}_1) + \sqrt{-1}\,\check{\mathfrak{a}}_1$ and, additionally, admits a meromorphic continuation to all of $\check{\mathfrak{a}}_1 + \sqrt{-1}\,\check{\mathfrak{a}}_1$. As such, the singularities lie along hyperplanes.

Suppose that

$$(P, S; A) \succcurlyeq \begin{cases} (P_1', S_1'; A_1') \\ (P_2', S_2'; A_2'). \end{cases}$$

It will be assumed that $('P_1, 'S_1)$ and $('P_2, 'S_2)$ are associate. Since there exists a canonical injection

$$W('A_2, 'A_1) \hookrightarrow W(A_2', A_1') \qquad ('w_{21} = w_{21}')$$

with image the elements in $W(A_2', A_1')$ which induce the identity on A, it follows that (P_1', S_1') and (P_2', S_2') are also associate. Let $'w_{21} \in W('A_2, 'A_1)$ — then the c-function

$$c_{\text{cus}}('P_2 \mid 'A_2 : 'P_1 \mid 'A_1 : 'w_{21} : '\Lambda_1)$$

is a linear transformation from $\mathcal{E}_{\text{cus}}(\langle \delta \otimes \delta_M \rangle, \delta \otimes \mathcal{O}_1')$ to $\mathcal{E}_{\text{cus}}(\langle \delta \otimes \delta_M \rangle, \delta \otimes \mathcal{O}_2')$. The lemma of descent for c-functions then says that the diagram

$$\begin{array}{ccc} \mathcal{E}_{\text{cus}}(\delta, \mathcal{O}_1') & \xrightarrow{c_{\text{cus}}(P_2'|A_2' : P_1'|A_1' : w_{21}' : \Lambda_1')} & \mathcal{E}_{\text{cus}}(\delta, \mathcal{O}_2') \\ \swarrow & & \searrow \\ \mathcal{E}_{\text{cus}}(\langle \delta \otimes \delta_M \rangle, \delta \otimes \mathcal{O}_1') & \xrightarrow[c_{\text{cus}}('P_2|'A_2 : 'P_1|'A_1 : 'w_{21} : '\Lambda_1)]{} & \mathcal{E}_{\text{cus}}(\langle \delta \otimes \delta_M \rangle, \delta \otimes \mathcal{O}_2') \end{array}$$

is commutative.

Our objective will be to obtain a formula for

$$\left(Q^{\mathbf{H}} E(P_1 \mid A_1 : \Phi_1 : \Lambda_1 : ?), Q^{\mathbf{H}} E(P_2 \mid A_2 : \Phi_2 : \Lambda_2 : ?)\right).$$

For this purpose, it will be convenient to change our notation a little and study

$$Q^{\mathbf{H}} E(P^* \mid A^* : \Phi^* : \Lambda^* : ?).$$

Bearing in mind that

$$Q^{\mathbf{H}} = \sum_{P \in \mathcal{C}_\Gamma} (-1)^{\operatorname{rank}(P)} \ldots,$$

the first step is to investigate

$$E^P(P^* \mid A^* : \Phi^* : \Lambda^* : ?).$$

Observe that now, just as in §6, the triple $(P^*, S^*; A^*)$ represents the fixed data whereas the triple $(P, S; A)$ represents the variable data.

The relation of association breaks up the Γ-cuspidal split parabolic subgroups of G into equivalence classes. Let $\mathcal{C}^*$ be the class containing P^* — then we distinguish two cases:

$$\begin{cases} \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P) = \varnothing \\ \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P) \neq \varnothing. \end{cases}$$

Suppose that the first eventuality is in force — then it is a well-known simple fact that

$$E^P(P^* \mid A^* : \Phi^* : \Lambda^* : ?) = 0.$$

Accordingly, it is the second eventuality which is of primary interest. To treat it, some preparation will be needed.

Let P_i' $(1 \le i \le r_P)$ be a set of representatives for

$$P \backslash \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P).$$

Then

$$\mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P) = \coprod_i \mathcal{C}_i^*(P)$$

where

$$\mathcal{C}_i^*(P) = P \cdot \{P_i'\} \cap \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P).$$

Let $P_{i\mu}'$ $(1 \le \mu \le {}'r_i)$ be a set of representatives for $\Gamma \cap P \backslash \mathcal{C}_i^*(P)$ — then

$$\{P_{i\mu}' \colon 1 \le i \le r_P,\ 1 \le \mu \le {}'r_i\}$$

is a set of representatives for

$$\Gamma \cap P \backslash \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P).$$

Finally, extend the $P_{i\mu}'$ $(1 \le \mu \le {}'r_i)$ to a set of representatives $P_{i\mu}'$ $(1 \le \mu \le r_i')$ for

$$\Gamma \backslash G \cdot \{P_i'\} \cap \mathcal{C}^*.$$

The functional equations for the c-functions admit a ready description in terms of these choices. Given $w_i^* \in W(A_i', A^*)$, put

$$w_{i\mu}^* = I(P_{i\mu}' \mid A_{i\mu}' : P_i' \mid A_i') w_i^* \in W(A_{i\mu}', A^*).$$

Let

$$P' \in \mathcal{C}^* \cap \operatorname{Dom}_\Gamma(P),$$

A' the special split component of (P', S'). Given $w_i' \in W(A', A_i')$, put

$$w_{i\mu}' = w_i' I(P_i' \mid A_i' : P_{i\mu}' \mid A_{i\mu}') \in W(A', A_{i\mu}').$$

There is then an equality of meromorphic functions

$$\begin{aligned} c_{\text{cus}}(P' \mid A' &: P^* \mid A^* : w_i' w_i^* : \Lambda^*) \\ &= \sum_{\mu=1}^{r_i} c_{\text{cus}}(P' \mid A' : P_{i\mu}' \mid A_{i\mu}' : w_{i\mu}' : w_{i\mu}^* \Lambda^*) \\ &\quad \circ c_{\text{cus}}(P_{i\mu}' \mid A_{i\mu}' : P^* \mid A^* : w_{i\mu}^* : \Lambda^*). \end{aligned}$$

Actually, in what follows, we shall be primarily interested in the case when w_i' is in the image of the canonical injection

$$W('A, 'A_i) \hookrightarrow W(A', A_i')$$

so $w_i' = 'w_i$, say. Suppose that $1 \leq \mu \leq 'r_i$ — then $w_{i\mu}'$ is in the image of the canonical injection

$$W('A, 'A_{i\mu}) \hookrightarrow W(A', A_{i\mu}')$$

so $w_{i\mu}' = 'w_{i\mu}$, say. Set, for simplicity,

$$\Lambda_{i\mu}'(w^*) = w_{i\mu}^* \Lambda^* \qquad (\Lambda_{i\mu}'(w^*) = '\Lambda_{i\mu}(w^*) + \Lambda_{i\mu}(w^*)).$$

Then, in view of the descent property supra,

$$\begin{aligned} c_{\text{cus}}(P' \mid A' &: P_{i\mu}' \mid A_{i\mu}' : w_{i\mu}' : \Lambda_{i\mu}'(w^*))?' \\ &= c_{\text{cus}}('P \mid 'A : 'P_{i\mu} \mid 'A_{i\mu} : 'w_{i\mu} : '\Lambda_{i\mu}(w^*))'?. \end{aligned}$$

As for the other possibility, namely that $'r_i < \mu \leq r_i'$, there is an $x \in G$ such that

$$xP_i'x^{-1} = P_{i\mu}'$$

but this time $P_{i\mu}'$ is not a dominated predecessor of P. We claim that P and xPx^{-1} are not Γ-conjugate. Assume the contrary, e.g.

$$P = \gamma(xPx^{-1})\gamma^{-1}.$$

Then it would follow that $\gamma P_{i\mu}' \gamma^{-1}$ is a dominated predecessor of P. However, in view of the fact that G and P conjugacy are one and the same on $\operatorname{Dom}_\Gamma(P)$, there must be an index $\nu \leq 'r_i$ with the property that $P_{i\nu}'$ is

$\Gamma \cap P$-conjugate to $\gamma P'_{i\mu}\gamma^{-1}$, an impossibility. Hence the claim. Owing to our hypothesis on w'_i,

$$w'_i \circ I(P \mid A : xPx^{-1} \mid xAx^{-1}) = I(P \mid A : xPx^{-1} \mid xAx^{-1}).$$

Consequently, for standard reasons,

$$c_{\text{cus}}(P' \mid A' : P'_{i\mu} \mid A'_{i\mu} : w'_{i\mu} : w^*_{i\mu}\Lambda^*) = 0.$$

In summary, therefore, when

$$w'_i \in \operatorname{Im}(W('A, 'A_i)),$$

the terms in the functional equation indexed by $\mu = 1, \ldots, 'r_i$ can be interpreted in terms of the daggered picture whereas those indexed by $\mu = 'r_i + 1, \ldots, r'_i$ drop out altogether.

Apart from the facts just mentioned, the investigation of

$$E^P(P^* \mid A^* : \Phi^* : \Lambda^* : ?)$$

also depends on a lemma of decomposition for $W(A', A^*)$, itself a variant on a well-known theme.

In terms of the domination

$$(P, S; A) \geqslant (P'_i, S'_i; A'_i),$$

represent P per P'_i, i.e. write

$$P = (P'_i)_{F'_i}.$$

Put

$$W^P(A'_i, A^*) = \{w^*_i \in W(A'_i, A^*) : w_i^{-*}\lambda'_i > 0 \ \forall \ \lambda'_i \in F'_i\}.$$

LEMMA 9.1. *There is a disjoint decomposition*

$$W(A', A^*) = \coprod_{i=1}^{r_P} \ \coprod_{'w_i \in W('A, 'A_i)} w'_i \cdot W^P(A'_i, A^*),$$

that is

$$\forall\, w^* \in W(A', A^*), \quad \exists$$

$$\begin{cases} \text{a unique index } i \\ \text{a unique } 'w_i \in W('A, 'A_i) \\ \text{a unique } w^*_i \in W^P(A'_i, A^*) \end{cases}$$

such that

$$w^* = w'_i w^*_i.$$

Proof. Uniqueness. Deny the contention — then $w_i'w_i^* = w_j'w_j^*$ where either $i \neq j$ or $i = j$, $w_i' \neq w_j'$. Let $w_{ij}' = w_i^{-\prime}w_j'$, so that $w_i^* = w_{ij}'w_j^*$. Since $w_{ij}' \in \operatorname{Im}(W('A_i, 'A_j))$, there is a $\lambda_{ij}' \in F_j'$ with $w_{ij}'\lambda_{ij}' < 0$ (cf. [**3.a**]), thus

$$w_{ij}'\lambda_{ij}' = -\sum n_k \lambda_k',$$

the n_k being non-negative for all values of k and strictly positive for at least one value of k. Noting that $w_i^{-*}w_{ij}' = w_j^{-*}$, we have now, on the one hand,

$$w_j^* \in W^P(A_j', A^*) \Rightarrow w_j^{-*}\lambda_{ij}' > 0$$

while, on the other,

$$w_i^* \in W^P(A_i', A^*) \Rightarrow w_i^{-*}w_{ij}'\lambda_{ij}' = -\sum n_k w_i^{-*}\lambda_k' < 0,$$

a contradiction.

Existence. Take a $w^* \in W(A', A^*)$. Consider

$$\{\lambda \in \Sigma_{'P}(\mathfrak{m}, '\mathfrak{a}) : w^{-*}\lambda > 0\}.$$

This set evidently determines a chamber in $'\mathfrak{a}$, call it $\mathcal{C}(w^*)$. On general grounds, there exists an index i and a $'w_i \in W('A, 'A_i)$ such that $\mathcal{C}(w^*) = 'w_i \cdot \mathcal{C}_{'P_i}('\mathfrak{a}_i)$ (cf. [**3.a**]). Let $w_i^* = w_i^{-\prime}w^*$ — then $w^* = w_i'w_i^*$, so we have only to show that $w_i^* \in W^P(A_i', A^*)$, which, however, is immediate, $w_i'\lambda_i'$ being positive on $\mathcal{C}(w^*)$ for all $\lambda_i' \in F_i'$.

Hence the lemma. □

Here is the result governing

$$E^P(P^* \mid A^* : \Phi^* : \Lambda^* : ?).$$

PROPOSITION 9.2. *Retain the preceding assumptions and notations — then*

$$E^P(P^* \mid A^* : \Phi^* : \Lambda^* : x)$$

is equal to

$$\sum_{i=1}^{r_P} \sum_{\mu=1}^{'r_i} \sum_{w_{i\mu}^* \in W^P(A_{i\mu}', A^*)} a_x^{(\Lambda_{i\mu}(w^*) - \rho)}$$

$$\times E\big('P_{i\mu} \mid 'A_{i\mu} : c_{\text{cus}}\big(P_{i\mu}' \mid A_{i\mu}' : P^* \mid A^* : w_{i\mu}^* : \Lambda^*\big)\Phi^* : '\Lambda_{i\mu}(w^*) : x\big)$$

[Note: We are allowing ourselves a slight solecism....]

The sense of equality is, needless to say, that of meromorphic functions. A literal version will be given below.

To make the verification, observe that either side of the claimed equality is slowly increasing. Accordingly, we shall employ the familiar principle of negligibility. Let $[\ldots]$ stand for the difference between the two — then, viewed as a function on $K \times M \times A$, as is permissible,

$$[\ldots]^{\prime P} \sim 0$$

unless $P' \in \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P)$. This said, fix a $P' \in \mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P)$ — then it need only be shown that

$$[\ldots]^{\prime P} = 0$$

to draw the desired conclusion, viz.

$$[\ldots] = 0.$$

But we have

$$\begin{aligned} &\left[E^P(P^* \mid A^* : \Phi^* : \Lambda^* : ?)\right]^{\prime P}(k, m', a') \\ &\quad = E^{P'}(P^* \mid A^* : \Phi^* : \Lambda^* : km'a') \\ &\quad = \sum_{w^* \in W(A', A^*)} (a')^{(\Lambda'(w^*) - \rho')} \\ &\qquad \cdot \left(c_{\mathrm{cus}}(P' \mid A' : P^* \mid A^* : w^* : \Lambda^*)\Phi^*\right)(k, m') \left(\Lambda'(w^*) = w^*\Lambda\right). \end{aligned}$$

Decompose a' per $A' = {}'A \cdot A$ to get $a' = {}'aa$. Using Lemma 9.1, write $w^* = w_i' w_i^*$. Since

$$w_i' \in \mathrm{Im}(W('A, 'A_i)),$$

the functional equations supra for

$$c_{\mathrm{cus}}(P' \mid A' : P^* \mid A^* : w_i' w_i^* : \Lambda^*)$$

then lead us to

$$\begin{aligned} &\left[E^P(P^* \mid A^* : \Phi^* : \Lambda^* : ?)\right]^{\prime P}(k, m', a') \\ &= \sum_{i=1}^{r_P} \sum_{\mu=1}^{'r_i} \sum_{w_{i\mu}^* \in W^P(A'_{i\mu}, A^*)} \\ &\quad \times \Bigg\{ \sum_{'w_{i\mu} \in W('A, 'A_{i\mu})} ('a)^{('\Lambda_{i\mu}(w^*) - '\rho)} \cdot c_{\mathrm{cus}}('P \mid 'A : 'P_{i\mu} \mid 'A_{i\mu} : 'w_{i\mu} : '\Lambda_{i\mu}(w^*)) \\ &\qquad \times \left[a^{(\Lambda_{i\mu}(w^*) - \rho)} \cdot \left(c_{\mathrm{cus}}(P'_{i\mu} \mid A'_{i\mu} : P^* \mid A^* : w_{i\mu}^* : \Lambda^*)\Phi^*\right)(k, m')\right] \Bigg\} \end{aligned}$$

$$= \sum_{i=1}^{r_P} \sum_{\mu=1}^{'r_i} \sum_{w_{i\mu}^* \in W^P(A'_{i\mu}, A^*)} a^{(\Lambda_{i\mu}(w^*)-\rho)}$$

$$\times E^{'P}\big('P_{i\mu} \mid 'A_{i\mu} : c_{\mathrm{cus}}\big(P'_{i\mu} \mid A'_{i\mu} : P^* \mid A^* : w_{i\mu}^* : \Lambda^*\big)\Phi^* : '\Lambda_{i\mu}(w^*) : km''a\big)$$

which is exactly what we wanted to prove.

An Eisenstein series, qua a meromorphic function, is the continuation of another expression, itself a series with a well-defined region of convergence. It is therefore natural to try to find conditions on Λ^* which will serve to ensure that both sides of the equation appearing in the proposition supra fall within the appropriate domains.

It is plain that the projection of

$$w_{i\mu}^* \mathcal{C}_{P*}(\mathfrak{a}^*) \qquad \big(w_{i\mu}^* \in W^P(A'_{i\mu}, A^*)\big)$$

onto $'\mathfrak{a}_{i\mu}$ is contained in

$$\mathcal{C}_{'P_{i\mu}}('\mathfrak{a}_{i\mu})$$

and dually. That being so, put

$$'t_{i\mu} = \max_{'\lambda_{i\mu} \in \Sigma^0_{'P_{i\mu}}(\mathfrak{m}, '\mathfrak{a}_{i\mu})} \big(\langle '\lambda_{i\mu}, '\rho_{i\mu}\rangle / \langle '\lambda_{i\mu}, '\rho_{i\mu}(w^*)\rangle\big),$$

a positive real number. Suppose now that $\mathrm{Re}(\Lambda^*) + 't_{i\mu}\rho^*$ is in the negative* chamber — then a short calculation, which need not be reproduced, allows us to infer that $\mathrm{Re}('\Lambda_{i\mu}(w^*)) + '\rho_{i\mu}$ is in the 'negative chamber. Matters can certainly be arranged in such a way as to guarantee that these conditions are uniform with respect to the data. In other words, if Λ^* is sufficiently negative, then

$$E^P(P^* \mid A^* : \Phi^* : \Lambda^* : x)$$

$$= \sum_{\gamma \in \Gamma/\Gamma \cap P} \int_{N/N \cap \Gamma} \big(a_{xn\gamma}^*\big)^{(\Lambda^* - \rho^*)} \cdot \Phi^*(xn\gamma)\, d_N(n)$$

while

$$E\big('P_{i\mu} \mid 'A_{i\mu} : c_{\mathrm{cus}}\big(P'_{i\mu} \mid A'_{i\mu} : P^* \mid A^* : w_{i\mu}^* : \Lambda^*\big)\Phi^* : '\Lambda_{i\mu}(w^*) : x\big)$$

$$= \sum_{\gamma_{i\mu} \in \Gamma \cap P/\Gamma \cap P'_{i\mu}} \big('a_{x\gamma_{i\mu}}\big)^{('\Lambda_{i\mu}(w^*) - '\rho)}$$

$$\cdot \big(c_{\mathrm{cus}}\big(P'_{i\mu} \mid A'_{i\mu} : P^* \mid A^* : w_{i\mu}^* : \Lambda^*\big)\Phi^*\big)(x\gamma_{i\mu}).$$

With this understanding, it therefore follows that

$$T_P(\mathbf{H} : E(P^* \mid A^* : \Phi^* : \Lambda^* : ?))(x)$$

or still

$$\sum_{\gamma \in \Gamma/\Gamma \cap P} \chi_{P,A:\mathfrak{I}}\big(I_P(\mathbf{H}) - H_{P|A}(x\gamma)\big)\cdot E^P\big(P^* \,|\, A^* : \Phi : \Lambda^* : x\gamma\big)$$

is equal to

$$\sum_{i=1}^{r_P} \sum_{\mu=1}^{'r_i} \sum_{w_{i\mu}^* \in W^P(A'_{i\mu},\, A^*)} \sum_{\gamma_{i\mu} \in \Gamma/\Gamma \cap P'_{i\mu}} \big(a'_{x\gamma_{i\mu}}\big)^{(\Lambda'_{i\mu}(w^*) - \rho'_{i\mu})}$$

$$\times \chi^{F'_{i\mu},*}\Big(I_{P'_{i\mu}}(\mathbf{H}) - H_{P'_{i\mu}|A'_{i\mu}}(x\gamma_{i\mu})\Big)$$

$$\cdot\,\Big(c_{\text{cus}}\big(P'_{i\mu} \,|\, A'_{i\mu} : P^* \,|\, A^* : w_{i\mu}^* : \Lambda^*\big)\Phi^*\Big)(x\gamma_{i\mu})$$

where, of course,

$$P = \big(P'_{i\mu}\big)_{F'_{i\mu}}$$

a-la

$$(P, S;\, A) \succcurlyeq \big(P'_{i\mu},\, S'_{i\mu};\, A'_{i\mu}\big).$$

It is easy to check that our formula is substantially independent of the choice of the representatives $P'_{i\mu}$.

All that is needed now for the calculation of

$$Q^{\mathbf{H}}E(P^* \,|\, A^* : \Phi^* : \Lambda^* : ?)$$

is a little more notation, coupled with some simple combinatorial remarks. Let us agree to write $\mathrm{Dom}_\Gamma(\mathcal{C}^*)$ for the set of all Γ-cuspidals P with the property that

$$\mathcal{C}^* \cap \mathrm{Dom}_\Gamma(P) \neq \varnothing.$$

Then $\mathrm{Dom}_\Gamma(\mathcal{C}^*)$ is a union of Γ-conjugacy classes, say

$$\mathrm{Dom}_\Gamma(\mathcal{C}^*) = \coprod_{c=1}^{C^*} \mathcal{P}_c^*.$$

Fix an element P_c^* in $\mathcal{P}_c^*$. Since

$$\begin{aligned} Q^{\mathbf{H}} &= \sum_{P \in \mathcal{C}_\Gamma} (-1)^{\operatorname{rank}(P)} \ldots \\ &= \sum_{P \in \mathcal{C}_\Gamma - \mathrm{Dom}_\Gamma(\mathcal{C}^*)} (-1)^{\operatorname{rank}(P)} \ldots \\ &\quad + \sum_{P \in \mathrm{Dom}_\Gamma(\mathcal{C}^*)} (-1)^{\operatorname{rank}(P)} \ldots, \end{aligned}$$

the fact that

$$P \in \mathcal{C}_\Gamma - \mathrm{Dom}_\Gamma(\mathcal{C}^*)$$
$$\Rightarrow$$
$$E^P(P^* \,|\, A^* : \Phi^* : \Lambda^* : ?) = 0,$$

in conjunction with the preceding developments, serves to imply that

$$Q^{\mathbf{H}} E(P^* \,|\, A^* : \Phi^* : \Lambda^* : ?)$$

is equal to

$$\sum_{c=1}^{C^*} (-1)^{\mathrm{rank}(P_c^*)} \cdot T_{P_c^*}(\mathbf{H} : E(P^* \,|\, A^* : \Phi^* : \Lambda^* : ?)),$$

there being an explicit formula for

$$T_{P_c^*}(\mathbf{H} : E(P^* \,|\, A^* : \Phi^* : \Lambda^* : ?))$$

which is best dealt with by a little bookeeping. Let P_i^* $(1 \le i \le r^*)$ be a set of representatives for $G \backslash \mathcal{C}^*$ — then

$$\mathcal{C}^* = \coprod_i \mathcal{C}_i^*$$

where $\mathcal{C}_i^* = G \cdot \{P_i^*\} \cap \mathcal{C}^*$. Let $P_{i\mu}^*$ $(1 \le \mu \le r_i^*)$ be a set of representatives for $\Gamma \backslash \mathcal{C}_i^*$ — then

$$\{P_{i\mu}^* : 1 \le i \le r^*,\ 1 \le \mu \le r_i^*\}$$

is a set of representatives for $\Gamma \backslash \mathcal{C}^*$. Given $\mathcal{P}_c^*$, write $\mathcal{B}_c^*$ for the set of all pairs (i, μ) for which there exists an element of $\mathcal{P}_c^*$ dominating $P_{i\mu}^*$. With each pair $(i, \mu) \in \mathcal{B}_c^*$ there is associated a unique subset

$$F_{i\mu}^*(c) \subset \Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*)$$

such that

$$(P_{i\mu}^*)_{F_{i\mu}^*(c)} \in \mathcal{P}_c^*.$$

Call this latter parabolic $P_c^*(i, \mu)$. We then have that

$$T_{P_c^*}(\mathbf{H} : E(P^* \,|\, A^* : \Phi^* : \Lambda^* : ?))(x)$$

is equal to

$$\sum_{(i,\mu) \in \mathcal{B}_c^*} \ \sum_{w_{i\mu}^* \in W^{P_c^*(i,\mu)}(A_{i\mu}^*, A^*)} \ \sum_{\gamma_{i\mu} \in \Gamma / \Gamma \cap P_{i\mu}^*} (a_{x\gamma_{i\mu}}^*)^{(w_{i\mu}^* \Lambda^* - \rho_{i\mu}^*)}$$
$$\times \chi^{F_{i\mu}^*(c),*}\big(I_{P_{i\mu}^*}(\mathbf{H}) - H_{P_{i\mu}^* | A_{i\mu}^*}(x\gamma_{i\mu})\big)$$
$$\cdot \big(c_{\mathrm{cus}}(P_{i\mu}^* \,|\, A_{i\mu}^* : P^* \,|\, A^* : w_{i\mu}^* : \Lambda^*)\Phi^*\big)(x\gamma_{i\mu}).$$

Accordingly,

$$Q^{\mathrm{H}}E(P^* \mid A^* : \Phi^* : \Lambda^* : ?)(x)$$

can be written as the sum over c of these C^*-expressions provided we insert the factor

$$(-1)^{\operatorname{rank}(P_c^*(i,\mu))}.$$

This seemingly intractable conclusion admits a straightforward reduction to wit: The symbolic sums

$$(1) \qquad \sum_{c=1}^{C^*} \sum_{(i,\mu)\in\mathfrak{B}_c^*} \sum_{w_{i\mu}^* \in W^{P_c^*(i,\mu)}(A_{i\mu}^*, A^*)} f\big(i, \mu, F_{i\mu}^*(c), w_{i\mu}^*\big)$$

$$(2) \qquad \sum_{i=1}^{r^*} \sum_{\mu=1}^{r_i^*} \sum_{F_{i\mu}^* \subset \Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*)} \sum_{w_{i\mu}^* \in W^{(P_{i\mu}^*)_{F_{i\mu}^*}}(A_{i\mu}^*, A^*)} f\big(i, \mu, F_{i\mu}^*, w_{i\mu}^*\big),$$

$$(3) \qquad \sum_{i=1}^{r^*} \sum_{\mu=1}^{r_i^*} \sum_{w_{i\mu}^* \in W(A_{i\mu}^*, A^*)} \times \sum_{\{F_{i\mu}^* \subset \Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*) :\, \forall \lambda_{i\mu}^* \in F_{i\mu}^*,\, w_{i\mu}^{-*}\lambda_{i\mu}^* > 0\}} f\big(i, \mu, F_{i\mu}^*, w_{i\mu}^*\big)$$

are equal in the sense that if f is a function of four arguments with values in a vector space V over $\mathbf{C}$ such that for a given pair (i, μ)

$$f(i, \mu, ?, ?) : \mathcal{P}\big(\Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*)\big) \times W(A_{i\mu}^*, A^*) \to \mathbf{C},$$

then all the sums give the same value when applied to f. Indeed, this is clearly the case of the second and third so we need only deal explicitly with the first and the second. Let

$$\mathcal{I}^* = \Big\{\big(i, \mu, F_{i\mu}^*\big) : 1 \le i \le r^*,\ 1 \le \mu \le r_i^*,\ F_{i\mu}^* \in \mathcal{P}\big(\Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*)\big)\Big\}.$$

Then it is simply a question of establishing the truth of the following lemma.

LEMMA. *The map*

$$\begin{cases} \coprod_{c=1}^{C^*} \mathfrak{B}_c^* \to \mathcal{I}^* \\ (i, \mu) \in \mathfrak{B}_c^* \mapsto \big(i, \mu, F_{i\mu}^*(c)\big) \end{cases}$$

is a bijection.

Proof. Surjectivity. Let $(i, \mu, F_{i\mu}^*) \in \mathcal{I}^*$ be given — then

$$(P_{i\mu}^*)_{F_{i\mu}^*} \in \mathrm{Dom}_\Gamma(\mathcal{C}^*)$$

so there exists c, $1 \leq c \leq C^*$, such that

$$(P_{i\mu}^*)_{F_{i\mu}^*} \in \mathcal{P}_c^*.$$

By definition, therefore, $(i, \mu) \in \mathcal{B}_c^*$, implying that

$$F_{i\mu}^* = F_{i\mu}^*(c)$$

which is surjectivity.

Injectivity. If

$$\begin{cases} (i', \mu') \in \mathcal{B}_{c'}^* \\ (i'', \mu'') \in \mathcal{B}_{c''}^* \end{cases}$$

go to the same place, then $i' = i''$, $\mu' = \mu''$ because they appear in the image. Thus we must show that if

$$(i, \mu) \in \mathcal{B}_{c'}^* \cap \mathcal{B}_{c''}^* \quad \text{and} \quad F_{i\mu}^*(c') = F_{i\mu}^*(c''),$$

then $c' = c''$. But

$$P_{c'}^*(i, \mu) = P_{c''}^*(i, \mu) \in \mathcal{P}_{c'}^* \cap \mathcal{P}_{c''}^* = \varnothing$$

unless $c' = c''$ which is injectivity.

Hence the lemma. □

Coming back to

$$Q^{\mathbf{H}} E(P^* \mid A^* : \Phi^* : \Lambda^* : ?)(x),$$

initially given by a sum of the first kind, pass, via the second, to a sum of the third kind. Before we formulate a statement of recapitulation, there is a simplification to be made as regards the

$$\chi^{F_{i\mu}^*, *}\text{-term}$$

multiplied by -1 raised to the

$$\mathrm{rank}\big((P_{i\mu}^*)_{F_{i\mu}^*}\big) = \#\big(\Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*)\big) - \#(F_{i\mu}^*)$$

power. Given $w_{i\mu}^* \in W(A_{i\mu}^*, A^*)$, let

$$F(w_{i\mu}^*) = \big\{\lambda_{i\mu}^* \in \Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*) : w_{i\mu}^{-*}\lambda_{i\mu}^* < 0\big\}.$$

This done, place ourselves in the setting of Lemma 2.8, the role of F_0 there being played here by $F(w_{i\mu}^*)$. If, for brevity,

$$\tau_{*,*} \equiv \tau_{*,\mathfrak{L}},$$

then

$$\tau_{*,*}\big(F(w_{i\mu}^*):?\big)$$

is the characteristic function of the set of all $H_{i\mu}^* \in \mathfrak{a}_{i\mu}^*$ such that

$$\begin{cases} \lambda_{i\mu}^* \in F(w_{i\mu}^*) \Rightarrow \langle H_{i\mu}^*, \lambda_*^{i\mu}\rangle > 0 \\ \lambda_{i\mu}^* \notin F(w_{i\mu}^*) \Rightarrow \langle H_{i\mu}^*, \lambda_*^{i\mu}\rangle \leq 0. \end{cases}$$

Thanks to the lemma, then, the sum over the

$$F_{i\mu}^* \subset \Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*) \text{ such that } \forall\, \lambda_{i\mu}^* \in F_{i\mu}^*,\ w_{i\mu}^{-*}\lambda_{i\mu}^* > 0$$

of -1 raised to the

$$\operatorname{rank}\big((P_{i\mu}^*)_{F_{i\mu}^*}\big) = \#\big(\Sigma_{P_{i\mu}^*}^0(\mathfrak{g}, \mathfrak{a}_{i\mu}^*)\big) - \#(F_{i\mu}^*)$$

power times

$$\chi^{F_{i\mu}^*,*}\big(I_{P_{i\mu}^*}(\mathbf{H}) - H_{P_{i\mu}^*|A_{i\mu}^*}(x\gamma_{i\mu})\big)$$

equals -1 raised to the $\#(F(w_{i\mu}^*))$ power times

$$\tau_{*,*}\big(F(w_{i\mu}^*): I_{P_{i\mu}^*}(\mathbf{H}) - H_{P_{i\mu}^*|A_{i\mu}^*}(x\gamma_{i\mu})\big).$$

To summarize:

PROPOSITION 9.3. *Retain the preceding assumptions and notations — then*

$$Q^{\mathbf{H}}E(P^* \mid A^* : \Phi^* : \Lambda^* : ?)(x)$$

is equal to

$$\sum_{i=1}^{r^*} \sum_{\mu=1}^{r_i^*} \sum_{w_{i\mu}^* \in W(A_{i\mu}^*, A^*)} \sum_{\gamma_{i\mu} \in \Gamma/\Gamma \cap P_{i\mu}^*} \big(a_{x\gamma_{i\mu}}^*\big)^{(w_{i\mu}^*\Lambda^* - \rho_{i\mu}^*)}$$
$$\times (-1)^{\#(F(w_{i\mu}^*))}\tau_{*,*}\big(F(w_{i\mu}^*): I_{P_{i\mu}^*}(\mathbf{H}) - H_{P_{i\mu}^*|A_{i\mu}^*}(x\gamma_{i\mu})\big)$$
$$\times \big(c_{\text{cus}}(P_{i\mu}^* \mid A_{i\mu}^* : P^* \mid A^* : w_{i\mu}^* : \Lambda^*)\Phi^*\big)(x\gamma_{i\mu}).$$

Harmonic analysis now enters the picture. A brief review of the salient facts is therefore in order.

Let, as at the beginning, (P, S) be a Γ-cuspidal split parabolic subgroup of G with split component A which we take to be special. Put

$$l = \operatorname{rank}(P, S).$$

Let δ be a K-type, $\mathcal{O}$ an M-type. Denoting by $\mathcal{H}_A$ the space of Fourier-Laplace transforms of functions in $C_c^\infty(\sqrt{-1}\,\check{\mathfrak{a}})$, set

$$\mathcal{H}_A(\delta, \mathcal{O}) = \mathcal{H}_A \otimes \mathcal{E}_{\text{cus}}(\delta, \mathcal{O}).$$

If $\Phi \in \mathcal{H}_A(\delta, \mathcal{O})$, then Φ may be viewed as a differentiable function of (Λ, x) which, as a function of Λ, is entire and rapidly decreasing in vertical strips, and which, as a function of x, is a member of $\mathcal{E}_{\text{cus}}(\delta, \mathcal{O})$. Attached to Φ is the wave-packet

$$\Theta_\Phi(x) = \frac{1}{(2\pi)^l} \int_{\operatorname{Re}(\Lambda) = \Lambda_0} E(P \,|\, A : \Phi(\Lambda) : \Lambda : x) \,|\, d\Lambda \,|\, ,$$

Λ_0 a point in $\mathcal{T}_P(\check{\mathfrak{a}})$. The integral defining Θ_Φ is absolutely convergent and independent of the choice of Λ_0. Moreover, Θ_Φ is a rapidly decreasing differentiable function on G/Γ, thus lies in $L^2(G/\Gamma)$. Let R be a real number $> \|\rho\|$. By the R-tube, we mean the tube over the ball of radius R with center zero in $\check{\mathfrak{a}}$. Denoting by $\mathcal{H}_A(R)$ the space of all holomorphic functions in the R-tube which decay at infinity faster than the inverse of any polynomial, set

$$\mathcal{H}_A(\delta, \mathcal{O}; R) = \mathcal{H}_A(R) \otimes \mathcal{E}_{\text{cus}}(\delta, \mathcal{O}).$$

There is a strict inclusion

$$\mathcal{H}_A(\delta, \mathcal{O}) \hookrightarrow \bigcup_R \mathcal{H}_A(\delta, \mathcal{O}; R).$$

Let Φ belong to $\mathcal{H}_A(\delta, \mathcal{O}; R)$ — then, utilizing a limit process, one can show that it is possible to associate with Φ an element Θ_Φ in $L^2(G/\Gamma)$, which is, in fact, the L^2-limit of wave-packets formed from functions in $\mathcal{H}_A(\delta, \mathcal{O})$.

Let $(P_1, S_1), (P_2, S_2)$ be Γ-cuspidal split parabolic subgroups of G with special split components A_1, A_2. Let

$$\begin{cases} \delta_1, \delta_2 \text{ be } K\text{-types} \\ \mathcal{O}_1, \mathcal{O}_2 \text{ be } M_1, M_2\text{-types.} \end{cases}$$

Let

$$\begin{cases} \Phi_1 \in \mathcal{H}_{A_1}(\delta_1, \mathcal{O}_1) \\ \Phi_2 \in \mathcal{H}_{A_2}(\delta_2, \mathcal{O}_2). \end{cases}$$

It is then always true that

$$\int_{G/\Gamma} \Theta_{\Phi_1}(x)\overline{\Theta_{\Phi_2}(x)}d_G(x) = 0$$

unless (P_1, S_1) and (P_2, S_2) are associate (with common rank l, say), $\delta_1 = \delta_2$, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are associate, in which case

$$\int_{G/\Gamma} \Theta_{\Phi_1}(x)\overline{\Theta_{\Phi_2}(x)}d_G(x)$$

is equal to

$$\frac{1}{(2\pi)^l}\int_{\mathrm{Re}(\Lambda_1)=\Lambda_1^0} \big(c_{\mathrm{cus}}(P_2\,|\,A_2 : P_1\,|\,A_1 : w_{21} : \Lambda_1)\Phi_1(\Lambda_1), \Phi_2(-w_{21}\overline{\Lambda}_1)\big)\,|\,d\Lambda_1\,|$$

summed over the w_{21} in $W(A_2, A_1)$, Λ_1^0 being any point in $\mathcal{T}_{P_1}(\check{\mathfrak{a}}_1)$. Simple considerations of continuity imply that all this remains unaltered when only

$$\begin{cases} \Phi_1 \in \mathcal{H}_{A_1}(\delta_1, \mathcal{O}_1; R) \\ \Phi_2 \in \mathcal{H}_{A_2}(\delta_2, \mathcal{O}_2; R). \end{cases}$$

These points made, return to

$$Q^{\mathbf{H}}E(P^*\,|\,A^* : \Phi^* : \Lambda^* : ?)(x),$$

which, for Λ^* sufficiently negative, can be written, as has been seen above, in the form

$$\sum_{\gamma_{i\mu}\in\Gamma/\Gamma\cap P_{i\mu}^*} \varphi_{i\mu}(w_{i\mu}^* : \Lambda^* : x\gamma_{i\mu}),$$

summed over

$$\begin{cases} i,\ 1 \le i \le r^* \\ \mu,\ 1 \le \mu \le r_i^* \\ w_{i\mu}^*,\ w_{i\mu}^* \in W(A_{i\mu}^*, A^*), \end{cases}$$

where now, by definition,

$$\varphi_{i\mu}(w_{i\mu}^* : \Lambda^* : x)$$

is equal to

$$(a_x^*)^{(w_{i\mu}^*\Lambda^* - \rho_{i\mu}^*)} \times (-1)^{\#(F(w_{i\mu}^*))}\tau_{*,*}\big(F(w_{i\mu}^*) : I_{P_{i\mu}^*}(\mathbf{H}) - H_{P_{i\mu}^*|A_{i\mu}^*}(x)\big)$$
$$\times \big(c_{\mathrm{cus}}(P_{i\mu}^*\,|\,A_{i\mu}^* : P^*\,|\,A^* : w_{i\mu}^* : \Lambda^*)\Phi^*\big)(x).$$

Fix i and μ and a real number $R^*_{i\mu} > \|\rho^*_{i\mu}\|$ — then we intend to compute

$$\Phi_{i\mu}(w^*_{i\mu} : \Lambda^* : x)(\Lambda^*_{i\mu}) = \int_{A^*_{i\mu}} (a^*_{xa^*_{i\mu}})^{-(\Lambda^*_{i\mu} - \rho^*_{i\mu})} \varphi_{i\mu}(w^*_{i\mu} : \Lambda^* : xa^*_{i\mu}) d_{A^*_{i\mu}}(a^*_{i\mu})$$

for $\Lambda^*_{i\mu}$ belonging to the $R^*_{i\mu}$-tube. [Note: In so doing, it will be necessary to assume that Λ^* is ever more negative.] The outcome of this will be an expression for

$$Q^{\mathbf{H}} E(P^* \mid A^* : \Phi^* : \Lambda^* : ?)$$

in terms of wave-packets, thereby clearing the way for the final inner-product calculation.

We have

$$\int_{A^*_{i\mu}} (a^*_{xa^*_{i\mu}})^{-(\Lambda^*_{i\mu} - \rho^*_{i\mu})} \varphi_{i\mu}(w^*_{i\mu} : \Lambda^* : xa^*_{i\mu}) d_{A^*_{i\mu}}(a^*_{i\mu})$$
$$= (-1)^{\#(F(w^*_{i\mu}))} (a^*_x)^{(w^*_{i\mu}\Lambda^* - \Lambda^*_{i\mu})} (c_{\text{cus}}(P^*_{i\mu} \mid A^*_{i\mu} : P^* \mid A^* : w^*_{i\mu} : \Lambda^*) \Phi^*)(x)$$

times the integral over $\mathfrak{a}^*_{i\mu}$ of

$$\exp(\langle H^*_{i\mu}, w^*_{i\mu}\Lambda^* - \Lambda^*_{i\mu} \rangle)$$
$$\times \tau_{*,*}\left(F(w^*_{i\mu}) : I_{P^*_{i\mu}}(\mathbf{H}) - H_{P^*_{i\mu}|A^*_{i\mu}}(x) - H^*_{i\mu}\right).$$

This integral is best treated by passing to coordinates. To simplify the notation, let us make the following temporary changes in the data:

$$(P^*_{i\mu}, S^*_{i\mu}; A^*_{i\mu}) \to (P, S; A)$$
$$\begin{cases} w^*_{i\mu} \to w^* \\ H^*_{i\mu} \to H \\ \Lambda^*_{i\mu} \to \Lambda. \end{cases}$$

Put

$$J(P, A) = |\det[(\lambda_i, \lambda_j)]|^{1/2}.$$

LEMMA 9.4. *Suppose that* Λ *is in the R-tube — then, for* Λ^* *sufficiently negative, the integral over* $\mathfrak{a}$ *of*

$$\exp(\langle H, w^*\Lambda^* - \Lambda \rangle)$$
$$\times \tau_{*,*}(F(w^*) : I_P(\mathbf{H}) - H_{P|A}(x) - H)$$

is equal to

$$(-1)^{l-\#(F(w^*))}\cdot J(P,A)$$
$$\times\exp\left(\left\langle I_P(\mathbf{H})-H_{P|A}(x),w^*\Lambda^*-\Lambda\right\rangle\right)\cdot\left(1/\prod_{i=1}^{l}(w^*\Lambda^*-\Lambda,\lambda_i)\right).$$

Proof. Given λ_i, determine H_i by the requirement

$$\langle H,\lambda_i\rangle=(H,H_i)\qquad(H\in\mathfrak{a}).$$

Define a map $T\colon\mathbf{R}^l\to\mathfrak{a}$ via the rule

$$T(t_1,\ldots,t_l)=\sum_{i=1}^{l}t_iH_i.$$

Then

$$\left\langle T(t_1,\ldots,t_l),w^*\Lambda^*-\Lambda\right\rangle=\sum_{i=1}^{l}t_i(w^*\Lambda^*-\Lambda,\lambda_i).$$

Furthermore, by definition,

$$\tau_{*,*}\left(F(w^*)\colon I_P(\mathbf{H})-H_{P|A}(x)-T(t_1,\ldots,t_l)\right)=1$$

iff

$$\begin{cases}\lambda_i\in F(w^*)\Rightarrow\left\langle I_P(\mathbf{H})-H_{P|A}(x)-T(t_1,\ldots,t_l),\lambda^i\right\rangle>0\\ \lambda_i\notin F(w^*)\Rightarrow\left\langle I_P(\mathbf{H})-H_{P|A}(x)-T(t_1,\ldots,t_l),\lambda^i\right\rangle\le 0,\end{cases}$$

that is, $\forall\,\lambda_i$,

$$\left\langle I_P(\mathbf{H})-H_{P|A}(x),\lambda^i\right\rangle-t_i\text{ is }\begin{cases}>0 & \text{if }\lambda_i\in F(w^*)\\ \le 0 & \text{if }\lambda_i\notin F(w^*).\end{cases}$$

It can be supposed that

$$F(w^*)=\{\lambda_1,\ldots,\lambda_{l^*}\}.$$

Set

$$c_i=\left\langle I_P(\mathbf{H})-H_{P|A}(x),\lambda^i\right\rangle.$$

Then, in terms of these coordinates, our integral becomes

$$J(P,A)\cdot\int_{-\infty}^{c_1}\cdots\int_{-\infty}^{c_{l^*}}\int_{c_{l^*+1}}^{+\infty}\cdots\int_{c_l}^{+\infty}[\ldots]\,dt_1\ldots dt_l$$

where

$$[\ldots] = \prod_{i=1}^{l} \exp(t_i(w^*\Lambda^* - \Lambda, \lambda_i)).$$

Formally,

$$(-\infty)\colon \int_{-\infty}^{c_i} \exp(t_i(w^*\Lambda^* - \Lambda, \lambda_i))\, dt_i$$
$$= \exp(c_i(w^*\Lambda^* - \Lambda, \lambda_i))/(w^*\Lambda^* - \Lambda, \lambda_i),$$
$$(+\infty)\colon \int_{c_i}^{+\infty} \exp(t_i(w^*\Lambda^* - \Lambda, \lambda_i))\, dt_i$$
$$= -\exp(c_i(w^*\Lambda^* - \Lambda, \lambda_i))/(w^*\Lambda^* - \Lambda, \lambda_i).$$

But Λ^* is at our disposal in the sense that we can assume ahead of time that it is very negative. Since

$$\|\mathrm{Re}(\Lambda)\| \le R,$$

calculation $(-\infty)$ is valid provided

$$\mathrm{Re}(w^*\Lambda^* - \Lambda, \lambda_i) > 0$$

which will be the case, as

$$\lambda_i \in F(w^*) \Rightarrow w^{-*}\lambda_i < 0,$$

while calculation $(+\infty)$ is valid provided

$$\mathrm{Re}(w^*\Lambda^* - \Lambda, \lambda_i) < 0$$

which will be the case, as

$$\lambda_i \notin F(w^*) \Rightarrow w^{-*}\lambda_i > 0.$$

The value of the integral is therefore

$$(-1)^{l - \#(F(w^*))} \cdot J(P, A)$$
$$\times \prod_{i=1}^{l} \exp(c_i(w^*\Lambda^* - \Lambda, \lambda_i))/(w^*\Lambda^* - \Lambda, \lambda_i).$$

However,

$$\sum_{i=1}^{l} c_i(w^*\Lambda^* - \Lambda, \lambda_i) = \sum_{i=1}^{l} (w^*\Lambda^* - \Lambda, \lambda_i) \cdot \left\langle I_P(\mathbf{H}) - H_{P|A}(x), \lambda^i \right\rangle$$
$$= \left\langle I_P(\mathbf{H}) - H_{P|A}(x), w^*\Lambda^* - \Lambda \right\rangle$$

so we finally get

$$(-1)^{l-\#(F(w^*))}\cdot J(P,A)$$
$$\times\exp\big(\big\langle I_P(\mathbf{H})-H_{P|A}(x),w^*\Lambda^*-\Lambda\big\rangle\big)\cdot\left(1/\prod_{i=1}^{l}(w^*\Lambda^*-\Lambda,\lambda_i)\right),$$

as desired. □

Consequently, for $\Lambda^*_{i\mu}$ belonging to the $R^*_{i\mu}$-tube, if Λ^* is sufficiently negative, then

$$\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:x)(\Lambda^*_{i\mu})$$

is equal to

$$(-1)^{\operatorname{rank}(P^*_{i\mu})}\cdot J(P^*_{i\mu},A^*_{i\mu})$$
$$\times\exp\big(\big\langle I_{P^*_{i\mu}}(\mathbf{H}),w^*_{i\mu}\Lambda^*-\Lambda^*_{i\mu}\big\rangle\big)\cdot\left(1/\prod_{\lambda^*_{i\mu}}(w^*_{i\mu}\Lambda^*-\Lambda^*_{i\mu},\lambda^*_{i\mu})\right)$$
$$\times\big(c_{\mathrm{cus}}(P^*_{i\mu}\,|\,A^*_{i\mu}:P^*\,|\,A^*:w^*_{i\mu}:\Lambda^*)\Phi^*\big)(x).$$

Because

$$\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:x)(\Lambda^*_{i\mu})$$

needn't decay fast enough vertically, we cannot assert that it is in $\mathcal{H}_{A^*_{i\mu}}(\delta,\mathcal{O}^*_{i\mu};R^*_{i\mu})$. No real difficulty is present, though. Indeed, the function

$$\exp\big(\varepsilon(\Lambda^*_{i\mu},\Lambda^*_{i\mu})\big)\cdot\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:x)(\Lambda^*_{i\mu})\qquad(\varepsilon>0)$$

does fall off suitably at infinity, hence

$$\Theta_{\exp(\varepsilon(?,?))\cdot\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:?)}\in L^2(G/\Gamma)$$

Moreover,

$$\lim_{\varepsilon\downarrow 0}\Theta_{\exp(\varepsilon(?,?))\cdot\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:?)}$$

exists in $L^2(G/\Gamma)$,

$$\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:?)(\Lambda^*_{i\mu})$$

being L^2 on vertical lines. Actually, it is easy to identify this limit. For, as a moment's reflection shows,

$$\lim_{\varepsilon\downarrow 0}\Theta_{\exp(\varepsilon(?,?))\cdot\Phi_{i\mu}(w^*_{i\mu}:\Lambda^*:?)}(x)$$
$$=\sum_{\gamma_{i\mu}\in\Gamma/\Gamma\cap P^*_{i\mu}}\varphi_{i\mu}(w^*_{i\mu}:\Lambda^*:x\gamma_{i\mu})\qquad\text{a.e.}$$

This remark carries with it the a posteriori conclusion that

$$\sum_{\gamma_{i\mu} \in \Gamma/\Gamma \cap P^*_{i\mu}} \varphi_{i\mu}(w^*_{i\mu} : \Lambda^* : x\gamma_{i\mu})$$

is a square integrable function on G/Γ. The same is therefore true of

$$Q^{\mathbf{H}}E(P^* \mid A^* : \Phi^* : \Lambda^* : ?)(x).$$

In this connection, observe that the truncation parameter $\mathbf{H}$ is arbitrary.... [Note: Recall, by comparison, that Λ^* is sufficiently negative.] Following the customary practice, write

$$\Theta_{\Phi_{i\mu}(w^*_{i\mu} : \Lambda^* : ?)}$$

for

$$\lim_{\varepsilon \downarrow 0} \Theta_{\exp(\varepsilon(?,?)) \cdot \Phi_{i\mu}(w^*_{i\mu} : \Lambda^* : ?)}.$$

We have proved:

PROPOSITION 9.5. *Retain the preceding assumptions and notations — then*

$$Q^{\mathbf{H}}E(P^* \mid A^* : \Phi^* : \Lambda^* : ?)(x)$$

is equal to

$$\sum_{i=1}^{r^*} \sum_{\mu=1}^{r_i^*} \sum_{w^*_{i\mu} \in W(A^*_{i\mu}, A^*)} \Theta_{\Phi_{i\mu}(w^*_{i\mu} : \Lambda^* : ?)}(x).$$

A prenatal version of the inner product formula can now be given. For this purpose, it will be convenient to revert to the notation involving the subscripts 1 and 2. So let again $(P_1, S_1), (P_2, S_2)$ be Γ-cuspidal split parabolic subgroups of G with special split components A_1, A_2. Let

$$\begin{cases} \delta_1, \delta_2 \text{ be } K\text{-types} \\ \mathcal{O}_1, \mathcal{O}_2 \text{ be } M_1, M_2\text{-types.} \end{cases}$$

Let

$$\begin{cases} \Phi_1 \in \mathcal{E}_{\text{cus}}(\delta_1, \mathcal{O}_1) \\ \Phi_2 \in \mathcal{E}_{\text{cus}}(\delta_2, \mathcal{O}_2). \end{cases}$$

Then, supposing that Λ_1 and Λ_2 are sufficiently negative,

$$\big(Q^{\mathbf{H}}E(P_1 \mid A_1 : \Phi_1 : \Lambda_1 : ?), Q^{\mathbf{H}}E(P_2 \mid A_2 : \Phi_2 : \Lambda_2 : ?)\big)$$

is equal to

$$\sum_{i_1}\sum_{\mu_1}\sum_{i_2}\sum_{\mu_2}\sum_{W(A_{i_1\mu_1},A_1)}\sum_{W(A_{i_2\mu_2},A_2)}$$
$$\times\left(\Theta_{\Phi_{i_1\mu_1}(w_{i_1\mu_1}:\Lambda_1:?)},\Theta_{\Phi_{i_2\mu_2}(w_{i_2\mu_2}:\Lambda_2:?)}\right)$$

where

$$\begin{cases} w_{i_1\mu_1}\in W\left(A_{i_1\mu_1},A_1\right)\\ w_{i_2\mu_2}\in W\left(A_{i_2\mu_2},A_2\right).\end{cases}$$

This makes it plain that

$$\left(Q^{\mathbf{H}}E(P_1\,|\,A_1:\Phi_1:\Lambda_1:?),\,Q^{\mathbf{H}}E(P_2\,|\,A_2:\Phi_2:\Lambda_2:?)\right)$$

is null unless (P_1, S_1) and (P_1, S_2) are associate (with common rank l, say), $\delta_1=\delta_2$, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are associate, in which case the summand

$$\left(\Theta_{\Phi_{i_1\mu_1}(w_{i_1\mu_1}:\Lambda_1:?)},\Theta_{\Phi_{i_2\mu_2}(w_{i_2\mu_2}:\Lambda_2:?)}\right),$$

that is,

$$\int_{G/\Gamma}\Theta_{\Phi_{i_1\mu_1}(w_{i_1\mu_1}:\Lambda_1:?)}(x)\overline{\Theta_{\Phi_{i_2\mu_2}(w_{i_2\mu_2}:\Lambda_2:?)}(x)}d_G(x),$$

can be written as the sum over

$$w_{21}(i_2,\mu_2:i_1,\mu_1)\in W\left(A_{i_2\mu_2},A_{i_1\mu_1}\right)$$

of

$$\frac{1}{(2\pi)^l}\int_{\mathrm{Re}(\Lambda_{i_1\mu_1})=\Lambda^0_{i_1\mu_1}}[\ldots]\,|\,d\Lambda_{i_1\mu_1}\,|$$

where $[\ldots]$ is the inner product of

$$c_{\mathrm{cus}}\left(P_{i_2\mu_2}\,|\,A_{i_2\mu_2}:P_{i_1\mu_1}\,|\,A_{i_1\mu_1}:w_{21}(i_2,\mu_2:i_1,\mu_1):\Lambda_{i_1\mu_1}\right)$$

applied to

$$\Phi_{i_1\mu_1}\left(w_{i_1\mu_1}:\Lambda_1:?\right)\left(\Lambda_{i_1\mu_1}\right)$$

with

$$\Phi_{i_2\mu_2}\left(w_{i_2\mu_2}:\Lambda_2:?\right)\left(-w_{21}(i_2,\mu_2:i_1,\mu_1)\overline{\Lambda_{i_1\mu_1}}\right),$$

$\Lambda^0_{i_1\mu_1}$ being any point in $\mathcal{T}_{P_{i_1\mu_1}}(\check{\mathfrak{a}}_{i_1\mu_1})$.

On the basis of these considerations, there is no loss of generality in supposing henceforth that P_1, P_2 belong to a fixed association class $\mathcal{C}$, say. In turn, this enables us to simplify the notation so as to put it in line with that utilized in [**3.a**]. Thus let P_i $(1 \le i \le r)$ be a set of representatives for $G\backslash\mathcal{C}$ — then

$$\mathcal{C} = \coprod_i \mathcal{C}_i$$

where $\mathcal{C}_i = G \cdot \{P_i\} \cap \mathcal{C}$. Let $P_{i\mu}$ $(1 \le \mu \le r_i)$ be a set of representatives for $\Gamma\backslash\mathcal{C}_i$ — then

$$\{P_{i\mu}: 1 \le i \le r, 1 \le \mu \le r_i\}$$

is a set of representatives for $\Gamma\backslash\mathcal{C}$. We have, correspondingly, that

$$\left(Q^{\mathbf{H}}E(P_1 | A_1 : \Phi_1 : \Lambda_1 : ?), Q^{\mathbf{H}}E(P_2 | A_2 : \Phi_2 : \Lambda_2 : ?)\right)$$

is equal to

$$\sum_{i=1}^{r} \sum_{\mu=1}^{r_i} \sum_{j=1}^{r} \sum_{\nu=1}^{r_j} \sum_{w_{i\mu} \in W(A_{i\mu}, A_1)} \sum_{w_{j\nu} \in W(A_{j\nu}, A_2)} \times \left(\Theta_{\Phi_{i\mu}(w_{i\mu} : \Lambda_1 : ?)}, \Theta_{\Phi_{j\nu}(w_{j\nu} : \Lambda_2 : ?)}\right)$$

provided, of course, that Λ_1 and Λ_2 are sufficiently negative. Take now

$$\left(\Theta_{\Phi_{i\mu}(w_{i\mu} : \Lambda_1 : ?)}, \Theta_{\Phi_{j\nu}(w_{j\nu} : \Lambda_2 : ?)}\right)$$

and write it as the sum over

$$w(j, \nu : i, \mu) \in W(A_{j\nu}, A_{i\mu})$$

of

$$\frac{1}{(2\pi)^l} \int_{\mathrm{Re}(\Lambda_{i\mu}) = \Lambda_{i\mu}^0} [\ldots] \, |d\Lambda_{i\mu}|$$

where $[\ldots]$ is the inner product of

$$c_{\mathrm{cus}}\left(P_{j\nu} | A_{j\nu} : P_{i\mu} | A_{i\mu} : w(j, \nu : i, \mu) : \Lambda_{i\mu}\right)$$

applied to

$$\Phi_{i\mu}(w_{i\mu} : \Lambda_1 : ?)(\Lambda_{i\mu})$$

with

$$\Phi_{j\nu}(w_{j\nu} : \Lambda_2 : ?)\left(-w(j, \nu : i, \mu)\overline{\Lambda}_{i\mu}\right),$$

$\Lambda^0_{i\mu}$ being any point in $\mathcal{T}_{P_{i\mu}}(\check{\mathfrak{a}}_{i\mu})$. For definiteness, we can and will assume that

$$\left\{\begin{array}{l}\forall\, i, \forall\, \mu,\ R_{i\mu} = 2\|\rho_{i\mu}\| \\ \forall\, j, \forall\, \nu,\ R_{j\nu} = 2\|\rho_{j\nu}\|\end{array}\right. = R, \quad \text{say.}$$

In reality, there is no need to stress this point since it plays no role in what follows. We have yet to explicate the specific nature of $\Phi_{i\mu}$ and $\Phi_{j\nu}$. To begin with, recall that

$$\left\{\begin{array}{l}J(P_1, A_1) = \left|\det\left[\left(\lambda^1_i, \lambda^1_j\right)\right]\right|^{1/2} \\ J(P_2, A_2) = \left|\det\left[\left(\lambda^2_i, \lambda^2_j\right)\right]\right|^{1/2}.\end{array}\right.$$

These positive real numbers are actually equal. This is most easily seen by remarking that for any (P, S) in $\mathcal{C}$ with special split component A, $J(P, A)$ is the volume of $\check{\mathfrak{a}}$ modulo the lattice spanned by the elements of $\Sigma^0_P(\mathfrak{g}, \mathfrak{a})$ or still, the volume of $\check{\mathfrak{a}}$ modulo the lattice spanned by the elements of $\pm\Sigma_P(\mathfrak{g}, \mathfrak{a})$. But for every $w_{21} \in W(A_2, A_1)$,

$$w_{21} \cdot \left(\pm\Sigma_{P_1}(\mathfrak{g}, \mathfrak{a}_1)\right) = \pm\Sigma_{P_2}(\mathfrak{g}, \mathfrak{a}_2),$$

implying, therefore, that

$$J(P_1, A_1) = J(P_2, A_2).$$

We shall agree to write

$$\operatorname{vol}(\mathcal{C})$$

for their common value. Next, to avoid any confusion, observe that

$$\operatorname{rank}(P_{i\mu}) + \operatorname{rank}(P_{j\nu}) \equiv 0 \mod 2.$$

Accordingly, given

$$w(j, \nu : i, \mu) \in W(A_{j\nu}, A_{i\mu}),$$

the integral

$$\frac{1}{(2\pi)^l}\int_{\operatorname{Re}(\Lambda_{i\mu}) = \Lambda^0_{i\mu}} [\ldots]\, |d\Lambda_{i\mu}|\,,$$

upon expansion, becomes the integral

$$\frac{\operatorname{vol}(\mathcal{C})^2}{(2\pi)^l} \cdot \int_{\operatorname{Re}(\Lambda_{i\mu}) = \Lambda^0_{i\mu}} [\ldots]\, |d\Lambda_{i\mu}|\,,$$

the integrand being the product of

$$\exp\Big(\Big\langle I_{P_{i\mu}}(\mathbf{H}), w_{i\mu}\Lambda_1 - \Lambda_{i\mu}\Big\rangle\Big)\cdot\Big(1/\prod_{\lambda_{i\mu}}\big(w_{i\mu}\Lambda_1 - \Lambda_{i\mu}, \lambda_{i\mu}\big)\Big)$$

and

$$\exp\Big(\Big\langle I_{P_{j\nu}}(\mathbf{H}), w_{j\nu}\overline{\Lambda}_2 + w(j,\nu:i,\mu)\Lambda_{i\mu}\Big\rangle\Big)$$
$$\times\Big(1/\prod_{\lambda_{j\nu}}\big(w_{j\nu}\overline{\Lambda}_2 + w(j,\nu:i,\mu)\Lambda_{i\mu}, \lambda_{j\nu}\big)\Big)$$

and

$$\big(c_{\text{cus}}\big(P_{j\nu}\,|\,A_{j\nu}:P_{i\mu}\,|\,A_{i\mu}:w(j,\nu:i,\mu):\Lambda_{i\mu}\big)c_{\text{cus}}\big(P_{i\mu}\,|\,A_{i\mu}:P_1\,|\,A_1:w_{i\mu}:\Lambda_1\big)\Phi_1,$$
$$c_{\text{cus}}\big(P_{j\nu}\,|\,A_{j\nu}:P_2\,|\,A_2:w_{j\nu}:\Lambda_2\big)\Phi_2\big).$$

As we shall see, a substantial portion of this admittedly complicated expression can be collapsed after some additional manipulation.

Rewrite

$$\big(Q^{\mathbf{H}}E(P_1\,|\,A_1:\Phi_1:\Lambda_1:?), Q^{\mathbf{H}}E(P_2\,|\,A_2:\Phi_2:\Lambda_2:?)\big)$$

in the form

$$\sum_{i=1}^{r}\sum_{\mu=1}^{r_i}\sum_{w_{i\mu}\in W(A_{i\mu},A_1)}$$
$$\times\frac{\operatorname{vol}(\mathcal{C})^2}{(2\pi)^l}\cdot\int_{\operatorname{Re}(\Lambda_{i\mu})=\Lambda_{i\mu}^0}\exp\Big(\Big\langle I_{P_{i\mu}}(\mathbf{H}), w_{i\mu}\Lambda_1 - \Lambda_{i\mu}\Big\rangle\Big)$$
$$\cdot\Big(1/\prod_{\lambda_{i\mu}}\big(w_{i\mu}\Lambda_1 - \Lambda_{i\mu}, \lambda_{i\mu}\big)\Big)[\ldots]\,|d\Lambda_{i\mu}|$$

where $[\ldots]$ is the sum

$$\sum_{j=1}^{r}\sum_{\nu=1}^{r_j}\sum_{w_{j\nu}\in W(A_{j\nu},A_2)}\sum_{w(j,\nu:i,\mu)\in W(A_{j\nu},A_{i\mu})}$$

of the product of

$$\exp\Big(\Big\langle I_{P_{j\nu}}(\mathbf{H}), w_{j\nu}\overline{\Lambda}_2 + w(j,\nu:i,\mu)\Lambda_{i\mu}\Big\rangle\Big)$$
$$\times\Big(1/\prod_{\lambda_{j\nu}}\big(w_{j\nu}\overline{\Lambda}_2 + w(j,\nu:i,\mu)\Lambda_{i\mu}, \lambda_{j\nu}\big)\Big)$$

with

$$\big(c_{\text{cus}}(P_{j\nu} \mid A_{j\nu} : P_{i\mu} \mid A_{i\mu} : w(j, \nu : i, \mu) : \Lambda_{i\mu}) c_{\text{cus}}(P_{i\mu} \mid A_{i\mu} : P_1 \mid A_1 : w_{i\mu} : \Lambda_1)\Phi_1,$$
$$c_{\text{cus}}(P_{j\nu} \mid A_{j\nu} : P_2 \mid A_2 : w_{j\nu} : \Lambda_2)\Phi_2\big).$$

With i, μ, and $w_{i\mu}$ fixed, [...] is a function of $\Lambda_{i\mu}$. We intend to prove that it is holomorphic, qua a function in the tube over $\mathfrak{F}_{P_{i\mu}}(\check{\mathfrak{a}}_{i\mu})$.

Here is the argument. The domain of holomorphy of

$$c_{\text{cus}}(P_{j\nu} \mid A_{j\nu} : P_{i\mu} \mid A_{i\mu} : w(j, \nu : i, \mu) : \Lambda_{i\mu})$$

contains

$$\mathfrak{F}_{P_{i\mu}}(\check{\mathfrak{a}}_{i\mu}) + \sqrt{-1}\,\check{\mathfrak{a}}_{i\mu},$$

hence no singularities can come from it. As

$$\exp\left(\left\langle I_{P_{j\nu}}(\mathbf{H}), w_{j\nu}\overline{\Lambda}_2 + w(j, \nu : i, \mu)\Lambda_{i\mu}\right\rangle\right)$$

is obviously holomorphic, the only possible singularities arise from

$$1\Big/\prod_{\lambda_{j\nu}} \left(w_{j\nu}\overline{\Lambda}_2 + w(j, \nu : i, \mu)\Lambda_{i\mu}, \lambda_{j\nu}\right),$$

these occurring when

$$\left(w_{j\nu}\overline{\Lambda}_2 + w(j, \nu : i, \mu)\Lambda_{i\mu}, \lambda_{j\nu}\right) = 0$$

for some $\lambda_{j\nu}$, the corresponding hyperplanes being distinct. Our function thus has, at worst, simple singularities along hyperplanes. The singular hyperplanes associated with different terms in the sum may very well coincide but this will not raise the order of the singularity (it being a question of addition rather than multiplication). To prove, therefore, that the singularities have codimension ≥ 2, hence that our function is continuable along them, we proceed as follows. Since i, μ, and $w_{i\mu}$ are fixed, set for simplicity

$$\begin{cases} P = P_{i\mu},\ A = A_{i\mu},\ \Lambda = \Lambda_{i\mu} \\ \Phi = c_{\text{cus}}(P_{i\mu} \mid A_{i\mu} : P_1 \mid A_1 : w_{i\mu} : \Lambda_1)\Phi_1. \end{cases}$$

Choose $k_{j\nu} \in K$ with the property that $P_{j\nu} = k_{j\nu}P_j k_{j\nu}^{-1}$ — then $A_{j\nu} = k_{j\nu}A_j k_{j\nu}^{-1}$. In these notations, we must thus establish the holomorphicity of

$$\sum_{j=1}^{r} \sum_{w_{j2} \in W(A_j, A_2)} \sum_{w_j \in W(A_j, A)} \left(1 / \prod_{\lambda_j} \left(w_{j2}\bar{\Lambda}_2 + w_j\Lambda, \lambda_j\right)\right)$$

$$\times \sum_{\nu=1}^{r_j} \exp\left(\left\langle I_{P_{j\nu}}(\mathbf{H}), k_{j\nu} \cdot \left(w_{j2}\bar{\Lambda}_2 + w_j\Lambda\right)\right\rangle\right)$$

$$\times \left(c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P \mid A : k_{j\nu}w_j : \Lambda\right)\Phi, c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P_2 \mid A_2 : k_{j\nu}w_{j2} : \Lambda_2\right)\Phi_2\right)$$

as a function of Λ in the tube over $\mathfrak{T}_P(\check{\mathfrak{a}})$. Observe that $\Sigma_{\nu=1}^{r_j}\ldots$ is holomorphic so it is only the terms indexed by the triples (j, w_{j2}, w_j) which can cause a problem. Consider, then, a singular hyperplane

$$\left\{\Lambda: \left(w_{j2}\bar{\Lambda}_2 + w_j\Lambda, \lambda_j\right) = 0\right\}$$

determined by one of these triples. We may then attach to λ_j, in the usual way, a Γ-cuspidal parabolic $P_{\lambda_j} \succcurlyeq P_j$, the special split component of P_{λ_j} being, of course

$$A_{\lambda_j} = \exp\left(\operatorname{Ker}(\lambda_j)\right).$$

On the other hand, there is also attached to λ_j another Γ-cuspidal parabolic $P_{j'}$ in $\mathcal{C}$, itself a dominated predecessor of P_{λ_j}, arising from the simple reflection w_{λ_j} (cf. [**3.a**]). [Note: Strictly speaking $P_{j'}$ may not be one of our fixed representatives for $G\backslash\mathcal{C}$ but there is no real harm in pretending that it is.] We recall that

$$w_{\lambda_j} \in W\left(A_{j'}^{\dagger}, A_j^{\dagger}\right), w_{\lambda_j}(\lambda_j) = -\lambda_{j'}, \text{say},$$

so

$$-w_{\lambda_j}(\lambda_j) = \lambda_{j'} \in \Sigma_{P_{j'}}^0(\mathfrak{g}, \mathfrak{a}_{j'}).$$

The singular hyperplane

$$\left\{\Lambda: \left(w_{\lambda_j}w_{j2}\bar{\Lambda}_2 + w_{\lambda_j}w_j\Lambda, \lambda_{j'}\right) = 0\right\}$$

is the same as the one with which we started. Accordingly, the summands which are singular along a given hyperplane occur in pairs. To draw the required conclusion, it need only be shown that the residues add up to zero. The residue with respect to λ_j is

$$\left(1 / \prod_{\lambda \neq \lambda_j, \lambda \in \Sigma_{P_j}^0(\mathfrak{g}, \mathfrak{a}_j)} \left(w_{j2}\bar{\Lambda}_2 + w_j\Lambda, \lambda\right)\right)$$

$$\times \sum_{\nu=1}^{r_j} \exp\left(\left\langle I_{P_{j\nu}}(\mathbf{H}), k_{j\nu} \cdot \left(w_{j2}\bar{\Lambda}_2 + w_j\Lambda\right)\right\rangle\right)$$

$$\times \left(c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P \mid A : k_{j\nu}w_j : \Lambda\right)\Phi, c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P_2 \mid A_2 : k_{j\nu}w_{j2} : \Lambda_2\right)\Phi_2\right)$$

while the residue with respect to $\lambda_{j'}$ is

$$\Big(-1\Big/\prod_{\lambda\neq\lambda_{j'},\,\lambda\in\Sigma^0_{P_{j'}}(\mathfrak{g},\mathfrak{a}_{j'})}\big(w_{\lambda_j}w_{j2}\overline{\Lambda}_2+w_{\lambda_j}w_j\Lambda,\lambda\big)\Big)$$

$$\times\sum_{\nu=1}^{r_j}\exp\Big(\Big\langle I_{P_{j'\nu}}(\mathbf{H}),\,k_{j'\nu}\cdot\big(w_{\lambda_j}w_{j2}\overline{\Lambda}_2+w_{\lambda_j}w_j\Lambda\big)\Big\rangle\Big)$$

$$\times\Big(c_{\mathrm{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P\,|\,A:k_{j'\nu}w_{\lambda_j}w_j:\Lambda\big)\Phi,$$

$$c_{\mathrm{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P_2\,|\,A_2:k_{j'\nu}w_{\lambda_j}w_{j2}:\Lambda_2\big)\Phi_2\Big).$$

To show that

$$\mathrm{Res}(\lambda_j)+\mathrm{Res}(\lambda_{j'})=0,$$

look first at the products

$$\prod_{\lambda\neq\lambda_j},\ldots,\prod_{\lambda\neq\lambda_{j'}}\cdots.$$

Because

$$w_{j2}\overline{\Lambda}_2+w_j\Lambda\,|\,\mathfrak{a}_j^{\dagger}=0,$$

and the

$$\lambda\neq\lambda_j,\qquad\lambda\in\Sigma^0_{P_j}(\mathfrak{g},\mathfrak{a}_j),$$

when restricted to $\mathfrak{a}_{\lambda_j}$, give $\Sigma^0_{P_{\lambda_j}}(\mathfrak{g},\mathfrak{a}_j)$, we have

$$\prod_{\lambda\neq\lambda_j,\,\lambda\in\Sigma^0_{P_j}(\mathfrak{g},\mathfrak{a}_j)}\big(w_{j2}\overline{\Lambda}_2+w_j\Lambda,\lambda\big)$$

$$=\prod_{\lambda,\,\lambda\in\Sigma^0_{P_{\lambda_j}}(\mathfrak{g},\mathfrak{a}_{\lambda_j})}\big(w_{j2}\overline{\Lambda}_2+w_j\Lambda,\lambda\big)$$

or still, as $w_{\lambda_j}=1$ on $\mathfrak{a}_{\lambda_j}$,

$$\prod_{\lambda,\,\lambda\in\Sigma^0_{P_{\lambda_j}}(\mathfrak{g},\mathfrak{a}_{\lambda_j})}\big(w_{\lambda_j}w_{j2}\overline{\Lambda}_2+w_{\lambda_j}w_j\Lambda,\lambda\big)$$

$$=\prod_{\lambda\neq\lambda_{j'},\,\lambda\in\Sigma^0_{P_{j'}}(\mathfrak{g},\mathfrak{a}_{j'})}\big(w_{\lambda_j}w_{j2}\overline{\Lambda}_2+w_{\lambda_j}w_j\Lambda,\lambda\big).$$

It remains to establish the equality of the sums, i.e., that

$$\sum_{\nu=1}^{r_j}\exp\Big(\Big\langle I_{P_{j\nu}}(\mathbf{H}),\,k_{j\nu}\cdot\big(w_{j2}\overline{\Lambda}_2+w_j\Lambda\big)\Big\rangle\Big)$$

$$\times\Big(c_{\mathrm{cus}}\big(P_{j\nu}\,|\,A_{j\nu}:P\,|\,A:k_{j\nu}w_j:\Lambda\big)\Phi,\;c_{\mathrm{cus}}\big(P_{j\nu}\,|\,A_{j\nu}:P_2\,|\,A_2:k_{j\nu}w_{j2}:\Lambda_2\big)\Phi_2\Big)$$

is equal to

$$\sum_{\nu=1}^{r_j} \exp\Big(\Big\langle I_{P_{j'\nu}}(\mathbf{H}),\, k_{j'\nu}\cdot\big(w_{\lambda_j}w_{j2}\overline{\Lambda}_2 + w_{\lambda_j}w_j\Lambda\big)\Big\rangle\Big)$$
$$\times\Big(c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P\,|\,A:k_{j'\nu}w_{\lambda_j}w_j:\Lambda\big)\Phi,$$
$$c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P_2\,|\,A_2:k_{j'\nu}w_{\lambda_j}w_{j2}:\Lambda_2\big)\Phi_2\Big).$$

We shall work on the second term first. Thanks to the functional equations, we have

$$c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P_2\,|\,A_2:k_{j'\nu}w_{\lambda_j}w_{j2}:\Lambda_2\big)$$
$$=\sum_{\nu'=1}^{r_{j'}} c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P_{j\nu'}\,|\,A_{j\nu'}:k_{j'\nu}w_{\lambda_j}k_{j\nu'}^{-1}:k_{j\nu'}w_{j2}\Lambda_2\big)$$
$$\circ\, c_{\text{cus}}\big(P_{j\nu'}\,|\,A_{j\nu'}:P_2\,|\,A_2:k_{j\nu'}w_{j2}:\Lambda_2\big).$$

In reality, there is a small difficulty in making this assertion (and others of the same nature). What is the point? The functional equations provide an equality but only in the sense of meromorphic functions. We have assumed that Λ_2 is very negative which, however, does not rule out the possibility that some transform of it may hit a singular hyperplane of c_{cus}. Since the set of singularities is locally finite, the reader will agree that our equation is valid on a dense, open subset of Λ_2's, a set to which we tacitly confine ourselves from now on. This said, we claim that

$$\exp\Big(\Big\langle I_{P_{j'\nu}}(\mathbf{H}),\, k_{j'\nu}\cdot\big(w_{\lambda_j}w_{j2}\overline{\Lambda}_2 + w_{\lambda_j}w_j\Lambda\big)\Big\rangle\Big)$$
$$\times\Big(c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P\,|\,A:k_{j'\nu}w_{\lambda_j}w_j:\Lambda\big)\Phi\Big),$$
$$c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P_{j\nu'}\,|\,A_{j\nu'}:k_{j'\nu}w_{\lambda_j}k_{j\nu'}^{-1}:k_{j\nu'}w_{j2}\Lambda_2\big)$$
$$\times\Big(c_{\text{cus}}\big(P_{j\nu'}\,|\,A_{j\nu'}:P_2\,|\,A_2:k_{j\nu'}w_{j2}:\Lambda_2\big)\Phi_2\Big)$$

is equal to

$$\exp\Big(\Big\langle I_{P_{j\nu'}}(\mathbf{H}),\, k_{j\nu'}\cdot\big(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\big)\Big\rangle\Big)$$
$$\times\Big(c_{\text{cus}}\big(P_{j\nu'}\,|\,A_{j\nu'}:P_{j'\nu}\,|\,A_{j'\nu}:k_{j\nu'}w_{\lambda_j}^{-1}k_{j'\nu}^{-1}:k_{j'\nu}w_{\lambda_j}w_j\Lambda\big)$$
$$\times c_{\text{cus}}\big(P_{j'\nu}\,|\,A_{j'\nu}:P\,|\,A:k_{j'\nu}w_{\lambda_j}w_j:\Lambda\big)\Phi,$$
$$c_{\text{cus}}\big(P_{j\nu'}\,|\,A_{j\nu'}:P_2\,|\,A_2:k_{j\nu'}w_{j2}:\Lambda_2\big)\Phi_2\Big).$$

We start the verification by remarking that the equality of the inner products follows from an adjoint computation, namely

$$c_{\text{cus}}\left(P_{j'\nu} \mid A_{j'\nu} : P_{j\nu'} \mid A_{j\nu'} : k_{j'\nu} w_{\lambda_j} k_{j\nu'}^{-1} : k_{j\nu'} w_{j2} \Lambda_2\right)^*$$

$$= c_{\text{cus}}\left(P_{j\nu'} \mid A_{j\nu'} : P_{j'\nu} \mid A_{j'\nu} : k_{j\nu'} w_{\lambda_j}^{-1} k_{j'\nu}^{-1} : -k_{j'\nu} w_{\lambda_j} w_{j2} \overline{\Lambda}_2\right)$$

$$= c_{\text{cus}}\left(P_{j\nu'} \mid A_{j\nu'} : P_{j'\nu} \mid A_{j'\nu} : k_{j\nu'} w_{\lambda_j}^{-1} k_{j'\nu}^{-1} : k_{j'\nu} w_{\lambda_j} w_j \Lambda\right),$$

the passage to the last line being justified with the observation that

$$w_{\lambda_j} w_{j2} \overline{\Lambda}_2 + w_{\lambda_j} w_j \Lambda \in \check{\mathfrak{a}}_{\lambda_j} + \sqrt{-1}\, \check{\mathfrak{a}}_{\lambda_j},$$

hence that

$$\begin{cases} -w_{\lambda_j} w_{j2} \overline{\Lambda}_2 \\ w_{\lambda_j} w_j \Lambda \end{cases}$$

have the same projection onto $\check{\mathfrak{a}}_{j'}^{\dagger} + \sqrt{-1}\, \check{\mathfrak{a}}_{j'}^{\dagger}$ and the c-function depends only on these components. As for the exponentials, put

$$\begin{cases} P_{\lambda_{j'\nu}} = k_{j'\nu} P_{\lambda_j} k_{j'\nu}^{-1} \\ P_{\lambda_{j\nu'}} = k_{j\nu'} P_{\lambda_j} k_{j\nu'}^{-1}. \end{cases}$$

There are two possibilities:

(1) $P_{\lambda_{j'\nu}}$ and $P_{\lambda_{j\nu'}}$ are Γ-conjugate;

(2) $P_{\lambda_{j'\nu}}$ and $P_{\lambda_{j\nu'}}$ are not Γ-conjugate.

The second possibility can be ignored since then

$$\begin{cases} c_{\text{cus}}\left(P_{j'\nu} \mid A_{j'\nu} : P_{j\nu'} \mid A_{j\nu'} : k_{j'\nu} w_{\lambda_j} k_{j\nu'}^{-1} : k_{j\nu'} w_{j2} \Lambda_2\right) = 0 \\ c_{\text{cus}}\left(P_{j\nu'} \mid A_{j\nu'} : P_{j'\nu} \mid A_{j'\nu} : k_{j\nu'} w_{\lambda_j}^{-1} k_{j'\nu}^{-1} : k_{j'\nu} w_{\lambda_j} w_j \Lambda\right) = 0, \end{cases}$$

implying that the claimed equality is automatic in this case. Turning to the first possibility, assume that actually $P_{\lambda_{j'\nu}} = P_{\lambda_{j\nu'}}$ — then

$$k_{j'\nu} \cdot \left(w_{\lambda_j} w_{j2} \overline{\Lambda}_2 + w_{\lambda_j} w_j \Lambda\right) = k_{j\nu'} \cdot \left(w_{j2} \overline{\Lambda}_2 + w_j \Lambda\right),$$

so we have

$$\left\langle I_{P_{j'\nu}}(\mathbf{H}), k_{j'\nu} \cdot \left(w_{\lambda_j} w_{j2} \overline{\Lambda}_2 + w_{\lambda_j} w_j \Lambda\right)\right\rangle$$

$$= \left\langle I_{P_{\lambda_{j'\nu}}}(\mathbf{H}), k_{j'\nu} \cdot \left(w_{\lambda_j} w_{j2} \overline{\Lambda}_2 + w_{\lambda_j} w_j \Lambda\right)\right\rangle$$

$$= \left\langle I_{P_{\lambda_{j\nu'}}}(\mathbf{H}), k_{j\nu'} \cdot \left(w_{j2} \overline{\Lambda}_2 + w_j \Lambda\right)\right\rangle$$

$$= \left\langle I_{P_{j\nu'}}(\mathbf{H}), k_{j\nu'} \cdot \left(w_{j2} \overline{\Lambda}_2 + w_j \Lambda\right)\right\rangle,$$

thus the exponentials are surely equal when $P_{\lambda_{j'\nu}} = P_{\lambda_{j\nu'}}$. In general, therefore, all that need be done is to check that

$$\begin{aligned}
&\exp\Bigl(\Bigl\langle I_{P_{j\nu'}}(\mathbf{H}),\, k_{j\nu'}\cdot\bigl(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\bigr)\Bigr\rangle\Bigr)\\
&\qquad\times\Bigl(c_{\text{cus}}\bigl(P_{j\nu'}\,|\,A_{j\nu'} : P_{j'\nu}\,|\,A_{j'\nu} : k_{j\nu'}w_{\lambda_j}^{-1}k_{j'\nu}^{-1} : k_{j'\nu}w_{\lambda_j}w_j\Lambda\bigr)\\
&\qquad\times c_{\text{cus}}\bigl(P_{j'\nu}\,|\,A_{j'\nu} : P\,|\,A : k_{j'\nu}w_{\lambda_j}w_j : \Lambda\bigr)\Phi,\\
&\qquad\qquad c_{\text{cus}}\bigl(P_{j\nu'}\,|\,A_{j\nu'} : P_2\,|\,A_2 : k_{j\nu'}w_{j2} : \Lambda_2\bigr)\Phi_2\Bigr)
\end{aligned}$$

is unchanged when $P_{j\nu'}$ is replaced by a Γ-conjugate $\gamma P_{j\nu'}\gamma^{-1}$. Let k be the K-component of γ per $G = K\cdot(\gamma P_{j\nu'}\gamma^{-1})$ — then

$$\begin{aligned}
&\Bigl\langle I_{\gamma P_{j\nu'}\gamma^{-1}}(\mathbf{H}),\, kk_{j\nu'}\cdot\bigl(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\bigr)\Bigr\rangle\\
&\qquad= \Bigl\langle I_{P_{j\nu'}}(\mathbf{H}),\, k_{j\nu'}\cdot\bigl(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\bigr)\Bigr\rangle\\
&\qquad\quad - \Bigl\langle H_{P_{j\nu'}|A_{j\nu'}}(\gamma),\, k_{j\nu'}\cdot\bigl(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\bigr)\Bigr\rangle.
\end{aligned}$$

Moreover (cf. **[3.a]**),

$$\begin{aligned}
&\bigl(c_{\text{cus}}(\gamma P_{j\nu'}\gamma^{-1}\ldots)c_{\text{cus}}(P_{j'\nu}\ldots)\Phi,\, c_{\text{cus}}(\gamma P_{j\nu'}\gamma^{-1}\ldots)\Phi_2\bigr)\\
&\quad= \exp\Bigl(\Bigl\langle H_{P_{j\nu'}|A_{j\nu'}}(\gamma),\, k_{j\nu'}w_{j2}\overline{\Lambda}_2\Bigr\rangle\Bigr)\cdot\exp\Bigl(\Bigl\langle H_{P_{j\nu'}|A_{j\nu'}}(\gamma),\, k_{j\nu'}w_j\Lambda\Bigr\rangle\Bigr)\\
&\qquad\times\bigl(c_{\text{cus}}(P_{j'\nu}\ldots)c_{\text{cus}}(P_{j'\nu}\ldots)\Phi,\, c_{\text{cus}}(P_{j\nu'}\ldots)\Phi_2\bigr),
\end{aligned}$$

the '2-rho' factor being absorbed by the integration implicit in the inner product. This checks the invariance under Γ-conjugacy. The claim is thereby settled. Consequently, the second term

$$\begin{aligned}
&\sum_{\nu=1}^{r_j}\exp\Bigl(\Bigl\langle I_{P_{j'\nu}}(\mathbf{H}),\, k_{j'\nu}\cdot\bigl(w_{\lambda_j}w_{j2}\overline{\Lambda}_2 + w_{\lambda_j}w_j\Lambda\bigr)\Bigr\rangle\Bigr)\\
&\qquad\times\Bigl(c_{\text{cus}}\bigl(P_{j'\nu}\,|\,A_{j'\nu} : P\,|\,A : k_{j'\nu}w_{\lambda_j}w_j : \Lambda\bigr)\Phi,\\
&\qquad\qquad c_{\text{cus}}\bigl(P_{j'\nu}\,|\,A_{j'\nu} : P_2\,|\,A_2 : k_{j'\nu}w_{\lambda_j}w_{j2} : \Lambda_2\bigr)\Phi_2\Bigr)
\end{aligned}$$

can be written, after reordering, in the form

$$\sum_{\nu=1}^{r_j} \exp\Big(\Big\langle I_{P_{j\nu}}(\mathbf{H}),\, k_{j\nu}\cdot\big(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\big)\Big\rangle\Big)$$

$$\times \sum_{\nu'=1}^{r_{j'}} \Big(c_{\text{cus}}\big(P_{j\nu}\,|\,A_{j\nu} : P_{j'\nu'}\,|\,A_{j'\nu'} : k_{j\nu}w_{\lambda_j}^{-1}k_{j'\nu'}^{-1} : k_{j'\nu'}w_{\lambda_j}w_j\Lambda\big)$$

$$\times c_{\text{cus}}\big(P_{j'\nu'}\,|\,A_{j'\nu'} : P\,|\,A : k_{j'\nu'}w_{\lambda_j}w_j : \Lambda\big)\Phi,$$

$$c_{\text{cus}}\big(P_{j\nu}\,|\,A_{j\nu} : P_2\,|\,A_2 : k_{j\nu}w_{j2} : \Lambda_2\big)\Phi_2\Big)$$

or still, employing the functional equations once again, as

$$\sum_{\nu=1}^{r_j} \exp\Big(\Big\langle I_{P_{j\nu}}(\mathbf{H}),\, k_{j\nu}\cdot\big(w_{j2}\overline{\Lambda}_2 + w_j\Lambda\big)\Big\rangle\Big)$$

$$\times \Big(c_{\text{cus}}\big(P_{j\nu}\,|\,A_{j\nu} : P\,|\,A : k_{j\nu}w_j : \Lambda\big)\Phi,\ c_{\text{cus}}\big(P_{j\nu}\,|\,A_{j\nu} : P_2\,|\,A_2 : k_{j\nu}w_{j2} : \Lambda_2\big)\Phi_2\Big)$$

which, being the first term, serves to establish the equality of the sums. It follows that

$$\operatorname{Res}(\lambda_j) + \operatorname{Res}(\lambda_{j'}) = 0,$$

hence holomorphicity.

Let us reinforce our position. The inner product

$$\big(Q^{\mathbf{H}}E(P_1\,|\,A_1 : \Phi_1 : \Lambda_1 : ?),\, Q^{\mathbf{H}}E(P_2\,|\,A_2 : \Phi_2 : \Lambda_2 : ?)\big)$$

admits the representation

$$\sum_{i=1}^{r} \sum_{\mu=1}^{r_i} \sum_{w_{i\mu}\in W(A_{i\mu},\, A_1)}$$

$$\times \frac{\operatorname{vol}(\mathcal{C})^2}{(2\pi)^l}\cdot \int_{\operatorname{Re}(\Lambda_{i\mu})=\Lambda^0_{i\mu}} \exp\Big(\Big\langle I_{P_{i\mu}}(\mathbf{H}),\, w_{i\mu}\Lambda_1 - \Lambda_{i\mu}\Big\rangle\Big)$$

$$\times \Big(1\Big/\prod_{\lambda_{i\mu}}\big(w_{i\mu}\Lambda_1 - \Lambda_{i\mu},\, \lambda_{i\mu}\big)\Big)[\ldots]\,|d\Lambda_{i\mu}|$$

where now, as we know, $[\ldots]$, the sum

$$\sum_{j=1}^{r} \sum_{\nu=1}^{r_i} \sum_{w_{j\nu}\in W(A_{j\nu},\, A_2)} \sum_{w(j,\nu\,:\,i,\mu)\in W(A_{j\nu},\, A_{i\mu})}$$

of the product of

$$\exp\left(\left\langle I_{P_{j\nu}}(\mathbf{H}), w_{j\nu}\Lambda_2 + w(j,\nu:i,\mu)\Lambda_{i\mu}\right\rangle\right)$$
$$\times\left(1/\prod_{\lambda_{j\nu}}\left(w_{j\nu}\overline{\Lambda}_2 + w(j,\nu:i,\mu)\Lambda_{i\mu}, \lambda_{j\nu}\right)\right)$$

with

$$\left(c_{\text{cus}}\left(P_{j\nu}\,|\,A_{j\nu}:P_{i\mu}\,|\,A_{i\mu}:w(j,\nu:i,\mu):\Lambda_{i\mu}\right)c_{\text{cus}}\left(P_{i\mu}\,|\,A_{i\mu}:P_1\,|\,A_1:w_{i\mu}:\Lambda_1\right)\Phi_1,\right.$$
$$\left. c_{\text{cus}}\left(P_{j\nu}\,|\,A_{j\nu}:P_2\,|\,A_2:w_{j\nu}:\Lambda_2\right)\Phi_2\right),$$

is a holomorphic function of $\Lambda_{i\mu}$ in the tube over $\mathcal{T}_{P_{i\mu}}(\check{\mathfrak{a}}_{i\mu})$. It is of interest, although perhaps not of importance, that these conclusions have been reached with no assumption whatsoever on $\mathbf{H}$. To make further progress, however, it will at last be necessary to impose a condition on $\mathbf{H}$. Before doing this, we shall indicate the next step in the analysis. We are summing over triples $(i, \mu, w_{i\mu})$. Fix i and μ — then we intend to prove that if

$$w_{i\mu} \neq I\left(P_{i\mu}\,|\,A_{i\mu}:P_1\,|\,A_1\right),$$

that is, if

$$w_{i\mu}\left(\mathcal{C}_{P_1}(\mathfrak{a}_1)\right) \neq \mathcal{C}_{P_{i\mu}}(\mathfrak{a}_{i\mu}),$$

then

$$\int_{\text{Re}(\Lambda_{i\mu})=\Lambda^0_{i\mu}} \cdots = 0,$$

provided $\mathbf{H}$ is suitably restricted.

Here is the condition on $\mathbf{H}$. Fix $\mathbf{H}_0$ in $\mathfrak{a}$ — then it will be supposed that $\mathbf{H} \lessdot \mathbf{H}_{00}$ where

$$\mathbf{H}_{00} = \mathbf{H}_0 - t\mathbf{H}_\rho,$$

t a large real number determined via the following condsiderations. Replacing $\mathbf{H}$ by $\mathbf{H}_0$ in the formulae supra, put

$$\begin{cases} H^0_{i\mu} = I_{P_{i\mu}}(\mathbf{H}_0) \\ H^0_{j\nu} = I_{P_{j\nu}}(\mathbf{H}_0). \end{cases}$$

Then, qua a function of $\Lambda_{i\mu}$, the argument in the exponential is given by

$$\left\langle H^0_{j\nu}, w(j,\nu:i,\mu)\Lambda_{i\mu}\right\rangle - \left\langle H^0_{i\mu}, \Lambda_{i\mu}\right\rangle$$

where, without loss of generality, we may assume that the centralizer of $w(j, \nu : i, \mu)$ in $\check{\mathfrak{a}}_{i\mu}$ contains no dual roots. If now $\mathbf{H}_{00}$ is substituted for $\mathbf{H}_0$, then we obtain, accordingly,

$$\left\langle H_{j\nu}^0, w(j, \nu : i, \mu)\Lambda_{i\mu}\right\rangle - \left\langle H_{i\mu}^0, \Lambda_{i\mu}\right\rangle$$
$$-t\left(\left\langle H_{\rho_{j\nu}}, w(j, \nu : i, \mu)\Lambda_{i\mu}\right\rangle - \left\langle H_{\rho_{i\mu}}, \Lambda_{i\mu}\right\rangle\right)$$

where (cf. §6)

$$w(j, \nu : i, \mu)^{-1}\rho_{j\nu} - \rho_{i\mu} \in -\mathcal{G}_{P_{i\mu}}(\check{\mathfrak{a}}_{i\mu}).$$

Owing to an elementary estimate, due to Langlands **[2.b]**, there is a positive constant C and an element

$$H_{w(j, \nu : i, \mu)} \in \mathfrak{a}_{i\mu}$$

such that

$$\| c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P_{i\mu} \mid A_{i\mu} : w(j, \nu : i, \mu) : \Lambda_{i\mu}\right)\|_{\text{OP}}$$

is bounded by

$$C \cdot \exp\left(\left\langle H_{w(j, \nu : i, \mu)}, \operatorname{Re}(\Lambda_{i\mu}) - \rho_{i\mu}\right\rangle\right)$$

times

$$\left(1 / \prod_{\lambda_{i\mu}} \left(\operatorname{Re}(\Lambda_{i\mu}) + \rho_{i\mu}, \lambda_{i\mu}\right)\right).$$

We can and will assume that

$$\operatorname{Re}(\Lambda_{i\mu}) \in -\frac{3}{2}\rho_{i\mu} - \mathcal{C}_{P_{i\mu}}(\check{\mathfrak{a}}_{i\mu}).$$

Redefining the constant C, we then have the majorization

$$\| c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P_{i\mu} \mid A_{i\mu} : w(j, \nu : i, \mu) : \Lambda_{i\mu}\right)\|_{\text{OP}}$$
$$\leq C \cdot \exp\left(\left\langle H_{w(j, \nu : i, \mu)}, \operatorname{Re}(\Lambda_{i\mu})\right\rangle\right).$$

All this leads, therefore, to the estimate

$$\left| \exp\left(\left\langle I_{P_{i\mu}}(\mathbf{H}_{00}), -\Lambda_{i\mu}\right\rangle\right) \cdot \exp\left(\left\langle I_{P_{j\nu}}(\mathbf{H}_{00}), w(j, \nu : i, \mu)\Lambda_{i\mu}\right\rangle\right) \right|$$
$$\times \| c_{\text{cus}}\left(P_{j\nu} \mid A_{j\nu} : P_{i\mu} \mid A_{i\mu} : w(j, \nu : i, \mu) : \Lambda_{i\mu}\right)\|_{\text{OP}}$$
$$\leq C \cdot \exp\Big(\Big\langle w(j, \nu : i, \mu)^{-1} H_{j\nu}^0 - H_{i\mu}^0 + H_{w(j, \nu : i, \mu)}$$
$$+ t\left(H_{\rho_{i\mu}} - w(j, \nu : i, \mu)^{-1} H_{\rho_{j\nu}}\right), \operatorname{Re}(\Lambda_{i\mu})\Big\rangle\Big).$$

Bearing in mind that

$$H_{\rho_{i\mu}} - w(j,\nu:i,\mu)^{-1}H_{\rho_{j\nu}} \in \mathcal{D}_{P_{i\mu}}(\mathfrak{a}_{i\mu}),$$

we now fix $\mathbf{H}_{00}$ by requiring that t be large enough to secure

$$w(j,\nu:i,\mu)^{-1}H_{j\nu}^{0} - H_{i\mu}^{0} + H_{w(j,\nu:i,\mu)}$$
$$+t\left(H_{\rho_{i\mu}} - w(j,\nu:i,\mu)^{-1}H_{\rho_{j\nu}}\right) \in \mathcal{D}_{P_{i\mu}}(\mathfrak{a}_{i\mu})$$

for all i, μ and j, ν. Thanks to the remarks following Lemma 6.7, if $\mathbf{H} \lessdot \mathbf{H}_{00}$, then

$$\operatorname{Re}\left(\left\langle I_{P_{j\nu}}(\mathbf{H}), w(j,\nu:i,\mu)\Lambda_{i\mu}\right\rangle - \left\langle I_{P_{i\mu}}(\mathbf{H}), \Lambda_{i\mu}\right\rangle\right)$$
$$\leq \operatorname{Re}\left(\left\langle I_{P_{j\nu}}(\mathbf{H}_{00}), w(j,\nu:i,\mu)\Lambda_{i\mu}\right\rangle - \left\langle I_{P_{i\mu}}(\mathbf{H}_{00}), \Lambda_{i\mu}\right\rangle\right),$$

implying that the domination is controlled by $\mathbf{H}_{00}$ alone, the decay being, in fact, exponential in $\operatorname{Re}(\Lambda_{i\mu})$.

The formula above for the inner product of

$$Q^{\mathbf{H}}E(P_1 \mid A_1 : \Phi_1 : \Lambda_1 : ?)$$

with

$$Q^{\mathbf{H}}E(P_2 \mid A_2 : \Phi_2 : \Lambda_2 : ?)$$

has been obtained under the supposition that Λ_1 and Λ_2 are sufficiently negative, the precise sense in which this is so being a function of the radius R of the ambient tube (R fixed per the preceding agreements). In addition, to ensure holomorphicity in $\Lambda_{i\mu}$ of [...], we saw earlier that it was necessary to restrict Λ_2 to a certain dense, open subset of its initial domain of definition. Once these choices have been made, an expression is produced, an expression which is then open to modification, subject, of course, to the requisite justifications. This is where the condition on $\mathbf{H}$ comes in. Assuming that $\mathbf{H} \lessdot \mathbf{H}_{00}$, we shall prove, under the conditions mentioned earlier, that

$$\int_{\operatorname{Re}(\Lambda_{i\mu}) = \Lambda_{i\mu}^{0}} \cdots = 0$$

by shifting the contour of integration

In this connection, there is a little lemma which should be recalled as it will be helpful (cf. [**3.a**]).

LEMMA. *Let $\mathcal{S}$ be a connected, open region in $\mathbf{R}^n$. Let F be a holomorphic function in the tube over $\mathcal{S}$ such that for each bounded line segment σ in $\mathcal{S}$,*

$$F \in L^1(\sigma \times \mathbf{R}^n).$$

Then: The integral

$$\int_{\{x\}\times\mathbf{R}^n} F$$

is independent of the choice of x in $\mathcal{S}$.

Thanks to our hypothesis on $w_{i\mu}$, there is a $\lambda_{i\mu}$ such that $w_{i\mu}^{-1}\lambda_{i\mu} < 0$. Move the contour of integration from $\Lambda_{i\mu}^0$ to $\Lambda_{i\mu}^0 - t\lambda^{i\mu}(t \geq 0, t \to +\infty)$. That this is permissible is a consequence of the fact that

$$\exp(\ldots)[\ldots]$$

is holomorphic and none of the terms in the product vanish except, perhaps, for the one corresponding to $\lambda_{i\mu}$, but, for the one corresponding to $\lambda_{i\mu}$,

$$\begin{aligned} &\operatorname{Re}(w_{i\mu}\Lambda_1, \lambda_{i\mu}) > 0 \\ \Rightarrow \\ &\operatorname{Re}\big(w_{i\mu}\Lambda_1 - (\Lambda_{i\mu}^0 - t\lambda^{i\mu}), \lambda_{i\mu}\big) \\ &= \operatorname{Re}(w_{i\mu}\Lambda_1, \lambda_{i\mu}) - (\Lambda_{i\mu}^0, \lambda_{i\mu}) + t \\ &\geq |\operatorname{Re}(w_{i\mu}\Lambda_1, \lambda_{i\mu})| - |(\Lambda_{i\mu}^0, \lambda_{i\mu})| > 0, \end{aligned}$$

so it does not hit a zero either. Our lemma then tells us that the integral does not depend on t. To conclude that it is null, we can therefore let $t \to +\infty$. Since the integrand evidently eventually admits an L^1-majorant, hence $\to 0$ dominatedly, it follows that

$$\int_{\operatorname{Re}(\Lambda_{i\mu}) = \Lambda_{i\mu}^0} \cdots = 0,$$

as desired.

There is a unique i such that P_1 is G-conjugate to P_i and a unique μ such that P_1 is Γ-conjugate to $P_{i\mu}$. In the sum over the triples $(i, \mu, w_{i\mu})$, only the terms corresponding to this particular i and μ survive, there being a contribution when

$$w_{i\mu} = I(P_{i\mu} | A_{i\mu} : P_1 | A_1).$$

Changing the notation, we can then say that, under the standing hypotheses on Λ_1, Λ_2 and $\mathbf{H}$, the inner product

$$\left(Q^{\mathbf{H}}E(P_1 \mid A_1 : \Phi_1 : \Lambda_1 : ?),\, Q^{\mathbf{H}}E(P_2 \mid A_2 : \Phi_2 : \Lambda_2 : ?)\right)$$

is equal to

$$\frac{\operatorname{vol}(\mathcal{C})^2}{(2\pi)^l} \cdot \int_{\operatorname{Re}(\Lambda)=\Lambda_1^0} \exp\left(\left\langle I_{P_1}(\mathbf{H}),\, \Lambda_1 - \Lambda \right\rangle\right) \cdot \left(1 \Big/ \prod_{i=1}^{l} \left(\Lambda_1 - \Lambda,\, \lambda_i^1\right)\right)[\ldots] \mid d\Lambda \mid$$

where $[\ldots]$ is the sum

$$\sum_{i=1}^{r} \sum_{\mu=1}^{r_i} \sum_{w_{i\mu:2} \in W(A_{i\mu},\, A_2)} \sum_{w_{i\mu:1} \in W(A_{i\mu},\, A_1)}$$

of the product of

$$\exp\left(\left\langle I_{P_{i\mu}}(\mathbf{H}),\, w_{i\mu:2}\overline{\Lambda}_2 + w_{i\mu:1}\Lambda \right\rangle\right) \times \left(1 \Big/ \prod_{\lambda_{i\mu}} \left(w_{i\mu:2}\overline{\Lambda}_2 + w_{i\mu:1}\Lambda,\, \lambda_{i\mu}\right)\right)$$

with

$$\left(c_{\mathrm{cus}}\left(P_{i\mu} \mid A_{i\mu} : P_1 \mid A_1 : w_{i\mu:1} : \Lambda\right)\Phi_1,\, c_{\mathrm{cus}}\left(P_{i\mu} \mid A_{i\mu} : P_2 \mid A_2 : w_{i\mu:2} : \Lambda_2\right)\Phi_2\right),$$

in toto, a holomorphic function of Λ. We shall evaluate the integral by shifting the contours and computing residues.

The integral itself is taken over $\Lambda_1^0 + \sqrt{-1}\,\check{\mathfrak{a}}_1$. Pass to coordinates by means of the change of variables

$$\Lambda = \Lambda_1^0 + \sum_{i=1}^{l} z^i \lambda_1^i \qquad (z^i \in \mathbf{C}).$$

The corresponding Jacobian is

$$\left| \det\left[\left(\lambda_1^i,\, \lambda_1^j\right)\right] \right|^{1/2},$$

the inverse of

$$\left| \det\left[\left(\lambda_i^1,\, \lambda_j^1\right)\right] \right|^{1/2},$$

which is nothing more than vol($\mathcal{C}$). In these parameters, the term which will contribute residues is

$$\left(1/\prod_{i=1}^{l}\left[(\Lambda_1 - \Lambda_1^0, \lambda_i^1) - z^i\right]\right),$$

the integration being carried out according to the scheme

$$\int_{-\sqrt{-1}\,\infty}^{+\sqrt{-1}\,\infty}\cdots\int_{-\sqrt{-1}\,\infty}^{+\sqrt{-1}\,\infty} dz^l \ldots dz^1.$$

Now move the first line of the integration to

$$\int_{C-\sqrt{-1}\,\infty}^{C+\sqrt{-1}\,\infty} \qquad (C \ll 0).$$

Using an argument similar to that employed above, we see that the integral with respect to z^1 tends to 0 as $C \to -\infty$. We are therefore left with the residue which occurs at

$$(\Lambda_1 - \Lambda_1^0, \lambda_1^1).$$

But then we must evaluate

$$-\frac{\mathrm{vol}(\mathcal{C})}{(2\pi)^{l-1}}\cdot\int_{-\sqrt{-1}\,\infty}^{+\sqrt{-1}\,\infty}\cdots\int_{-\sqrt{-1}\,\infty}^{+\sqrt{-1}\,\infty}\{?\}\, dz^l \ldots dz^2$$

where $\{?\}$ is

$$\left(1/\prod_{i=2}^{l}\left[(\Lambda_1 - \Lambda_1^0, \lambda_i^1) - z^i\right]\right)\cdot[\ldots],$$

the Λ-variable implicit in $[\ldots]$ having the value

$$\Lambda_1^0 + (\Lambda_1 - \Lambda_1^0, \lambda_1^1) + \sum_{i=2}^{l} z^i\lambda_1^i.$$

Repeat the procedure per $z^2,\ldots,z^l$ — then Λ_1^0 will cancel in the end.

We thus arrive at the following conclusion, to wit: The inner product

$$\left(Q^{\mathbf{H}}E(P_1\,|\,A_1:\Phi_1:\Lambda_1:?), Q^{\mathbf{H}}E(P_2\,|\,A_2:\Phi_2:\Lambda_2:?)\right)$$

is equal to the sum

$$(-1)^l\cdot\mathrm{vol}(\mathcal{C})\cdot\sum_{i=1}^{r}\sum_{\mu=1}^{r_i}\sum_{w_{i\mu:2}\in W(A_{i\mu},A_2)}\sum_{w_{i\mu.1}\in W(A_{i\mu},A_1)}$$

of the product of

$$\exp\left(\left\langle I_{P_{i\mu}}(\mathbf{H}), w_{i\mu:2}\overline{\Lambda}_2 + w_{i\mu:1}\Lambda_1\right\rangle\right)$$
$$\times\left(1/\prod_{\lambda_{i\mu}}\left(w_{i\mu:2}\overline{\Lambda}_2 + w_{i\mu:1}\Lambda_1, \lambda_{i\mu}\right)\right)$$

with

$$\left(c_{\text{cus}}\left(P_{i\mu} \mid A_{i\mu} : P_1 \mid A_1 : w_{i\mu:1} : \Lambda_1\right)\Phi_1, c_{\text{cus}}\left(P_{i\mu} \mid A_{i\mu} : P_2 \mid A_2 : w_{i\mu:2} : \Lambda_2\right)\Phi_2\right).$$

While this result has been established with Λ_1 and Λ_2 subject to certain restrictions (and $\mathbf{H}$ too, of course), it is clear that both sides of the equation are meromorphic in $(\Lambda_1, \overline{\Lambda}_2)$. In other words, we have proved the following theorem.

THEOREM 9.6. *Fix* $\mathbf{H}_0$ *in* $\mathfrak{a}$ — *then there exists* $\mathbf{H}_{00} \lessdot \mathbf{H}_0$ *such that for all* $\mathbf{H} \lessdot \mathbf{H}_{00}$ *and*

$$\forall\, \mathcal{C}, \forall \begin{cases} P_1 \in \mathcal{C} \\ P_2 \in \mathcal{C} \end{cases}$$

the inner product

$$\left(Q^{\mathbf{H}}E(P_1 \mid A_1 : \Phi_1 : \Lambda_1 : ?), Q^{\mathbf{H}}E(P_2 \mid A_2 : \Phi_2 : \Lambda_2 : ?)\right)$$

is equal to the sum

$$(-1)^l \cdot \text{vol}(\mathcal{C}) \cdot \sum_{i=1}^{r} \sum_{\mu=1}^{r_i} \sum_{w_{i\mu.2} \in W(A_{i\mu}, A_2)} \sum_{w_{i\mu:1} \in W(A_{i\mu}, A_1)}$$

of the product of

$$\exp\left(\left\langle I_{P_{i\mu}}(\mathbf{H}), w_{i\mu:2}\overline{\Lambda}_2 + w_{i\mu:1}\Lambda_1\right\rangle\right)$$
$$\times\left(1/\prod_{\lambda_{i\mu}}\left(w_{i\mu:2}\overline{\Lambda}_2 + w_{i\mu:1}\Lambda_1, \lambda_{i\mu}\right)\right)$$

with

$$\left(c_{\text{cus}}\left(P_{i\mu} \mid A_{i\mu} : P_1 \mid A_1 : w_{i\mu:1} : \Lambda_1\right)\Phi_1, c_{\text{cus}}\left(P_{i\mu} \mid A_{i\mu} : P_2 \mid A_2 : w_{i\mu:2} : \Lambda_2\right)\Phi_2\right).$$

[Note: It is a question here of special split components.]

There is also a daggered version of this theorem which could be stated formally as Theorem 9.6 (bis). An informal statement will suffice, however.

Suppose that

$$(P, S; A) \succcurlyeq \begin{cases} (P_1', S_1'; A_1') \\ (P_2', S_2'; A_2'), \end{cases}$$

it being assumed that $('P_1, 'S_1)$ and $('P_2, 'S_2)$ are associate, belonging to $'\mathcal{C}$, say. Introduce the partial truncation operator $Q_P^{\mathbf{H}}$ (cf. §8) — then attached to

$$\begin{cases} \Phi_1' \\ \Phi_2' \end{cases}$$

are partial Eisenstein series

$$\begin{cases} E('P_1 \mid 'A_1 : '\Phi_1 : '\Lambda_1 : ?) \\ E('P_2 \mid 'A_2 : '\Phi_2 : '\Lambda_2 : ?), \end{cases}$$

themselves functions on $G/\Gamma \cap P$. The commutative diagram connecting $Q_P^{\mathbf{H}}$ and $1 \times Q^{I_M(\mathbf{H})} \times 1$ then allows us to assert that for $\mathbf{H}$ suitably restricted, the inner product

$$\left(Q_P^{\mathbf{H}} E('P_1 \mid 'A_1 : '\Phi_1 : '\Lambda_1 : ?), Q_P^{\mathbf{H}} E('P_2 \mid 'A_2 : '\Phi_2 : '\Lambda_2 : ?)\right)$$

is equal to the sum

$$(-)^{'l} \cdot \operatorname{vol}('\mathcal{C}) \cdot \sum_{i=1}^{'r} \sum_{\mu=1}^{'r_i} \sum_{'w_{i\mu} \cdot 2 \in W('A_{i\mu}, 'A_2)} \sum_{'w_{i\mu} : 1 \in W('A_{i\mu}, 'A_1)}$$

of the product of

$$\exp\left(\left\langle I_{'P_{i\mu}}(\mathbf{H}), 'w_{i\mu : 2}{}'\overline{\Lambda}_2 + 'w_{i\mu : 1}{}'\Lambda_1 \right\rangle\right)$$
$$\times \left(1 / \prod_{'\lambda_{i\mu}} \left('w_{i\mu : 2}{}'\overline{\Lambda}_2 + 'w_{i\mu : 1}{}'\Lambda_1, '\lambda_{i\mu}\right)\right)$$

with

$$\left(c_{\text{cus}}('P_{i\mu} \mid 'A_{i\mu} : 'P_1 \mid 'A_1 : 'w_{i\mu : 1} : '\Lambda_1)'\Phi_1,\right.$$
$$\left. c_{\text{cus}}('P_{i\mu} \mid 'A_{i\mu} : 'P_2 \mid 'A_2 : 'w_{i\mu : 2} : '\Lambda_2)'\Phi_2\right).$$

REMARK. In the next paper in this series, using Proposition 9.5 and methods from the theory of Eisenstein systems, we shall give a completely different proof of Theorem 9.6 (in a generalized form).

10. Recapitulation. The purpose of this section will be to provide a capsule overview of certain aspects of the present paper by way of a technical summary which can then serve as a convenient reference for later work. In so doing, we shall set up a list of axioms and show how the truncation operator admits a characterization in terms of them.

Let us first recall the definition of $\mathfrak{a}$. Thus fix a set of representatives

$$\{(P_m^{\max}, S_m^{\max})\}$$

for the Γ-conjugacy classes of maximal Γ-cuspidal split parabolic subgroups of G. Let $A_m^{\max}$ be the special split component of $(P_m^{\max}, S_m^{\max})$ — then, by definition,

$$\mathfrak{a} = \bigoplus_m \mathfrak{a}_m^{\max}.$$

If now (P, S) is a Γ-cuspidal split parabolic subgroup of G with special split component A, then we define a map

$$I_P\colon \mathfrak{a} \to \mathfrak{a}$$

by requiring that when

$$(P_\mu, S_\mu; A_\mu) \succcurlyeq (P, S; A),$$

P maximal Γ-cuspidal so that

$$\gamma_\mu P_\mu \gamma_\mu^{-1} = P_{m(\mu)}^{\max}$$

for some $\gamma_\mu \in \Gamma$ and some index $m(\mu)$, the orthogonal projection of

$$I_P(\mathbf{H}) \qquad (\mathbf{H} \in \mathfrak{a})$$

onto $\mathfrak{a}_\mu$ is

$$I(P_\mu \mid A_\mu\colon P_{m(\mu)}^{\max} \mid A_{m(\mu)}^{\max}) + H_{P_\mu|A_\mu}(\gamma_\mu).$$

On the other hand, if

$$(P, S; A) \succcurlyeq (P', S'; A'),$$

then

$$I_{'P}\colon \mathfrak{a} \to {}'\mathfrak{a}$$

is the dotted arrow rendering the triangle

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{I_{P'}} & \mathfrak{a}' \\ & \searrow & \downarrow \perp \\ & & {}'\mathfrak{a} \end{array} \qquad (\mathfrak{a}' = {}'\mathfrak{a} \oplus \mathfrak{a})$$

commutative.

It should also be kept in mind that $\mathfrak{a}$ comes supplied with a natural ordering, namely given $\mathbf{H}_1, \mathbf{H}_2$ in $\mathfrak{a}$, write

$$\mathbf{H}_1 < \mathbf{H}_2$$

if for every Γ-cuspidal split parabolic subgroup (P, S) of G with special split component A it is true that

$$I_P(\mathbf{H}_2) \in I_P(\mathbf{H}_1) + \mathcal{C}_P(\mathfrak{a}).$$

This relation partially orders and, in fact, directs $\mathfrak{a}$.

The canonically defined map

$$I_M\colon \mathfrak{a} \to \mathfrak{a}_M$$

has a cofinal image and is order preserving.

Proceeding axiomatically, we shall suppose that there is attached to each Γ-cuspidal P and each $\mathbf{H} \in \mathfrak{a}$ a linear operator

$$Q_P^{\mathbf{H}}\colon S(G/\Gamma \cap P) \to S(G/\Gamma \cap P)$$

subject to the following conditions, all of which are possessed, of course, by the partial truncation operator (or by the truncation operator itself if $P = G$).

AXIOM I. *$Q_P^{\mathbf{H}}(S(G/\Gamma \cap P)) \subset S(G/(\Gamma \cap P)\cdot N)$ and the triangle*

$$\begin{array}{ccc} S(G/\Gamma \cap P) & \overset{f\mapsto f^P}{\longrightarrow} & S(G/(\Gamma \cap P)\cdot N) \\ Q_P^{\mathbf{H}}\searrow & & \swarrow Q_P^{\mathbf{H}} \\ & S(G/(\Gamma \cap P)\cdot N) & \end{array}$$

commutes.

AXIOM II. *The diagram*

$$\begin{array}{ccc} S(G/(\Gamma \cap P)\cdot N) & \overset{\mathfrak{F}_{P,N}}{\longrightarrow} & S(K \times M/\Gamma_M \times A) \\ Q_P^{\mathbf{H}}\downarrow & & \downarrow 1 \times Q^{I_M(\mathbf{H})} \times 1 \\ S(G/(\Gamma \cap P)\cdot N) & \underset{\mathfrak{F}_{P,N}}{\longrightarrow} & S(K \times M/\Gamma_M \times A) \end{array}$$

commutes.

AXIOM III. $\forall f \in S(G/\Gamma \cap P)$

$$f^P(x) = \sum_{P' \in \mathrm{Dom}_\Gamma(P)} \chi_{'P,'A:\mathcal{C}}\big(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x)\big)\cdot Q_{P'}^{\mathbf{H}} f(x).$$

We pause at this point to make two comments. First, Axioms I and II imply that if $Q^{\mathbf{H}}$ is known for any reductive group, then so are the $Q_P^{\mathbf{H}}$, this being the reason that the focus is on $Q^{\mathbf{H}}$ alone in what follows. Second, Axiom III implies that the $Q_P^{\mathbf{H}}$ are uniquely determined as can be seen by downward induction on rank(P), noting that $Q_P^{\mathbf{H}} f = f^P$ when P is Γ-percuspidal while, for all other P,

$$Q_P^{\mathbf{H}} f(x) = f^P(x) - \sum_{\substack{P' \in \mathrm{Dom}_\Gamma(P) \\ P' \neq P}} \chi_{'P,'A:\mathcal{C}}\big(I_{'P}(\mathbf{H}) - H_{'P|'A}(m_x)\big) \cdot Q_{P'}^{\mathbf{H}} f(x).$$

AXIOM IV. *$\forall\, r < -1 \exists\, r' < r$ such that*

$$Q^{\mathbf{H}}(S_r(G/\Gamma)) \subset S_{r'}(G/\Gamma),$$

the operation

$$Q^{\mathbf{H}}\colon S_r(G/\Gamma) \to S_{r'}(G/\Gamma)$$

being continuous. Furthermore, $\forall\, \mathbf{H}_0 \in \mathfrak{a}$,

$$\{Q^{\mathbf{H}}\colon \mathbf{H} < \mathbf{H}_0\}$$

is equicontinuous.

AXIOM V. *If $f \in R(G/\Gamma)$, then for all $g \in S(G/\Gamma)$,*

$$(Q^{\mathbf{H}} f)\cdot g \in L^1(G/\Gamma)$$

and

$$(Q^{\mathbf{H}} f, g) = (f, Q^{\mathbf{H}} g).$$

AXIOM VI. *If f has compact support, then so does $Q^{\mathbf{H}} f$. Moreover, if C is a compact subset of G/Γ, then there exists $\mathbf{H}(C) \in \mathfrak{a}$ such that*

$$\mathbf{H} < \mathbf{H}(C) \Rightarrow Q^{\mathbf{H}} f = f \quad \text{on } C.$$

The remaining axioms will be true only on a non-empty subset $\mathfrak{a}_Q$ of $\mathfrak{a}$ which is cofinal in $\mathfrak{a}$ (per $<$), hence is itself directed. Two properties are required.

(i) $\forall\, P$, $I_M(\mathfrak{a}_Q) \subset (\mathfrak{a}_M)_Q$.

(ii) $\forall\, \mathbf{H}_1, \forall\, \mathbf{H}_2$,

$$\mathbf{H}_1 \lessdot \mathbf{H}_2, \qquad \mathbf{H}_2 \in \mathfrak{a}_Q \Rightarrow \mathbf{H}_1 \in \mathfrak{a}_Q.$$

In connection with (ii), let us remind ourselves that given $\mathbf{H}_1, \mathbf{H}_2$ in $\mathfrak{a}$, we write

$$\mathbf{H}_1 \lessdot \mathbf{H}_2$$

if there exists an $H_0 \in \mathcal{C}_{P_0}(\mathfrak{a}_0)$ such that

$$I(P_0 | A_0 : P_{i_0} | A_{i_0})\big(I_{P_{i_0}}(\mathbf{H}_2) - I_{P_{i_0}}(\mathbf{H}_1)\big) = H_0$$

for all $i_0 = 1,\dots,r_0$. This relation partially orders $\mathfrak{a}$ and

$$\mathbf{H}_1 \lessdot \mathbf{H}_2 \Rightarrow \mathbf{H}_1 < \mathbf{H}_2.$$

The significance of (i) is that it implies that again we need only deal explicitly with the $Q^{\mathbf{H}}$.

AXIOM VII. $\forall\, \mathbf{H} \in \mathfrak{a}_Q$,

$$Q^{\mathbf{H}}(R(G/\Gamma)) \subset R(G/\Gamma).$$

AXIOM VIII. $\forall\, \mathbf{H} \in \mathfrak{a}_Q$,

$$Q^{\mathbf{H}} \circ Q^{\mathbf{H}} = Q^{\mathbf{H}}.$$

Furthermore, $\forall\, \mathbf{H}'$, $\forall\, \mathbf{H}''$,

$$\mathbf{H}'' < \mathbf{H}', \mathbf{H}' \in \mathfrak{a}_Q \Rightarrow Q^{\mathbf{H}''} \circ Q^{\mathbf{H}'} = Q^{\mathbf{H}'}.$$

The axioms thus entail that $Q^{\mathbf{H}}(\mathbf{H} \in \mathfrak{a}_Q)$ defines an orthogonal projection on $L^2(G/\Gamma)$, there being coincidence on

$$S(G/\Gamma) \cap L^2(G/\Gamma).$$

In addition,

$$\lim_{\mathbf{H} \to -\infty} Q^{\mathbf{H}} = \mathrm{ID} \qquad (\mathbf{H} \in \mathfrak{a}_Q)$$

in the strong operator topology.

AXIOM IX. $\forall\, \mathbf{H} \in \mathfrak{a}_Q$,

$$Q^{\mathbf{H}}(S_r^{\infty}(G/\Gamma)) \subset R(G/\Gamma).$$

Apropos of this axiom, observe that

$$Q^{\mathbf{H}}\colon S_r^{\infty}(G/\Gamma) \to R(G/\Gamma)$$

is continuous, as follows from the closed graph theorem.

AXIOM X. *Let* (P_1, S_1), (P_2, S_2) *be* Γ*-cuspidal split parabolic subgroups of* G *with special split components* A_1, A_2. *Let*

$$\begin{cases} \delta_1, \delta_2 \text{ be } K\text{-types} \\ \mathcal{O}_1, \mathcal{O}_2 \text{ be } M_1, M_2\text{-types}. \end{cases}$$

Let

$$\begin{cases}\Phi_1 \in \mathcal{E}_{\text{cus}}(\delta_1, \mathcal{O}_1)\\ \Phi_2 \in \mathcal{E}_{\text{cus}}(\delta_2, \mathcal{O}_2).\end{cases}$$

Then, for all $\mathbf{H} \in \mathfrak{a}_Q$,

$$\big(Q^{\mathbf{H}}E(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : ?), Q^{\mathbf{H}}E(P_2 \,|\, A_2 : \Phi_2 : \Lambda_2 : ?)\big)$$

is null unless (P_1, S_1) *and* (P_2, S_2) *are associate, in* $\mathcal{C}$ (say), $\delta_1 = \delta_2$, *and* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are associate, in which case*

$$\big(Q^{\mathbf{H}}E(P_1 \,|\, A_1 : \Phi_1 : \Lambda_1 : ?), Q^{\mathbf{H}}E(P_2 \,|\, A_2 : \Phi_2 : \Lambda_2 : ?)\big)$$

is equal to the sum

$$(-1)^{l} \cdot \operatorname{vol}(\mathcal{C}) \cdot \sum_{\iota=1}^{r} \sum_{\mu=1}^{r_\iota} \sum_{w_{\iota\mu\,:\,2} \in W(A_{\iota\mu}, A_2)} \sum_{w_{\iota\mu\,:\,1} \in W(A_{\iota\mu}, A_1)}$$

of the product of

$$\exp\Big(\Big\langle I_{P_{\iota\mu}}(\mathbf{H}), w_{i\mu\,:\,2}\overline{\Lambda}_2 + w_{i\mu\,:\,1}\Lambda_1\Big\rangle\Big)$$

$$\times \left(1 \Big/ \prod_{\lambda_{\iota\mu}} \left(w_{\iota\mu\,:\,2}\overline{\Lambda}_2 + w_{i\mu\,:\,1}\Lambda_1, \lambda_{i\mu}\right)\right)$$

with

$$\big(c_{\text{cus}}(P_{\iota\mu} \,|\, A_{\iota\mu} : P_1 \,|\, A_1 : w_{\iota\mu\,:\,1} : \Lambda_1)\Phi_1, c_{\text{cus}}(P_{i\mu} \,|\, A_{i\mu} : P_2 \,|\, A_2 : w_{i\mu\,:\,2} : \Lambda_2)\Phi_2\big).$$

We stress once more that these axioms are actual properties of the truncation operator. Indeed, they characterize it in the following sense. Fix an $\mathbf{H} \in \mathfrak{a}_Q$ and suppose that we are given a linear operator

$$T\colon S(G/\Gamma) \to S(G/\Gamma)$$

satisfying Axioms IV–X — then,

$$\forall f \in S(G/\Gamma),$$
$$Tf = Q^{\mathbf{H}}f \quad \text{a.e.}$$

The proof hinges on the familiar principle that a rapidly decreasing function which is orthogonal to all Eisenstein series associated with cusp forms (including the case when $P = G$) must, of necessity, vanish a.e. This being so, let now E_1 and E_2 be cuspidal Eisenstein series — then

$$\begin{aligned}(TE_1, E_2) &= (TE_1, TE_2) \quad \text{(by VIII)}\\ &= \big(Q^{\mathbf{H}}E_1, Q^{\mathbf{H}}E_2\big) \quad \text{(by X)} = \big(Q^{\mathbf{H}}E_1, E_2\big) \quad \text{(by VIII)}.\end{aligned}$$

Changing the notation, it therefore follows that

$$TE = Q^{\mathbf{H}}E \quad \text{a.e.}$$

for any Eisenstein series E associated with a cusp form. The other axioms will be needed to force this conclusion for an arbitrary $f \in S(G/\Gamma)$. Let Θ_Φ be a wave-packet — then, $\forall\, E$,

$$\begin{aligned}(T\Theta_\Phi, E) &= (\Theta_\Phi, TE) \quad \text{(by V)}\\ &= \left(\Theta_\Phi, Q^{\mathbf{H}}E\right) \quad \text{(cf. supra)}\\ &= \left(Q^{\mathbf{H}}\Theta_\Phi, E\right) \quad \text{(by V)}.\end{aligned}$$

Hence

$$T\Theta_\Phi = Q^{\mathbf{H}}\Theta_\Phi \quad \text{a.e.}$$

But the axioms certainly imply that T defines an orthogonal projection on $L^2(G/\Gamma)$, there being coincidence on

$$S(G/\Gamma) \cap L^2(G/\Gamma).$$

Accordingly, since the wave-packets are dense in $L^2(G/\Gamma)$,

$$\begin{aligned}&\forall\, f \in S(G/\Gamma) \cap L^2(G/\Gamma),\\ &Tf = Q^{\mathbf{H}}f \quad \text{a.e.}\end{aligned}$$

Finally, write

$$S(G/\Gamma) = \bigcup_{r<-1} S_r(G/\Gamma)$$

and fix an $f \in S_r(G/\Gamma)$. Using the fundamental theorem of reduction, write

$$G = \bigcup_{i_0=1}^{r_0} \mathfrak{S}_{t_0,\, w_0}\kappa_{i_0} \cdot \Gamma$$

or still

$$G = \bigcup_{i_0=1}^{r_0} K \cdot A_{i_0}[t_0] \cdot \omega_{i_0} \cdot \Gamma.$$

Let

$$A_{i_0}^n[t_0] = \left\{a \in A_{i_0} \colon e^{-n} \le \xi_\lambda(a) \le t_0 \,\forall\, \lambda \in \Sigma_{P_{i_0}}^0(\mathfrak{g}, \mathfrak{a}_{i_0})\right\}.$$

Call C_n the image in G/Γ of the Γ-saturation $C_n(\Gamma)$ of

$$\bigcup_{i_0=1}^{r_0} K \cdot A_{i_0}^n[t_0] \cdot \omega_{i_0}$$

in G — then C_n is compact with $C_n \subset C_{n+1}$ and

$$G/\Gamma = \bigcup C_n.$$

On $\mathfrak{S}_{t_0, \omega_0}\kappa_{i_0}$,

$$|f| \leq C_f \cdot \Xi_{P_{i_0}}^r.$$

Suppose that $x \in \mathfrak{S}_{t_0, \omega_0}\kappa_{i_0}$ but $x \notin C_n(\Gamma)$ — then

$$x \notin K \cdot A_{i_0}^n[t_0] \cdot \omega_{i_0},$$

so

$$|f(x)| \leq C_f \cdot \Xi_{P_{i_0}}(x) \Xi_{P_{i_0}}(x)^{r-1} \leq C_f e^{-n} \cdot \Xi_{P_{i_0}}(x)^{r-1}.$$

This means that for all $x \in \mathfrak{S}_{t_0, \omega_0}\kappa_{i_0}$,

$$|f(x) - \chi_{C_n} f(x)| \leq C_f e^{-n} \cdot \Xi_{P_{i_0}}(x)^{r-1}.$$

Consequently,

$$\chi_{C_n} f \to f \quad \text{in } S_{r-1}(G/\Gamma).$$

However,

$$\chi_{C_n} f \in S(G/\Gamma) \cap L^2(G/\Gamma)$$

and, by what has been ascertained above,

$$T(\chi_{C_n} f) = Q^{\mathbf{H}}(\chi_{C_n} f) \quad \text{a.e.}$$

Thanks to IV, then,

$$Tf = Q^{\mathbf{H}} f \quad \text{a.e.},$$

as desired.

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