A NOTE ON TAMELY RAMIFIED EXTENSIONS OF RINGS

MARTIN LLOYD BROWN
A NOTE ON TAMELY RAMIFIED
EXTENSIONS OF RINGS

M. L. BROWN

Buhler gave a criterion for a class of finite free extensions of
discrete valuation rings to be tamely ramified 1-dimensional regular
rings. In this note, we extend this criterion to finite free extensions of
general local rings and, in the final section, indicate the extension to
schemes.

1. Introduction. To set the notation, let $A$ be a noetherian local
ring of Krull dimension $n$ and let $A \to B$ be a finite free extension of
rings; denote by $d_{B/A}$ the discriminant of this extension, defined as
\[ \det(\text{tr}(b_i b_j)) \]
where $b_1, \ldots, b_m$ is a free basis of $B$ over $A$ and $\text{tr}: B \to A$
denotes the trace morphism. Let $m_A$ be the maximal ideal of $A$ and define
a function $v_{m_A}$ on $A$ by $v_{m_A}(x) = r$ where $r$ is the largest integer with
$x \in m_A'$ and $v_{m_A}(0) = \infty$. Note that $v_{m_A}$ is a valuation if $\text{gr}_A(m_A)$ has no
zero divisors, in particular if $A$ is regular [2].

If $n_1, \ldots, n_s$ are the maximal ideals of $B$ lying over $m_A$, define the
ramification index $e_{n_i/m_A}$ to be $l_B(B_{n_i}/m_A B_{n_i})$ where $l_B(M)$ denotes the
length (of a composition series) of the artin $B$-module $M$. If $A$ is a discrete
valuation ring, the $e_{n_i/m_A}$ clearly coincide with the usual ramification
indices of algebraic number theory. Recall that the embedding dimension
$\text{ed}(B)$ of the semi-local ring $B$ is max $\dim_{\kappa(n_i)} n_i/n_i^2$ where $n_i$ runs
through all maximal ideals of $B$. With the above notation the main result
of this paper is:

**Theorem 1.** If $A$ is regular (resp. $\text{gr}_A(m_A)$ has no zero divisors) and if
$B = A[X]/\langle f(X) \rangle$ where $f(X)$ is a monic polynomial and $\kappa(m) \to \kappa(n_i)$ is
separable for all $i = 1, \ldots, s$, then

\[ v_{m_A}(d_{B/A}) \geq \sum_{i=1}^s \left( e_{n_i/m_A} - 1 \right) \left[ \kappa(n_i) : \kappa(m_A) \right] \]

with equality if and only if (resp. only if) $\text{ed}(B) = \text{ed}(A)$ and $B$ is tamely
ramified over $A$ in that $p \nmid e_{n_i/m_A}$ for all $i$, where $p$ is the characteristic of
$\kappa(m_A)$. 

71
2. Proof of Theorem 1. We begin by some reductions. Observe that the conditions and conclusions of the theorem remain unchanged on base change by the \(m_A\)-adic completion of \(A: A \to B \otimes_A \hat{A} = \hat{B}\) so we may assume \(A\) is complete. Thus \(B\) is a product of local rings \(\prod B_i\) where each \(A \to B_i\) satisfies the conditions of Theorem 1. Since \(\nu_{m_A}(b_{B/A}) = \sum_i \nu_{m_A}(b_{B_i/A})\), it is easy to see that it is enough to prove Theorem 1 when \(B\) is local with maximal ideal \(m_B\), say.

Let \(e\) be the ramification index of \(B\) over \(A\) and \(a_1, \ldots, a_n\) form a basis of the cotangent space \(m_A/m_A^2\) over \(\kappa(m_A)\). There is a monic polynomial \(g \in A[X]\) with \(f = g^e + \sum_i a_i h_i\) where \(h_i \in A[X]\) for all \(i\). Letting \(R(p(X), q(X))\) denote the resultant of the polynomials \(p\) and \(q\) (see [3] or [5] for the properties of resultants we will use), then

\[
\nu_{m_A}(b_{B/A}) = R\left(f, f'\right) = R\left(f, eg^e g' + \sum_i a_i h'_i\right).
\]

If \(p \mid e\) where \(p = \text{char } \kappa(m_A)\), then \(e \equiv m_A\) and so

\[
\nu_{m_A}(b_{B/A}) = \nu_{m_A}\left(R\left(f, eg^e g' + \sum_i a_i h'_i\right)\right) \geq \text{degree } f = e\left[\kappa(m_B) : \kappa(m_A)\right].
\]

This completes the proof for the case of wild ramification.

Assume from now on that \(e \notin m_A\). Since \(f' \equiv eg^e g' \mod m_A\) and \(\kappa(m_A) \to \kappa(m_B)\) is separable, \(eg'\) and \(g^e g'\) are relatively prime in \(\kappa(m_A)[X]\). Thus by Hensel's lemma

\[
f' = \left(eg' + \sum_i a_i p_i\right)\left(g^e + \sum_i a_i q_i\right)
\]

where \(p_i, q_i \in A[X]\) with \(\text{deg}(p_i) < \text{deg}(g')\), \(\text{deg}(q_i) < \text{deg}(g^e g')\) for all \(i\).

Since \(\nu_{m_A}(R(g^e, eg')) = e\nu_{m_A}(R(g, g')) = 0\) we have

\[
\nu_{m_A}\left(R\left(f, eg' + \sum_i a_i p_i\right)\right) = 0.
\]

Thus

\[
\nu_{m_A}(b_{B/A}) = \nu_{m_A}\left(R\left(f, eg' + \sum_i a_i p_i\right)\right) + \nu_{m_A}\left(R\left(f, g^e + \sum_i a_i q_i\right)\right)
\]

\[
= \nu_{m_A}\left(R\left(f, g^e + \sum_i a_i q_i\right)\right);
\]

we conclude that if \(e = 1\) then \(\nu_{m_A}(b_{B/A}) = 0\) and \(m_AB = m_B\) proving the theorem for the unramified case \(e = 1\).
Assume from now on that \( e \geq 2 \) and put \( r = g^{e-1} + \sum a_i q_i \). Then

\[
\nu_{m_A}(R_{B/A}) = \nu_{m_A}(R(f - gr, r)) = \nu_{m_A}\left(R \left( \sum a_i (h_i - gq_i), r \right) \right)
\geq \deg r = (e - 1)[\kappa(m_A): \kappa(m_A)]
\]

and equality holds if and only if (resp. only if)

\[
I = \langle h_1 - gq_1, \ldots, h_n - gq_n, r \rangle \kappa(m_A)[X] = \kappa(m_A)[X]
\]

by Lemma 1 below. This completes the proof, for \( I = \kappa(m_A)[X] \) if and only if some \( h_i \) is invertible in \( \kappa(m_B) \) and so if and only if \( \Sigma a_i h_i = f - g^e \equiv 0 \mod m_B^2 \) gives a non-trivial linear relation between the \( a_i \)'s and \( g \) in \( m_B/m_B^2 \).

**Lemma 1.** If \( A \) is a regular local ring (resp. a local ring) and \( a_1, \ldots, a_n \) a basis of \( m_A \) and \( p_0, \ldots, p_n \in A[X] \) with \( p_0 \) monic, then

\[
R\left( \sum_{i=1}^n a_i p_i, p_0 \right) \not\in m_A^{1+\deg p_0}
\]

if and only if (resp. only if) \( p_0, \ldots, p_n \) are coprime in \( \kappa(m_A)[X] \).

**Proof.** If \( A \) is an arbitrary local ring, let \( m_A \in A[X] \) be a monic polynomial with residue in \( \kappa(m_A)[X] \) the highest common factor of \( p_0, \ldots, p_n \). Then for some \( q_i \in A[X] \) with \( q_0 \) monic and \( \deg mq_0 = \deg p_0 \), \( R(\Sigma_i a_i p_i, p_0) = R(m \Sigma_i a_i q_i, mq_0) \mod m_A^{\deg p_0+1} \) so if \( \deg m \geq 1 \), \( R(\Sigma_i a_i p_i, p_0) \in m_A^{\deg p_0+1} \) as required.

Conversely, if \( A \) is regular \( gr_A(m_A) \) is a polynomial ring \( \kappa(m_A)[X_1, \ldots, X_n] \) \([2]\), with the usual grading, so that monomials of total degree \( d \) in \( a_1, \ldots, a_n \) are linearly independent in \( A/m_A^{d+1} \). Since \( R(\Sigma_i a_i p_i, p_0) \) is a homogeneous polynomial of degree \( \deg p_0 \) in the \( a_i \) in \( A \), \( \nu_{m_A}(R(\Sigma_i a_i p_i, p_0)) = 1 + \deg p_0 \) if and only if \( R(\Sigma_i Z_i p_i, p_0) \) is the zero polynomial in the ring \( \kappa(m_A)[Z_1, \ldots, Z_n] \) where the \( Z_i \)'s are indeterminates.

Now if \( p_i \) are coprime in \( \kappa(m_A)[X] \) then \( \Sigma_{i=0}^n c_i p_i \equiv 1 \mod m_A \) for some \( c_i \in A[X] \). Thus

\[
\nu_{m_A}\left(R \left( \sum_{i=1}^n c_i p_i, p_0 \right) \right) = \nu_{m_A}\left(R \left( \sum_{i=0}^n c_i p_i, p_0 \right) \right) = 0
\]

so \( R(\Sigma_i Z_i p_i, p_0) \) is not the zero polynomial in \( \kappa(m_A)[Z] \) proving the lemma.
3. The obstruction for non-regular rings. Throughout this section the local ring $A$ is assumed to have no zero divisors in $\text{gr}_A(m_A)$.

For regular rings, Theorem 1 gives a necessary and sufficient numerical criterion for $A \to A[X]/\langle f(X) \rangle$ to be tamely ramified with $\text{ed}(f) = \text{ed}(A)$. The failure of this criterion to be necessary for non-regular rings is examined in this section; we will see that the obstruction lies in the equations defining the tangent cone $\text{gr}_A(m_A)$. Indeed, we construct a cohomology group $H^2(C_g^\ast)$ so that the numerical criterion is necessary and sufficient for all polynomials with a fixed reduction $g \mod m_A$, say, if and only if $H^2(C_g^\ast)$ is isomorphic to the vector space of homogeneous equations defining the tangent cone of degree equal to that of $g(X)$.

In the sense of Hilbert schemes classifying polynomials over $A$, this failure is not exceptional: “almost all” tamely ramified polynomials $f(X)$ over a non-regular ring $A$ with $\nu(\text{discr } f) > \sum_i (e_i - 1)[\kappa(m_i); \kappa(m_A)]$ have $\text{ed}(f(X)) = \text{ed}(A)$.

Nevertheless, for polynomials which are unramified or totally ramified or have degree $\leq 3$, the numerical criterion is necessary and sufficient over arbitrary local rings.

Fix a monic polynomial $g(X) = X^m + \sum_{i=0}^{m-1} b_i X^i$ in $\kappa(m_A)[X]$ and let $b_i \in A$ be elements with residue $\tilde{b}_i$ for all $i$. Then $g(X)$ factorizes as $\prod_i \tilde{g}_i(X)^{c_i}$ over $\kappa(m_A)$ where we assume $\tilde{g}_i(X)$ are distinct separable polynomials over $\kappa(m_A)$.

The Hilbert scheme $H_g = \text{Spec } A[X_1, \ldots, X_m, x_1, \ldots, x_m]_{x_1-b_1, \ldots, x_m-b_m, m}$ classifies the monic polynomials with reduction $g \mod m_A$ in that there is a bijection:

$$H_g(\text{Spec } A) \sim \{ \text{Monic polynomials } f(X) \text{ over } A \\ \text{with } f(X) \equiv g(X) \mod m_A \}$$

given by

$$\{ A[X_1, \ldots, X_m] \to A : X_i \mapsto c_i + b_i \} \to X^m + \sum_{i=0}^{m-1} (c_{i+1} + b_{i+1}) X^i.$$ 

Let $T$ be the tangent cone of Spec $A$, by definition $T = \text{Proj } \text{gr}_A(m_A)$ where $\text{gr}_A(m_A) = \bigoplus_{i=0}^{\infty} m_A^n / m_A^{i+1}$; fix a basis, once and for all, $a_1, \ldots, a_n$ of $m_A$ so that $n = \text{ed}(A)$. Let $T' = A_k^n \times_A T$ where $\kappa = \kappa(m_A) = A / m_A$ and $A_k^n = \text{Spec } \kappa[X_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ is affine $nm$-space over $\kappa$. Regarding Spec $\kappa$ as a $T$-scheme, via projection onto the 1st component $\text{gr}_A(m_A) \to A / m_A = \kappa$, there is a bijection:

$$H_g(\text{Spec } A / m_A^2) \sim T\text{-sch}(\text{Spec } \kappa, T')$$
NOTE ON TAMELY RAMIFIED EXTENSIONS OF RINGS 75

\{f: X_i \to c_i + b_i \} \to \left\{ X_{ij} \to x_{ij} \in \kappa \text{ all } i, j \text{ where } c_i = \sum_{j=1}^{n} a_j x_{ij} \mod m_A^{2} \right\}.

Denote by * the composite of the maps:

\[ H_g(Spec A)^{\text{natural}} \to H_g(Spec A/m_A^{2}) \xrightarrow{\sim} T\text{-sch}(Spec \kappa, T'). \]

PROPOSITION 1. (1) The integer \( s = \sum (e_i - 1)[\kappa(m_A)] \) is the same for all polynomials in \( H_g(Spec A) \).

(2) There are closed subschemes \( V \supset V' \) of \( T' \) so that for any \( h \in H_g(Spec A) \) with associated polynomial \( f(X) \),

(a) \( v_{m_A}(\text{discr } f(X)) > s \) if and only if \( h^* \in T\text{-sch}(Spec \kappa, V) \),
(b) \( \text{ed}(f(X)) > \text{ed}(A) \) if and only if \( h^* \in T\text{-sch}(Spec \kappa, V') \).

(3) \( V \) is a proper closed subscheme of \( T \) if and only if all polynomials in \( H_g(Spec A) \) are tamely ramified.

Proof. (1) Clear.

(2) Recall \( g(X) \) factorises as \( \prod g_i^{e_i} \) in \( \kappa[X] \) and choose representative monic polynomials \( g_i(X) \in A[X] \) with residue \( \tilde{g}_i(X) \mod m_A \) for all \( i \).

As in the proof of Theorem 1, it is not difficult to see that

\[ v_{m_A}(\text{discr } f(X)) > s \] if and only if

\[ v_{m_A}\left( R\left( f(X), \sum_{i} e_i g_i^{e_i-1} \prod_{j \neq i} g_j^{e_j} \right) \right) > s. \]

Now

\[ R\left( f(X), \sum_{i} e_i g_i^{e_i-1} \prod_{j \neq i} g_j^{e_j} \right) = R\left( f, \prod_{i} g_i^{e_i-1} \right) R\left( f, \sum_{i} e_i g_i' \prod_{j \neq i} g_j \right); \]

as in the proof of Theorem 1, \( v_{m_A}(R(f(X), \sum_{i} e_i g_i' \prod_{j \neq i} g_j)) = 0 \) if and only if \( p \) does not divide \( e_i \) for all \( i \) where \( p \) is the characteristic of \( \kappa(m_A) \).

It follows from Theorem 1 that if \( p \mid e_i \) for some \( e_i \), then \( v_{m_A}(\text{discr } f) > s \) for any \( f \) with reduction \( g \) so for this wildly ramified case \( V = T' \) has the required properties. If now \( p \nmid e_i \) for all \( i \) then \( v_{m_A}(\text{discr } f(X)) > s \) if and only if \( v_{m_A}(R(f, g_j)) > \deg g_j(X) \) for some \( j \). Putting

\[ f(X) = X^m + \sum_{i=0}^{m-1} \left( b_i + \sum_{j=1}^{n} a_j X_{ij} \right) X^i, \]

with the notation as previously, \( f(X) \) is the general polynomial of \( H_g(Spec A); R(f(X), g_j(X)) \) is a homogeneous polynomial in the \( X_{ij} \)'s of
degree \( \deg g_j(X) \). Moreover, the coefficient of each monomial in the \( X_i \)'s is a monomial in the \( a_j \)'s of degree \( \deg g_j(X) \). Let \( p_k(X_i) \); all \( i, j \) be the polynomial \( R(f(X), g_k(X)) \)-regarded as an element of \( \mathfrak{g}(\mathfrak{m}_A)[X_i] \) of degree \( \deg g_k \) and put \( p(X_i) = \prod_k p_k(X_i) \); all \( i, j \) \( \alpha \)'s. The ideal of \( \mathfrak{g}(\mathfrak{m}_A) \); all \( i, j \) generated by \( p(X_i) \) clearly defines the closed subscheme \( V \) of \( T' \).

(2b) With the notation above, let \( f^*(X) \) be a polynomial from \( H_0(\text{Spec } A) \) then \( f^*(X) = \prod_i \left( g_i^{e_i} + \sum_{j=1}^n a_{ij} p_j(X) \right) \) for some \( p_j(X) \in A[X] \). We assert \( \deg(f^*(X)) = \deg(A) \) if and only if \( \prod_i g_i^{e_i}, p_1(X), \ldots, p_n(X) \) have no common factor in the residue ring \( \kappa(\mathfrak{m}_A)[X] \). For, without loss of generality \( A \) is complete as in the proof of Theorem 1, so \( f^*(X) = \prod_i \left( g_i^{e_i} + \sum_{j=1}^n a_{ij} p_j(X) \right) \) for some polynomials \( p_j(X) \in A[X] \) by Hensel's lemma. By the proof of Theorem 1 and Lemma 1, \( \deg(f^*(X)) = \deg(A) \) if and only if \( g_i(X), p_{i1}(X), \ldots, p_{in}(X) \) have no common factor in \( \kappa(\mathfrak{m}_A)[X] \) for all \( i \). The assertion easily follows on expanding the product for \( f^*(X) \).

For a general polynomial \( f(X) \) in \( H_0(\text{Spec } A) \) put, as before, \( f(X) = X^m + \sum_j \left( b_j + \sum_{i=1}^t a_{ij} X_i \right) X^j \). Let \( f^*(X) \) denote the specialisation of \( f(X) \) under \( X_{ij} \rightarrow x_{ij} \in \kappa \), then \( \deg(f^*(X)) = \deg(A) \) if and only if \( \prod_i g_i^{e_i}, \sum_i x_{ij} X^j, j = 1, \ldots, n, \) have no common factor in \( \kappa(\mathfrak{m}_A)[X] \). Introducing arbitrary parameters \( Z_1, \ldots, Z_n \), then \( \deg(f^*(X)) > \deg(A) \) if and only if, by Lemma 1, \( R(g(X), \sum_{j=1}^n Z_j \sum_i x_{ij} X^j) \) is the zero polynomial, regarded as a polynomial in \( \kappa(\mathfrak{m}_A)[Z_1, \ldots, Z_n] \) by taking it mod \( \mathfrak{m}_A^{\deg g} \).

Thus \( R(g(X), \sum_{j=1}^n Z_j \sum_{i=0}^m X_i X^j) \mod \mathfrak{m}_A^{m+\deg g} \) is a homogeneous polynomial of degree \( \deg g(X) = m \) in the \( Z_i \), assuming it is non-zero.

Write \( R(g(X), \sum_{i,j} Z_i q_i(X_{ij}) \mod \mathfrak{m}_A^{m+\deg g} \) where \( Z_i \) runs over all monomials in \( Z_i \) of degree \( m \) and \( q_i(X_{ij}) \in \kappa(\mathfrak{m}_A)[X_{ij} \in \kappa(\mathfrak{m}_A)[X_i] \); \( 0 \leq i \leq n - 1 \), \( 1 \leq j \leq n \) is an homogeneous polynomial of degree \( m \). Thus \( \deg(f^*(X)) > \deg(A) \) if and only if \( q_i(x_{ij}) = 0 \) for all \( i \).

Let \( V' \) be the closed subscheme of \( T' \) defined by the ideal \( \langle q_i(X_{ij}) \rangle \); all \( i > 0 \), then clearly \( V' \) has the required properties. \( \square \)

From the above proof we deduce:

**Corollary 1.** (1) Either \( V = T' \) or \( V \) is a union of hypersurfaces of \( T' \) of degree \( t_k = [\kappa(\mathfrak{m}_k) : \kappa(\mathfrak{m}_A)] \), with multiplicity \( e_k \), for all \( k \), and is defined by an homogeneous equation \( \prod_k f_k(X_i) \); all \( ij \) of degree \( m = \deg g(x) \) with coefficients of \( X_{ij} \) in \( f_k \) homogeneous polynomials in \( a_1, \ldots, a_n \) of degree \( t_k \).

(2) \( V' \) is defined in \( T' \) by most \( \binom{n-1}{m} \) equations of degree \( m \) in the variables \( X_{ij} \) and with coefficients in \( \kappa(\mathfrak{m}_A) \).
We relate the equations defining $V, V'$ to those defining the tangent cone $T$ in its embedding $T \to \mathbb{P}^n_k$ given by the very ample sheaf $O_T(1)$. Let $S^m(m_A/m_A^2)$ denote the $m$th symmetric power of $m_A/m_A^2$ and let $K_m$ be the kernel of the natural map $S^m(m_A/m_A^2) \to m_A^m/m_A^{m+1}$; thus $K_m$ is the set of “equations of degree $m$ defining $T$”.

Suppose $g$ is tamely ramified and let $m = \deg g(x)$ and $s = \sum_i (e_i - 1)[\kappa(\pi_i)]: \kappa(m_A)]$, then there is a complex $C^-$:

$$0 \to T - \text{sch}(\kappa, V') \to T - \text{sch}(\kappa, V) \to S^m(m_A/m_A^2)^k \to m_A^m/m_A^{m+1}$$

where $i$ is the natural inclusion (‘complex’ meaning that composites of successive maps are zero: note that each component of the complex has a distinguished zero element).

To define the complex it is only necessary to define $j$. Let $f(X_{ij}; 0 \leq i \leq m - 1, 1 \leq j \leq n) = 0$ be the equation defining $V$. Since $\text{gr}_A(m_A)$ is without zero divisors, by Corollary 1(1) the coefficients of $f(X_{ij})$ are polynomials in the $a_i$'s of degree $m$. The proof of Proposition 1(2b) actually constructs a polynomial $f^*(X_{ij})$ in $S^*(m_A/m_A^2)[X_{ij}; 0 \leq i \leq m - 1, 1 \leq j \leq n]$, $S^*(m_A/m_A^2)$ denoting the symmetric algebra, whose image in $\text{gr}(m_A)[X]$ is $f(X_{ij})$ under the canonical map. Denote by $a_1^*, \ldots, a_n^*$ the unique liftings of $a_1, \ldots, a_n$ in $S^*(m_A/m_A^2)$. Let $z \in T\text{-sch}(\kappa, V')$ be given by $\{X_{ij} \to x_{ij} \in \kappa$ for all $i, j$ with $f(x_{ij}) = 0\}$ and define $j(z) = f^*(x_{ij}) \in S^m(m_A/m_A^2)$. Clearly $k \circ j = 0$ since $k \circ f(z) = k(f^*(x_{ij})) = f(x_{ij}) = 0$. Note that the coefficients of $f^*(X_{ij})$, regarded as a polynomial in $a_1^*, \ldots, a_n^*$ in $S^*(m_A/m_A^2)$ with coefficients in $\kappa(m_A)[X_{ij}]: all i j$, are precisely the equations defining $V'$. Thus $j(z) = 0$ if and only if $z = i(y)$ for some $y \in T\text{-sch}(\kappa, V')$ thus showing $j \circ i = 0$, and $C^-$ is a complex. taking cohomology, we deduce $H^0(C^-) = H^1(C^-) = 0$.

From Proposition 1, $\{\nu(\text{discr } f) = s$ if and only if $\text{ed}(f) = \text{ed}(A)$, for every $f(X)$ in $H_g(\text{Spec } A)$\} if and only if $i$ is surjective, thus if and only if $j$ is the zero map. We deduce:

**Proposition 2.** $H^2(C^-) \cong K_m$ if and only if $\{\nu(\text{discr } f(x)) = s \iff \text{ed}(f(x)) = \text{ed}(A)$, for all $f(x)$ in $H_g(\text{Spec } A)$\}.

**Corollary 2.** Suppose $f(X)$ has reduction $\Pi_i g_i(X)^{e_i} \mod m_A$ which has one of the following:

1. $f$ is totally ramified i.e. $\deg g_i = 1$ for all $i$,
2. $f$ is unramified i.e. $e_i = 1$ for all $i$,
(3) \( K_m = 0 \) where \( m = \deg f(x) \),
(4) \( \deg f(X) \leq 3 \),
then \( v(\text{discr}(f(X))) = \sum_i (e_i - 1)[\kappa(\pi_i): \kappa(\mathfrak{m}_A)] \) if and only if \( f \) is tamely ramified and \( \text{ed}(f(X)) = \text{ed}(A) \).

**Proof.** In any case, if \( f(X) \) is wildly ramified the result follows so we assume \( f \) is tamely ramified.

(1) Since \( f \) is totally ramified, the equation \( p(X_{ij}) \) defining \( V \) is, by Corollary 1, a product \( \prod_k p_k(X_{ij}) \) of factors linear in the \( X_{ij} \)’s and \( a_i \)'s. Let \( z \in T\text{-sch}(\kappa, V) \) be given by \( X_{ij} \rightarrow x_{ij} \in \kappa \) for all \( i, j \), then \( p(X_{ij}) = 0 \) implies \( p_k(x_{ij}) = 0 \) for some \( k \) since \( \text{gr}_A(\mathfrak{m}_A) \) has no zero divisors. Thus \( p_k(x_{ij}) = 0 \) is a linear relation between the linearly independent \( a_i \)'s so \( f(z) = 0 \). Since \( H^1(C') = 0, z = \iota(y) \) for some \( y \in T - \text{sch}(\kappa, V') \) proving the corollary in view of Proposition 1.

(2) If \( f \) is unramified, then obviously \( \nu_{\mathfrak{m}_A}(\text{discr}(f)) = 0 \) since \( f(X) = 0 \) has distinct roots mod \( m_A \). Thus \( \text{ed}(f(X)) = \text{ed}(A) \) by Theorem 1.

(3) The Corollary follows immediately from Proposition 2.

(4) If \( \deg f(X) \leq 3 \) then the only possibilities are that \( f \) is totally ramified or is unramified whence the result from (1) and (2). \( \square \)

Since resultants are ‘universally’ defined it easily follows from the proof of Proposition 1 that the subschemes \( V, V' \) of \( T' \) have a ‘universal’ construction in that they are induced from \( \mathbf{Z} \)-schemes independent of \( T' \):

**Proposition 3.** Given non-negative integers \( n, f_1, \ldots, f_r, e_1, \ldots, e_r \), there are affine \( \mathbf{Z} \)-schemes \( Z, Z' \) which are closed subschemes of \( \mathbf{A}_Z^n \), where \( w = n + \sum_{i=1}^r f_i(e_i + 1) \), with the following property. For any local ring \( A \) of embedding dimension \( n \), any monic polynomial \( g(X) \in A[X] \) with \( g(X) \equiv \prod_{i=1}^r g_i(X)^{e_i} \mod m_A \) where \( g_i(X) \) are distinct separable polynomials of degree \( f_i \), there is a map \( T' \rightarrow \mathbf{A}_Z^n \) so that \( V = Z \times_{\mathbf{A}_Z^n} T' \) and \( V' = Z' \times_{\mathbf{A}_Z^n} T' \).

**Example.** We construct a quartic \( f(X) \) over a 1-dimensional local ring \( A \) with \( f \) tamely ramified, \( \text{ed}(f) = \text{ed}(A) \), \( \nu_{\mathfrak{m}_A}(\text{discr}(f)) = 3 \) and \( s = \sum_i (e_i - 1)[\kappa(\pi_i): \kappa(\mathfrak{m}_A)] = 2 \).

Let \( Q \) be the field of rational numbers and let \( a_1, a_2 \) be independent transcendentals over \( Q \). Put \( A = Q[a_1, a_2]/\langle a_1^2 + a_2^2 \rangle \). Let \( f(X) = (X^2 + 1)^2 + a_2 X + a_1 \) be a polynomial over \( A \).
We claim $A$ and $f(X)$ are our example.

For $f(X) \equiv (X^2 + 1)^2 \mod \mathfrak{m}_A$ so $f(X)$ is tamely ramified with ramification index $2$ and $s = 2$. The discriminant of $f(X)$ is

$$(256a_1^3 - 27a_2^4 + 288a_1a_2^2 + 256(a_1^3 + a_2^3))/256 = (-32a_1^3 - 27a_2^4)/256$$

so that $v_{\mathfrak{m}_A}(\text{discr } f) = 3$.

$A$ is a $1$-dimensional local ring of embedding dimension $2$ and $\text{gr}_A(\mathfrak{m}_A) \cong Q[ X_1, X_2]/\langle X_1^2 + X_2^2 \rangle$ has no zero divisors.

The maximal ideal of $A[X]/(f(X))$ is $\langle a_1, a_2, X_1^2 + 1 \rangle$ since $a_1 = -(X_2 + 1)^2 - a_2 X_1 \in \langle a_2, X_1^2 + 1 \rangle$ in $A[X]/(f(X))$. Thus $\text{ed}(f(X)) = 2 = \text{ed}(A)$.

4. Applications. For the translation of Theorem 1 to schemes, we have:

**Theorem 2.** Let $Y$ be a regular scheme and $\mathbb{P}_Y^1$ the projective line bundle over $Y[4]$. Let $X$ be a closed subscheme of $\mathbb{P}_Y^1$ so that for every irreducible component $X_i$ of $X$, the induced $f_i: X_i \to Y$ is dominating and finite and all residue field extensions are separable. Then:

1. $X \to Y$ is flat;
2. $X$ is regular and tamely ramified over $Y$ if and only if $v_{\mathfrak{m}_Y}(\text{discr } X_{X_i}) = \Sigma_i (e_{X_i} - 1)[\kappa(\mathfrak{m}_{X_i}): \kappa(\mathfrak{m}_Y)]$ for all points $x, \ldots, x_n$ in the fibre $f^{-1}(y)$ and for all points $y$ of $Y$.

By $\text{discr } X_{X_i}$, we here mean the local discriminant of the finite free extension $O_{Y,y} \to \Gamma(X_x, \text{Spec } O_{Y,y}, O_{X,x}, \text{Spec } O_{Y,y})$; $e_{X_i}$ is defined similarly.

For the proof of the theorem, note that the question is local on $Y$ so we may assume $Y$ is affine. Moreover, we may replace $Y$ by $\text{Spec } O_{Y,y}$, by flat base change, and prove the theorem when $Y = \text{Spec } A$ with $A$ a regular local ring. In this case, $\mathbb{P}_A^1 = \text{Proj } A[X_0, X_1]$ has every finite subset of points contained in an open affine subscheme isomorphic to $\text{Spec } A[X]$; since condition (2) of the theorem is applied to such finite sets, we may assume $X$ is a closed subscheme of some $\text{Spec } A[X]$. Let $X = \text{Spec } A[X]/I$.

Let $I = a_1 \cap a_2 \cdot \cdot \cdot \cap a_n$ be the primary decomposition of $I$ in $A[X]$ and let $\mathfrak{p}_i = \sqrt{a_i}, i = 1, \ldots, n$, be the prime ideals associated to $a_i$. Then $\text{Spec } A[X]/\mathfrak{p}_i \to \text{Spec } A$ is dominating and finite for each $i$. Therefore
\( \pi_i \cap A = \{0\} \) so height \( \pi_i = 1 \) for each \( i \). It easily follows that \( \pi_i \) is a principal ideal, generated by \( p_i(X) \), say where \( p_i(X) \) is a non-constant polynomial in \( A[X] \).

Since \( \pi_i \cap A = \{0\} \) for all \( i \), we have \( a_i = \langle p_i(X)^{n_i} \rangle \) for some integers \( n_i \). Therefore \( \langle \prod_i p_i(X)^{n_i} \rangle \subseteq I = \cap_i a_i \). In the fibre \( A[X] \otimes_A \operatorname{fract}(A) \), \( I \) and \( \langle \prod_i p_i(X)^{n_i} \rangle \) coincide since \( A[X] \otimes_A \operatorname{fract}(A) \) is a principal ideal domain. It follows that for every \( q(X) \in I \) there are \( a, b \in A \) with \( aq(X) = bp(X) \) where \( p(X) = \prod_i p_i(X)^{n_i} \).

But \( A[X]/\langle p(X) \rangle \) is a finite \( A \)-module and, since \( A \) is normal, Kronecker's Theorem [1] shows that \( p_i(X) \) has invertible leading coefficient for all \( i \); thus we may suppose \( p(X), p_1(X), \ldots, p_n(X) \) are monic polynomials. Consequently, if \( aq(X) = bp(X) \) then \( q(X) \in \langle p(X) \rangle \) thus \( I = \langle p(X) \rangle \) and so \( X = \operatorname{Spec} A[X]/\langle p(X) \rangle \) where \( p(X) \) is a monic polynomial; consequently \( X \to Y \) is flat. The second part of the theorem now follows from Theorem 1.

**Corollary.** With \( f: X \to Y \) as in Theorem 2. Suppose that \( \operatorname{Reg}(Y) \) is open (resp. contains a non-empty open set). Then the set of points \( \{ x \in X \mid X \) is regular and tamely ramified over \( Y \) at every point of the fibre \( f^{-1}(f(x)) \) \) is open (resp. contains a non-empty open set).

**Proof.** By replacing \( Y \) by a regular open subscheme we may assume \( Y \) is regular. Now, \( \nu_{\mathfrak{m}_y}(\mathfrak{m}_y) \), \( \Sigma_i [\kappa(\mathfrak{m}_x): \kappa(\mathfrak{m}_y)] \) are upper semi-continuous on \( Y \) and \( \Sigma_i e_i [\kappa(\mathfrak{m}_x): \kappa(\mathfrak{m}_y)] \) is locally constant; the corollary now follows.

**References**


Received October 5, 1981.

**UNIVERSITY COLLEGE**
**CARDIFF**
**SOUTH GLAMORGAN**
**WALES, GB**
John Kelly Beem and Phillip E. Parker, Klein-Gordon solvability and the geometry of geodesics ........................................ 1
David Borwein and Amnon Jakimovski, Transformations of certain sequences of random variables by generalized Hausdorff matrices ....... 15
Willy Brandal and Erol Barbut, Localizations of torsion theories ........ 27
John David Brillhart, Paul Erdős and Richard Patrick Morton, On sums of Rudin-Shapiro coefficients. II ........................................ 39
Martin Lloyd Brown, A note on tamely ramified extensions of rings ...... 71
Chang P’ao Ch’én, A generalization of the Gleason-Kahane-Żelazko theorem ............................................................. 81
I. P. de Guzman, Annihilator alternative algebras .......................... 89
Patrick Ronald Halpin, Some Poincaré series related to identities of $2 \times 2$ matrices .......................................................... 107
Fumio Hiai, Masanori Ohya and Makoto Tsukada, Sufficiency and relative entropy in $\ast$-algebras with applications in quantum systems ...... 117
Dean Robert Hickerson, Splittings of finite groups .......................... 141
Jon Lee Johnson, Integral closure and generalized transforms in graded domains .......................................................... 173
Maria Grazia Marinari, Francesco Odetti and Mario Raimondo, Affine curves over an algebraically nonclosed field ....................... 179
Douglas Shelby Meadows, Explicit PL self-knottings and the structure of PL homotopy complex projective spaces ......................... 189
Charles Kimbrough Megibben, III, Crawley’s problem on the unique $\omega$-elongation of $p$-groups is undecidable .............................. 205
Mary Elizabeth Schaps, Versal determinantal deformations ............... 213
Stephen Scheinberg, Gauthier’s localization theorem on meromorphic uniform approximation ............................................... 223
Peter Frederick Stiller, On the uniformization of certain curves .......... 229
Ernest Lester Stitzinger, Engel’s theorem for a class of algebras ........ 245
Emery Thomas, On the zeta function for function fields over $F_p$ .......... 251