TRANSFORMATIONS OF CERTAIN SEQUENCES OF RANDOM VARIABLES BY GENERALIZED HAUSDORFF MATRICES

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Sufficient conditions are established for a generalized Hausdorff matrix to transform certain sequences of random variables into almost surely convergent sequences.

1. Introduction. Suppose that \( \{X_n\}(n = 0, 1, \ldots) \) is a sequence of random variables defined on a probability space \((\Omega, \mathcal{F}, P)\), and that \( A = \{a_{nk}\}(n, k = 0, 1, \ldots) \) is an infinite matrix. Let

\[
T_n = \sum_{k=0}^{\infty} a_{nk} X_k.
\]

The following theorem concerning the almost sure convergence to zero of the sequence \( \{T_n\} \) is due to Borwein [1].

**Theorem A.** If \( 1 < p \leq 2, 0 < M < \infty \) and

1. \( |X_n| \leq M \) a.s. for \( n = 0, 1, \ldots \),
2. \( \sum_{0 \leq i_1 < i_2 < \cdots < i_n} |E(X_{i_1}X_{i_2} \cdots X_{i_n})|^{p/(p-1)} \leq M^n \) for \( n = 1, 2, \ldots \),
3. \( \sum_{k=0}^{\infty} |a_{nk}| < \infty \) for \( n = 0, 1, \ldots \), and

\[
\lim_{n \to \infty} \log n \left( \sum_{k=0}^{\infty} |a_{nk}|^p \right)^{1/(p-n)} = 0,
\]

then \( T_n \to 0 \) a.s.

The sequence \( \{X_n\} \) is said to be multiplicative if the expectation \( E(X_{i_1}X_{i_2} \cdots X_{i_n}) = 0 \) whenever \( 0 \leq i_1 < i_2 < \cdots < i_n \); in particular, it is multiplicative if it is independent with \( EX_n = 0 \) for \( n = 0, 1, \ldots \). Condition (2) is trivially satisfied when \( \{X_n\} \) is multiplicative. The nature of Theorem A is clarified by comparison with Kolmogorov’s classical strong law of large numbers which states that if \( \{X_n\} \) is independent with \( EX_n = 0 \) for \( n = 0, 1, \ldots \), and if

\[
\sum_{k=0}^{\infty} \frac{EX_k^2}{(k + 1)^2} < \infty, \quad \text{then} \quad \frac{1}{n+1} \sum_{k=0}^{n} X_k \to 0 \quad \text{a.s.}
\]

We shall denote by \( \Gamma_p \) the set of matrices \( A \) such that \( T_n \to 0 \) a.s. whenever the sequence \( \{X_n\} \) satisfies conditions (1) and (2). Our primary
object in this paper is to establish conditions which are both sufficient and easy to verify for generalization Hausdorff matrices to be in $\Gamma_p$. Included in the class of generalized Hausdorff matrices are the matrices of such well-known methods of summability as the Cesàro, the Euler, and the weighted mean methods.

The matrix $A$ is said to have the \textit{Borel property} and we write $A \in (BP)$, if almost all sequences of zeros and ones are $A$-convergent to $1/2$. This amounts to (see \cite{5})

$$\frac{1}{2} \sum_{k=0}^{\infty} a_{nk} (1 - X_k) \to \frac{1}{2} \text{ a.s.}$$

when \{${X}_n$\} is the sequence of Rademacher functions on $\Omega = [0, 1]$ and $P$ is Lebesgue measure. Since, in this case, \{${X}_n$\} satisfies conditions (1) and (2), it follows that

if $\sum_{k=0}^{\infty} a_{nk}$ is convergent for $n = 0, 1, \ldots$ and $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1$, and if $A \in \Gamma_p$, then $A \in (BP)$.

**Generalized Hausdorff matrices.** Suppose in all that follows that $\lambda = \{\lambda_n\}$ is a sequence of real numbers satisfying

$$\lambda_0 \geq 0, \quad \lambda_n > 0 \quad \text{for } n = 1, 2, \ldots, \lambda_n \to \infty, \quad \sum_{r=1}^{\infty} \frac{1}{\lambda_n} = \infty,$$

and that $\alpha$ is a function of bounded variation on $[0, 1]$.

For $0 \leq k \leq n$, $0 < t \leq 1$, let

$$\lambda_{nk}(t) = \lambda_{k+1} \cdots \lambda_n \frac{1}{2 \pi i} \int_C \frac{t^2 \, dz}{(\lambda_k - z) \cdots (\lambda_n - z)};$$

$$\lambda_{nk}(0) = \lambda_{nk}(0 +),$$

$C$ being a positively sensed closed Jordan contour enclosing $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$. We observe the convention that products such as $\lambda_{k+1} \cdots \lambda_n = 1$ when $k = n$. Let

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t) \, d\alpha(t) \quad \text{for } 0 \leq k \leq n; \quad \lambda_{nk} = 0 \quad \text{for } k > n,$$

and denote the triangular matrix $\{\lambda_{nk}\}$ by $H(\lambda, \alpha)$. This is called a \textit{generalized Hausdorff matrix}.

Let

$$D_0 = (1 + \lambda_0) \, d_0 = 1,$$

$$D_n = \left(1 + \frac{1}{\lambda_1}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right) = (1 + \lambda_n) \, d_n \quad \text{for } n \geq 1.$$
Then, for $n \geq 0$,

$$D_n = \lambda_{n+1} d_{n+1} = \frac{\lambda_0}{1 + \lambda_0} + \sum_{k=0}^{n} d_k.$$

It is known (see [3]) that

$$(6) \quad 0 \leq \lambda_{nj}(t) \leq \sum_{k=0}^{n} \lambda_{nk}(t) \leq 1 \quad \text{for} \quad 0 \leq t \leq 1, \quad 0 \leq j \leq n,$$

$$(7) \quad \int_{0}^{1} \lambda_{nk}(t) \, dt = \frac{d_k}{D_n} \quad \text{for} \quad 0 \leq k \leq n,$$

$$(8) \quad \sum_{k=0}^{n} |\lambda_{nk}| \leq \int_{0}^{1} |\alpha(t)|.$$ 

Let

$$(9) \quad \rho_{nk} = \sum_{j=k}^{n} \frac{1}{\lambda_j}, \quad \sigma_{nk} = \left( \sum_{j=k}^{n} \frac{1}{\lambda_j^2} \right)^{1/2} \quad \text{for} \quad 1 \leq k \leq n.$$ 

We shall prove the following theorems.

**Theorem 1.** Let $M, m$ be positive constants. If $\alpha(0 +) = \alpha(0)$ and $\alpha(1-) = \alpha(1)$, and if $\lambda$ satisfies either

$$(10) \quad M \log \lambda_k \geq \lambda_{k+1} - \lambda_k \geq m \quad \text{for all sufficiently large} \quad k$$

or

$$(11) \quad M \geq \lambda_{k+1} - \lambda_k > 0 \quad \text{for all sufficiently large} \quad k \quad \text{and} \quad \log n / \sqrt[12]{\lambda_n} = o(1),$$

then $H(\lambda, \alpha) \in \Gamma_2$. If, in addition, $\alpha(1) - \alpha(0) = 1$, then $H(\lambda, \alpha) \in (BP)$.

**Theorem 2.** Let $\alpha(t) = \int_{0}^{t} \beta(u) \, du$ for $0 \leq t < 1$, and let $1 < p \leq 2$. If either

$$(12) \quad \beta \in L^p[0, 1] \quad \text{and} \quad \max_{0 \leq k \leq n} d_k \cdot \frac{\log n}{D_n} = o(1),$$

or

$$(13) \quad \beta \in L^\infty[0, 1] \quad \text{and} \quad \log n \left( \sum_{k=0}^{n} \left( \frac{d_k}{D_n} \right)^p \right)^{1/(p-1)} = o(1),$$

then $H(\lambda, \alpha) \in \Gamma_p$. If, in addition, $\{\lambda_n\}$ is non-decreasing and $\alpha(1) = 1$, then $H(\lambda, \alpha) \in (BP)$. 
It is known that $H(\lambda, \alpha) \in (BP)$ when $\alpha$ satisfies the conditions of Theorem 1 and $\lambda_n = n + c$, the case $c = 0$ of this result being due to Hill [6] and the case $c > 0$ to Liu and Rhoades [9]. On the other hand, Borwein and Cass [2] have shown that $H(\lambda, \alpha) \notin (BP)$ when $\alpha(t) = t$ and $\lambda_n = c \log(n + 1)$, $0 < c < 1/\log 4$. Borwein and Cass [2] have also shown Theorem 2 to hold in the case $p = 2$, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$.

2. Preliminary results.

**Lemma 1.** If $1 \leq k \leq n$, $0 < \lambda_k \leq \lambda_{k+1} \leq \cdots \leq \lambda_n$ and $0 \leq t \leq 1$, then

\[
\lambda_{nk}(t) \leq \frac{\sqrt{2}}{\lambda_k \sigma_{nk}}.
\]

**Proof.** Since $0 \leq \lambda_{nk}(t) \leq 1$, we may suppose that

\[(14) \quad \lambda_k^2 \sum_{j=k}^{n} \frac{1}{\lambda_j^2} > 2.\]

Jakimovski [6, Lemma 2.1] has shown that, for $u > 0$,

\[
\lambda_{nk}(e^{-u}) = \frac{1}{2\pi \lambda_k} \int_{-\infty}^{\infty} \frac{e^{iuv}}{\prod_{j=k}^{n}(1 + iv/\lambda_j)} dv,
\]

from which it follows that

\[
\lambda_{nk}(e^{-u}) \leq \frac{1}{2\pi \lambda_k} \int_{-\infty}^{\infty} \frac{dv}{\prod_{j=k}^{n}(1 + v^2/\lambda_j^2)^{1/2}}.
\]

Next, we have, by (14), that

\[
\prod_{j=k}^{n} \left(1 + \frac{v^2}{\lambda_j^2}\right) \geq 1 + v^2 \sum_{j=k}^{n} \frac{1}{\lambda_j^2} + \frac{v^4}{4} \sum_{r=k}^{n} \sum_{j=k}^{n} \frac{1}{\lambda_r^2} \left(\sum_{j=k}^{n} \frac{1}{\lambda_j^2} - \frac{1}{\lambda_r^2}\right)
\]

\[
\geq 1 + v^2 \sum_{j=k}^{n} \frac{1}{\lambda_j^2} + \frac{v^4}{4} \sum_{r=k}^{n} \sum_{j=k}^{n} \frac{1}{\lambda_r^2} \frac{1}{\lambda_j^2} = \left(1 + \frac{v^2}{2} \sigma_{nk}^2\right)^2.
\]

Hence, for $u > 0$,

\[
\lambda_{nk}(e^{-u}) \leq \frac{1}{2\pi \lambda_k} \int_{-\infty}^{\infty} \frac{dv}{1 + v^2 \sigma_{nk}^2/2} < \frac{\sqrt{2}}{\lambda_k \sigma_{nk}},
\]

and this completes the proof of Lemma 1.

The case $s = 0$, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ of the following lemma is due to Hausdorff [4].
LEMMA 2. Let \( \{\lambda_n\} \) be non-decreasing, and let \( s \) be a non-negative integer. Then

\[
\lim_{n \to \infty} \sum_{k=s}^{n} \lambda_{nk} = \begin{cases} 
\alpha(1) - \alpha(0 +) & \text{if } \lambda_s > 0, \\
\alpha(1) - \alpha(0) & \text{if } \lambda_s = 0;
\end{cases}
\]

and

\[
\lim_{n \to \infty} \lambda_{ns} = \begin{cases} 
0 & \text{if } \lambda_s > 0, \\
\alpha(0 +) - \alpha(0) & \text{if } \lambda_s = 0.
\end{cases}
\]

Proof. It is known [3, Theorem 1(iv) and Theorem 2] that (15) holds with \( s = 0 \) when \( \alpha(t) \) is non-decreasing, and the general case of (15) with \( s = 0 \) follows by expressing \( \alpha(t) \) as the difference of two non-decreasing functions.

Next, suppose \( s \geq 1 \) and let \( \tilde{\lambda}_k = \lambda_{k+s} \) for \( k = 0, 1, \ldots \). Then, for \( s \leq k \leq n \),

\[
\lambda_{nk} = \tilde{\lambda}_{n-s, k-s},
\]

\( \tilde{\lambda}_{nk} \) being defined by (4) and (5) with \( \{\lambda_k\} \) replaced by \( \{\tilde{\lambda}_k\} \). Hence, as \( n \to \infty \),

\[
\sum_{k=s}^{n} \lambda_{nk} = \sum_{k=0}^{n-s} \tilde{\lambda}_{n-s, k} \to \alpha(1) - \alpha(0 +)
\]

by (15) with \( s = 0 \), since \( \tilde{\lambda}_0 = \lambda_s > 0 \). This establishes (15) with \( s \geq 0 \).

To complete the proof of Lemma 2 we can deduce (16) from (15) by observing that, for \( n > s \geq 0 \),

\[
\lambda_{ns} = \sum_{k=s}^{n} \lambda_{nk} - \sum_{k=s+1}^{n} \lambda_{nk}.
\]

LEMMA 3. Let \( 0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots \), \( 0 < \delta < 1/2 \), and let \( s \) be a positive integer. Then there is an integer \( N \) and a positive constant \( M \) such that, for \( n \geq N \),

\[
\left( \sum_{k=s}^{n} \left( \int_{\delta}^{1-\delta} \lambda_{nk}(t) \, |d\alpha(t)| \right)^2 \right)^{1/2} \leq M \max \{M_1(n, s), M_2(n, s)\}
\]

where

\[
M_1(n, s) = \max_{s \leq k \leq n} \frac{e^{-\lambda_s \rho_{nk}}}{\lambda_k}
\]
and

\[ M_2(n, s) = \max_{\delta/2 \leq e^{\delta n} \leq 1 - \delta/2} \frac{1}{\lambda_k \sigma_{nk}}. \]

**Proof. Case 1.** Suppose that \( \lambda_0 = 0, s = 1 \). Let

\[ \omega_{nk} = \left( \left( 1 - \frac{\lambda_1}{\lambda_{k+1}} \right) \cdots \left( 1 - \frac{\lambda_1}{\lambda_n} \right) \right)^{1/\lambda_1} \quad \text{for } 0 \leq k < n, \omega_{nn} = 1. \]

Then, in view of (6), we have

\[
\sum_{k=1}^{n} \left( \int_{\delta}^{1-\delta} \lambda_{nk}(t) |d\alpha(t)| \right)^2 \leq \int_{\delta}^{1-\delta} |d\alpha(t)| \cdot \max_{1 \leq k \leq n} \int_{\delta}^{1-\delta} \lambda_{nk}(t) |d\alpha(t)| \leq V_{\delta} \max(I_1, I_2)
\]

where \( V_{\delta} = \int_{\delta}^{1-\delta} |d\alpha(t)| \),

\[ I_1 = \max_{|\omega_{nk} - 1/2| \leq 1/2 - 3\delta/4} \int_{\delta}^{1-\delta} \lambda_{nk}(t) |d\alpha(t)|, \]

and

\[ I_2 = \max_{|\omega_{nk} - 1/2| \leq 1/2 - 3\delta/4} \int_{\delta}^{1-\delta} \lambda_{nk}(t) |d\alpha(t)|. \]

To deal with \( I_1 \), let \( f(t) \) be a twice continuously differentiable function on \([0, 1]\) satisfying \( 0 \leq f(t) \leq 1 \), \( f(t) = 1 \) for \( |t - \frac{1}{2}| \geq \frac{3\delta}{4} \), \( f(t) = 0 \) for \( \delta \leq t \leq 1 - \delta \), and let

\[ B_n(f, t) = \sum_{k=0}^{n} \lambda_{nk}(t)f(\omega_{nk}). \]

Then, by a result proved by Leviatan [8, Theorem 7],

\[ I_1 \leq V_{\delta} \max_{\delta \leq t \leq 1-\delta} |B_n(f, t) - f(t)| \leq V_{\delta} K M_1(n, 1) \]

where \( K \) is a constant.

To deal with \( I_2 \) we note that, by Lemma 1,

\[ I_2 \leq \max_{|\omega_{nk} - 1/2| \leq 1/2 - 3\delta/4} \frac{V_{\delta}/2}{\lambda_k \sigma_{nk}} = \frac{V_{\delta}/2}{\lambda_k(n) \sigma_{n,k(n)}} \]

where \( k(n) \) is an integer satisfying \( 1 \leq k(n) \leq n \), \( 3\delta/4 < \omega_{n,k(n)} < 1 - 3\delta/4 \). Since \( \sum_{j=1}^{\infty} 1/\lambda_j = \infty \), it follows that, for every fixed integer \( j \), \( \lim_{n \to \infty} \omega_{nj} = 0 \) and hence that \( \lim_{n \to \infty} k(n) = \infty \). Further, since

\[ \log(1 - x) = x + O(x^2) \quad \text{for } |x| \leq 1/2, \]
we have that, for $k = k(n)$,
\[
\omega_{nk} \sim \omega_{n,k-1} = e^{-\rho_{nk} + O(\sigma_n^2)} = e^{-\rho_{nk} + O(\rho_{nk}/\lambda_k)} = e^{-\rho_{nk}(1 + o(1))}.
\]
Hence, for $n$ sufficiently large,
\[
\delta/2 < e^{-\rho_{n,k(n)}} < 1 - \delta/2,
\]
and thus
\[
I_2 \leq V_{\delta/2} M_2(n, 1).
\]
This completes the proof of Case 1.

Case 2. Suppose that $\lambda_0 \geq 0, s \geq 1$. Let
\[
\tilde{\lambda}_0 = 0, \quad \tilde{\lambda}_k = \lambda_{k+s-1} \quad \text{for } k = 1, 2, \ldots,
\]
and define $\tilde{\lambda}_{n,k}(t)$, $\tilde{M}_1(n, s)$, $\tilde{M}_2(n, s)$ by means of (4), (9), (17) and (18) with $\{\lambda_k\}$ replaced by $\{\tilde{\lambda}_k\}$. Then, for $n \geq k \geq s$, $0 \leq t \leq 1$, we have
\[
\tilde{\lambda}_{n-s+1,k-s+1}(t) = \lambda_{nk}(t),
\]
and hence, by Case 1,
\[
\sum_{k=s}^{n} \left( \int_{\delta}^{1-\delta} \tilde{\lambda}_{nk}(t) \, d\alpha(t) \right)^2 = \sum_{r=1}^{n-s+1} \left( \int_{\delta}^{1-\delta} \tilde{\lambda}_{n-s+1,r}(t) \, d\alpha(t) \right)^2
\leq M \max(\tilde{M}_1(n - s + 1, 1), \tilde{M}_2(n - s + 1, 1))
= M \max(M_1(n, s), M_2(n, s)).
\]
This completes the proof of Lemma 3.

**Lemma 4.** Let $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, $0 < \delta < 1/2$, $s \geq 2$, $\lambda_s > M + 1$, and let $\lambda$ satisfy either (10) or (11) with the same $M$ for $k \geq s - 1$. Then
\[
\lim_{n \to \infty} \log n \sum_{k=s}^{n} \left( \int_{\delta}^{1-\delta} \lambda_{nk}(t) \, d\alpha(t) \right)^2 = 0.
\]

**Proof.** Case 1. Suppose that $\lambda$ satisfies (10) for $k \geq s - 1$, and that $n \geq k \geq s$. Then $\lambda_n \geq \lambda_s + m(n - s)$, and
\[
M \rho_{nk} \geq \sum_{j=k}^{n} \frac{\lambda_{j+1} - \lambda_j}{\lambda_j \log \lambda_j} \geq \sum_{j=k}^{n} \int_{\lambda_j}^{\lambda_{j+1}} \frac{dx}{x \log x} = \log \frac{\log \lambda_{n+1}}{\log \lambda_k}.
\]
Hence
\[ \frac{e^{-\lambda_p n_k}}{\lambda_k} \leq 1 \cdot \left( \frac{\log \lambda_k}{\log \lambda_{n+1}} \right)^{\lambda_p/M}, \]
and so
\[ M_1(n, s) = O((\log \lambda_{n+1})^{-\lambda_p/M}) = o\left(\frac{1}{\log n}\right). \]

Suppose now that
\[ \frac{\delta}{2} \leq e^{-p n_k} \leq 1 - \frac{\delta}{2}. \]

Then
\[ m \log \frac{2}{2 - \delta} \leq m n_k \leq \sum_{j=k}^n \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \leq \sum_{j=k}^n \int_{\lambda_{j-1}}^{\lambda_j} \frac{dx}{x} = \log \frac{\lambda_n}{\lambda_{k-1}}, \]
so that \( \lambda_{k-1} \leq (1 - \delta/2)^m \lambda_n \) and hence, by (10), we have that
\[ \lambda_k \leq \lambda_{k-1} + M \log \lambda_k \leq \left(1 - \frac{\delta}{2}\right)^m \lambda_n + M \log \lambda_n. \]

Further, by (19) and (21),
\[ M \log \frac{2}{\delta} \geq \log \frac{\log \lambda_{n+1}}{\log \lambda_k}, \]
and so
\[ \lambda_k \geq \lambda^\varepsilon_{n+1} \]
where \( \varepsilon = (\delta/2)^M. \)

Next, let \( f(x) = 1/\log x \) so that
\[ f'(x) = \frac{1}{x^2 \log x} \left(1 + \frac{1}{\log x}\right) \leq \frac{c}{x^2 \log x} \]
for \( x \geq \lambda_s \) where \( c = 1 + 1/\log \lambda_s > 0. \) Hence, by (10), (22) and (23),
\[ cM(\lambda_k \sigma_n k)^2 \geq c \lambda_k^2 \sum_{j=k}^n \frac{\lambda_{j+1} - \lambda_j}{\lambda_j^2 \log \lambda_j} \geq \lambda_k^2 \sum_{j=k}^n \int_{\lambda_j}^{\lambda_{j+1}} \frac{c dx}{x^2 \log x} \geq \lambda_k^2 \int_{\lambda_k}^{\lambda_{n+1}} f'(x) dx \]
\[ = \frac{\lambda_k}{\log \lambda_k} \left(1 - \frac{\lambda_k \log \lambda_k}{\lambda_{n+1} \log \lambda_{n+1}}\right) \geq \frac{\lambda_k}{\log \lambda_n} \left(1 - (1 - \delta/2)^m - M \log \lambda_n / \lambda_n\right). \]
Consequently
\[ M_2(n, s) = O\left(\lambda_n^{-\epsilon/2} \log^{1/2} \lambda_n\right) = O\left(\lambda_n^{-\epsilon/4}\right) = O\left(n^{-\epsilon/4}\right) \]
\[ = o\left(\frac{1}{\log n}\right) . \]

The desired conclusion in Case 1 now follows from (20) and (24), by Lemma 3.

Case 2. Suppose that \( \lambda \) satisfies (11) for \( k \geq s - 1 \) and that \( n \geq k \geq s \). Then
\[ \rho_{nk} \geq \sum_{j=k}^{n} \frac{\lambda_{j+1} - \lambda_j}{\lambda_j} \geq \sum_{j=k}^{n} \int_{\lambda_j}^{\lambda_{j+1}} \frac{dx}{x} = \log \frac{\lambda_{n+1}}{\lambda_k} . \]

Hence, since \( \lambda_s > M + 1 \),
\[ e^{-\lambda_{s} \rho_{nk}} \leq \frac{1}{\lambda_k} \left(\frac{\lambda_k}{\lambda_{n+1}}\right)^{\lambda_{s}/M} \leq \frac{1}{\lambda^s} \]
and so
\[ M_1(n, s) \leq \frac{1}{\lambda^s} = o\left(\frac{1}{\log n}\right) . \]

Suppose now that (21) holds. Then, by (25),
\[ \lambda_k \geq \lambda_{n+1}(\delta/2)^M , \]
and hence
\[ \lambda_k \sigma_{nk} \geq \lambda_k \left(\frac{\rho_{nk}}{\lambda_n}\right)^{1/2} \geq \lambda_k \left(\frac{1}{\lambda_n} \log \frac{2}{2-\delta}\right)^{1/2} \]
\[ \geq \left(\frac{\delta}{2}\right)^M \left(\log \frac{2}{2-\delta}\right)^{1/2} \lambda_n^{1/2} . \]
Consequently
\[ M_2(n, s) = O\left(\lambda_{n+1}^{-1/2}\right) = o\left(\frac{1}{\log n}\right) . \]

The desired conclusion now follows from (26) and (27), by Lemma 3, and this completes the proof of Lemma 4.

3. Proof of Theorem 1. Suppose that \( n \geq k \geq s \) and that \( r = 3, 4, \ldots \). Let
\[ \lambda_{nk} = \int_{1/r}^{1-1/r} \lambda_{nk}(t) \, d\alpha(t) . \]
Let \( \{X_n\} \) be a sequence of random variables satisfying (1) and (2) with \( p = 2 \), and let
\[
T_n = \sum_{k=s}^{n} \lambda_{nk} X_k, \quad T'_n = \sum_{k=s}^{n} \lambda'_{nk} X_k.
\]

By Lemma 4, we have, subject to either (10) or (11), that
\[
\log n \sum_{k=s}^{n} (\lambda'_{nk})^2 \to 0 \quad \text{as } n \to \infty.
\]

Hence, by Theorem A,
\[
T'_n \to 0 \text{ a.s. as } n \to \infty.
\]

Let \( \Omega_r \) be the subset of \( \Omega \) on which \( T'_n \to 0 \) and \( |X_r| \leq M \), and let \( \Omega_0 = \bigcap_{r=3}^{\infty} \Omega_r \). Then
\[
T_n - T'_n = \sum_{k=s}^{n} X_k \left( \int_0^{1/r} \lambda_{nk}(t) \, d\alpha(t) - \int_{1/r}^{1} \lambda_{nk}(t) \, d\alpha(t) \right)
= \sum_{k=s}^{n} X_k \left( \int_0^{1/r} + \int_{1-1/r}^{1} \right) \lambda_{nk}(t) \, d\alpha(t),
\]
and hence, in view of (6), on \( \Omega_0 \)
\[
|T_n - T'_n| \leq M \left( \int_0^{1/r} + \int_{1-1/r}^{1} \right) |d\alpha(t)| \to 0 \quad \text{as } r \to \infty,
\]
since \( \alpha(0 +) = \alpha(0) \) and \( \alpha(1-) = \alpha(1) \). Thus
\[
\lim_{r \to \infty} T'_n = T_n \quad \text{on } \Omega_0 \text{ uniformly in } n \text{ for } n \geq s.
\]

On the other hand
\[
\lim_{n \to \infty} T'_n = 0 \quad \text{on } \Omega_0 \text{ for } r \geq 3.
\]

It follows that
\[
\lim_{n \to \infty} T_n = \lim_{n \to \infty} \lim_{r \to \infty} T'_n = \lim_{r \to \infty} \lim_{n \to \infty} T'_n = 0 \quad \text{on } \Omega_0.
\]
i.e., \( T_n \to 0 \) a.s.

Since \( \alpha(0) = \alpha(0+) \) we have, by Lemma 2, that \( \lim_{n \to \infty} \lambda_{nk} = 0 \) for \( k \geq 0 \). Consequently
\[
\sum_{k=0}^{n} \lambda_{nk} X_k \to 0 \quad \text{a.s.}
\]
and so \( H(\lambda, \alpha) \in \Gamma_2 \).
Finally, the additional condition $\alpha(1) - \alpha(0) = 1$ ensures, by Lemma 2, that
\[ \lim_{n \to \infty} \sum_{k=0}^{n} \lambda_{nk} = 1, \]
and hence that $H(\lambda, \alpha) \in (BP)$.

4. Proof of Theorem 2. Let $0 \leq k \leq n$. By (5), we have that
\[ \lambda_{nk} = \int_{0}^{1} \lambda_{nk}(t)\beta(t) \, dt. \]

First, suppose that (12) holds. Then, by Hölder's inequality and (7),
\[ |\lambda_{nk}|^p \leq \left( \int_{0}^{1} \lambda_{nk}(t) |\beta(t)|^p \, dt \right) \left( \int_{0}^{1} \lambda_{nk}(t) \, dt \right)^{p-1} = \left( \frac{d_k}{D_n} \right)^{p-1} \int_{0}^{1} \lambda_{nk}(t) |\beta(t)|^p \, dt. \]

Hence, by (6) and (12),
\[ \left( \sum_{k=0}^{n} |\lambda_{nk}|^p \right)^{1/(p-1)} \leq \frac{1}{D_n} \left( \int_{0}^{1} |\beta(t)|^p \, dt \sum_{k=0}^{n} d_k^{p-1} \lambda_{nk}(t) \right)^{1/(p-1)} \leq \max_{0 \leq k \leq n} d_k \cdot \frac{\|\beta\|^{p/(p-1)}}{D_n} = o \left( \frac{1}{\log n} \right). \]

It follows, by Theorem A, that $H(\lambda, \alpha) \in \Gamma_p$.

Next, suppose that (13) holds. Then, by (7),
\[ |\lambda_{nk}| \leq \|\beta\|_\infty \int_{0}^{1} \lambda_{nk}(t) \, dt = \|\beta\|_\infty \frac{d_k}{D_n}, \]
and hence
\[ \left( \sum_{k=0}^{n} |\lambda_{nk}|^p \right)^{1/(p-1)} \leq \|\beta\|_\infty^{p/(p-1)} \left( \sum_{k=0}^{n} \left( \frac{d_k}{D_n} \right)^p \right)^{1/(p-1)} = o \left( \frac{1}{\log n} \right). \]

Thus, by Theorem A, we have that $H(\lambda, \alpha) \in \Gamma_p$.

In view of Lemma 2, the additional conditions $\{\lambda_n\}$ monotonic and $\alpha(1) = 1$, ensure that
\[ \lim_{n \to \infty} \sum_{k=0}^{n} \lambda_{nk} = 1, \]
and hence that $H(\lambda, \alpha) \in (BP)$.

This completes the proof of Theorem 2.
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