Pacific Journal of Mathematics

A GENERALIZATION OF THE GLEASON-KAHANE-ŻELAZKO THEOREM

Chang P'ao Ch'ên

Vol. 107, No. 1 January 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 1, 1983

A GENERALIZATION OF THE GLEASON-KAHANE-ZELAZKO THEOREM

CHANG-PAO CHEN

In this paper, we consider two classes of commutative Banach algebras, which include $C^n(T)$, $\mathrm{Lip}_\alpha(T)$, $BV(T)$, $L^1 \cap L^p(G)$, $A^p(G)$, $L^1 \cap C_0(G)$, l^p, c_0, and $C_0(S)$. We characterize ideals of finite codimension in these two classes of algebras and thereby partially answer a question suggested by C. R. Warner and R. Whitley.

In [**5**] and [**9**], A. M. Gleason, J. P. Kahane and W. Zelazko gave independently the following characterization of maximal ideals: Let A be a commutative Banach algebra with unit element. Then a linear subspace M of codimension 1 in A is a maximal ideal in A if and only if it consists of noninvertible elements, or equivalently, each element of M belongs to some maximal ideal. This interesting result as first proved depended on the Hadamard Factorization Theorem.

This characterization of maximal ideals was extended in [**15**] and [**16**] to algebras without identity. In [**16**], C. R. Warner and R. Whitley also gave a characterization of ideals of finite codimension in $L^1(R)$ and $C[0, 1]$. They showed: Let A be any one of $L^1(R)$ and $C(S)$, where S is a compact subset of R. If M is a closed subspace of codimension n in A with the property that each element in M belongs to at least n regular maximal ideals, then M is an ideal. In fact, M is the intersection of n regular maximal ideals. Also in [**16**], C. R. Warner and R. Whitley suggested the following question: For what locally compact abelian group G does $L^1(G)$ have the property of $L^1(R)$ described above?

In this paper, we partially answer this question and generalize the work of C. R. Warner and R. Whitley. In this paper, two methods are introduced; One uses the Baire category theorem and the other generalizes the ideas of Theorems 2 and 4 in [**16**].

THEOREM 1. *Let A be a commutative Banach algebra with a countable maximal ideal space $\mathfrak{M}$. If M is a closed subspace of codimension n in A with the property that each element in M belongs to at least n regular maximal ideals, then M is an ideal, which is the intersection of n regular maximal ideals.*

Proof. From the hypothesis, we know that $M \subset \bigcup I_{s_1 s_2 \cdots s_n}$, where $I_{s_1 s_2 \cdots s_n}$ denotes the space $\{x \in A : \hat{x}$ vanishes at $s_1, s_2, \ldots, s_n\}$ and the union is taken over all sets of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$. Since $\mathfrak{M}$ is countable, the union is a countable union. By the Baire category theorem, $M \subset I_{s_1 s_2 \cdots s_n}$ for some set of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$. If not, for any set of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$, we have $M \cap I_{s_1 s_2 \cdots s_n} \subsetneq M$. By the open mapping theorem, we find that $M \cap I_{s_1 s_2 \cdots s_n}$ is of first category in M and so the union $\bigcup (M \cap I_{s_1 s_2 \cdots s_n})$ is of first category in M. This implies that M is of first category in itself and contradicts the fact that M is a Banach space. Therefore $M \subset I_{s_1 s_2 \cdots s_n}$ for some set of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$. Since M and $I_{s_1 s_2 \cdots s_n}$ are of codimension n in A, $M = I_{s_1 s_2 \cdots s_n}$. We have completed the proof.

EXAMPLE 2. Any of the following spaces has the property described in Theorem 1: $C^n(T)$; $\mathrm{Lip}_\alpha(T)$, $0 < \alpha \leq 1$; $BV(T)$; $L^p(G)$, $1 \leq p \leq \infty$, or $A^p(G)$ or $C(G)$, or any normed ideal in $L^1(G)$, where G is a metrizable compact abelian group; l^p, $1 \leq p < \infty$, and c_0 (cf. [**1, 2, 4, 7, 8, 10, 11, 12, 14**]).

REMARK 3. The structure of a metrizable compact abelian group can be found in [**12**, Theorem 2.2.6]. It is well-known that the maximal ideal space of l^∞ coincides with the Stone-Čech compactification βZ^+, whose cardinal number is uncountable. (See [**2**, pp. 58] and [**3**, pp. 244].) Therefore Theorem 1 cannot be applied to this case. Theorem 1 answers the question suggested by C. R. Warner and R. Whitley for $L^1(G)$ in the case G is compact and metrizable.

The following theorem extends the results presented in Theorem 1 to another kind of algebra while not hypothesizing that M be closed. (Compare this with Theorem 1 and [**16**, Theorems 2 and 4].) This theorem generalizes Theorems 2 and 4 in [**16**].

THEOREM 4. *Let A be a commutative Banach algebra with involution $x \to x^*$ satisfying $\hat{x}^* = \hat{x}^-$. Suppose that there is an element x_0 in A, with $\hat{x}_0$ never zero, and that there is a one-to-one real-valued function ϕ on the maximal ideal space $\mathfrak{M}$ of A such that $\hat{x}_0 \phi^j = \hat{x}_j$ for some x_j in A $(1 \leq j \leq n)$. If M is a subspace (not a priori closed) of codimension n in A with the property that each element in M belongs to at least n regular maximal ideals, then M is an ideal which is the intersection of n regular maximal ideals.*

Proof. Without loss of generality, we may assume that $\hat{x}_0$ is real-valued. Let $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-1}$ denote the cosets in the quotient space A/M corresponding to $x_0, x_1, \ldots, x_{n-1}$. If $\lambda_0\bar{x}_0 + \lambda_1\bar{x}_1 + \cdots + \lambda_{n-1}\bar{x}_{n-1} = \bar{0}$, then $\lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{n-1}x_{n-1} \in M$ and so the equation $\lambda_0 + \lambda_1\phi(s) + \cdots + \lambda_{n-1}\phi(s)^{n-1} = 0$ has n distinct solutions in s. This implies that the polynomial $\lambda_0 + \lambda_1 t + \cdots + \lambda_{n-1}t^{n-1}$ has n distinct zeros, which occurs only if all λ_j's are zero. Hence $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-1}$ form a basis for A/M.

There exist scalars $\lambda_0, \ldots, \lambda_{n-1}$ such that $x_n - \lambda_0 x_0 - \cdots - \lambda_{n-1}x_{n-1}$ is in M. Denote this element of M by m_0. We claim that $\hat{m}_0$ is real-valued. By hypothesis and since $m_0 \in M$, we find that the equation $\lambda_0 + \lambda_1\phi(s) + \cdots + \lambda_{n-1}\phi(s)^{n-1} = \phi(s)^n$ has n distinct solutions, say $s_1, s_2, \ldots, s_n$. We write down these relations as follows:

$$\lambda_0 + \lambda_1\phi(s_1) + \cdots + \lambda_{n-1}\phi(s_1)^{n-1} = \phi(s_1)^n,$$
$$\vdots$$
$$\lambda_0 + \lambda_1\phi(s_n) + \cdots + \lambda_{n-1}\phi(s_n)^{n-1} = \phi(s_n)^n.$$

By hypothesis, we know that $\phi(s_1), \phi(s_2), \ldots, \phi(s_n)$ are n distinct real numbers. By Cramer's rule, we find that $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are all real and so $\hat{m}_0$ is real-valued. As we saw above, $\hat{m}_0$ vanishes exactly at $s_1, s_2, \ldots, s_n$.

Let m be an element in M with $\hat{m}$ real-valued. We have $m + im_0 \in M$ and so the equation $\hat{m}(s) + i\hat{m}_0(s) = 0$ has n distinct solutions in s. This implies that $\hat{m}(s_1) = \cdots = \hat{m}(s_n) = 0$, because $\hat{m}_0$ vanishes exactly at $s_1, s_2, \ldots, s_n$.

Fix m in M. There exist scalars $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ such that $m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1}x_{n-1}$ is in M. We have $m + m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1}x_{n-1} \in M$ and so the equation $2\operatorname{Re}\hat{m}(s) - \lambda_0\hat{x}_0(s) - \cdots - \lambda_{n-1}\hat{x}_0(s)\phi(s)^{n-1} = 0$ has n distinct solutions in s. By Cramer's rule, we find that $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are all real. On the other hand, we have $-m + m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1}x_{n-1} \in M$ and so the equation $-2i\operatorname{Im}\hat{m}(s) - \lambda_0\hat{x}_0(s) - \cdots - \lambda_{n-1}\hat{x}_0(s)\phi(s)^{n-1} = 0$ has n distinct solutions in s. By Cramer's rule, we find that $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are all pure imaginary. Combining these two results we find that all λ_j's are zero. This shows that m^* is in M.

We know that

$$m = 2^{-1}(m + m^*) + i\left[(2i)^{-1}(m - m^*)\right],$$

where the Fourier-Gelfand transforms of $m + m^*$ and $(2i)^{-1}(m - m^*)$ are real-valued. From the results presented in the preceding two paragraphs, we find that $\hat{m}$ vanishes at $s_1, s_2, \ldots, s_n$ for every m in M. This says that $M \subset I_{s_1 s_2 \cdots s_n}$, where $I_{s_1 s_2 \cdots s_n}$ denotes the space $\{x \in A : \hat{x}$ vanishes at $s_1, s_2, \ldots, s_n\}$. Since M and $I_{s_1 s_2 \cdots s_n}$ are of codimension n in A, $M = I_{s_1 s_2 \cdots s_n}$. We have completed the proof.

EXAMPLE 5. Any of the following spaces has the property described in Theorem 4: $C^n(T)$; $\mathrm{Lip}_\alpha(T)$, $0 < \alpha \leq 1$; $BV(T)$; $L^1 \cap L^p(G)$, $1 \leq p \leq \infty$, or $A^p(G)$ or $L^1 \cap C_0(G)$, or any normed ideal in $L^1(G)$ which is invariant under involution, where G is either a metrizable compact abelian group or the direct product of the real line R and a metrizable compact abelian group; l^p, $1 \leq p < \infty$, and $C_0(S)$, where S is any closed subset of $R \times Z^\infty$.

Example 5 follows immediately from the following lemma:

LEMMA 6. *The following two types of algebras have the property described in Theorem* 4:

(i) *Any normed ideal in $L^1(G)$ which is invariant under involution, where G is a metrizable compact abelian group or the direct product of R and such a G.*

(ii) $C_0(S)$, *where S is any closed subset of $R \times Z^\infty$.*

Proof. Let A be a normed ideal in $L^1(G)$ which is invariant under involution, where G is either a metrizable compact abelian group or the direct product of the real line R and a metrizable compact abelian group. From Theorems 2.2.2 and 2.2.6 in **[12]** we find that Γ is of the form $\Gamma_1 \times \Gamma_2$, where Γ_1 is $\{0\}$ or R and Γ_2 is countable. Write Γ_2 as $\{\gamma_1, \gamma_2, \ldots\}$. Define a function ϕ on Γ as follows:

$$\phi(\gamma_m) = m \quad \text{if } \Gamma_1 = \{0\},$$

$$\phi(x, \gamma_m) = \frac{x}{(1 + 4\pi^2 x^2)^{1/2}} + m \quad \text{if } \Gamma_1 = R,$$

then ϕ is a one-to-one real-valued function on Γ.

Choose an integrable function h_0 on G with the following property:

$$\hat{h}_0(\gamma_m) = e^{-m^2} \quad \text{if } \Gamma_1 = \{0\},$$

$$\hat{h}_0(x, \gamma_m) = e^{-(x^2+m^2)} \quad \text{if } \Gamma_1 = R.$$

It is well-known that Γ is sigma-compact, say $\Gamma = \bigcup_{j=1}^{\infty} K_j$, where K_j are compact subsets of Γ. There exists functions g_j in A such that $\hat{g}_j \geq 0$ on Γ and $\hat{g}_j = 1$ on K_j. Define

$$g_0 = \sum_{j=1}^{\infty} \frac{g_j}{j^2 \| g_j \|_A} \quad \text{and} \quad f_0 = g_0 * h_0,$$

then f_0 is in A and $\hat{f}_0$ is never zero.

For the case $\Gamma_1 = R$ we have

$$\begin{aligned}
\hat{f}_0(x, \gamma_m)\phi(x, \gamma_m)^j &= \hat{g}_0(x, \gamma_m)e^{-(x^2+m^2)}\left[\frac{x}{(1+4\pi^2x^2)^{1/2}} + m\right]^j \\
&= \hat{g}_0(x, \gamma_m)e^{-(x^2+m^2)} \sum_{k=0}^{j} \binom{j}{k} x^k \hat{G}_1(x)^k m^{j-k} \\
&= \hat{g}_0(x, \gamma_m) \sum_{k=0}^{j} \binom{j}{k} e^{-x^2} x^k \hat{G}_1(x)^k e^{-m^2} m^{j-k} \\
&= \hat{g}_0(x, \gamma_m) \sum_{k=0}^{j} \binom{j}{k} \hat{H}_k(x) \hat{G}_1(x)^k e^{-m^2} m^{j-k} \\
&= \hat{g}_0(x, \gamma_m)\hat{F}_j(x, \gamma_m) \\
&= \hat{f}_j(x, \gamma_m)
\end{aligned}$$

where

$$\binom{j}{k} = \frac{j(j-1)(j-2)\cdots(j-k+1)}{k!}, \qquad \binom{j}{0} = 1,$$

$$G_1(x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\Gamma(1/2)} \int_0^{\infty} e^{-\pi x^2/\delta} e^{-\delta/4\pi} \frac{d\delta}{\delta},$$

$$\hat{H}_k(x) = e^{-x^2} x^k,$$

$$F_j = \sum_{k=0}^{j} \binom{j}{k} \Big(H_k * \underbrace{G_1 * \cdots * G_1}_{k \text{ terms}} \Big) \left(\sum_{m=1}^{\infty} e^{-m^2} m^{j-k} \gamma_m \right),$$

$$f_j = g_0 * F_j.$$

The definition of G_1 can be found in [**13**, pp. 132]. The existence of integrable functions H_k on R is based on the fact that the function e^{-x^2} is rapidly decreasing. We have $G_1 \in L^1(R)$, $H_k \in L^1(R)$ and the functions

$$\sum_{m=1}^{\infty} e^{-m^2} m^{j-k} \gamma_m$$

are integrable. This implies that $F_j \in L^1(G)$ and so f_j is in A. This result is also true for the case $\Gamma_1 = \{0\}$; with minor modifications the preceding proof applies.

It remains to show (ii). Let S be any closed subset of the space $R \times Z^\infty$. From Theorem XI.6.5 in **[3]** we find that S is locally compact. It is well-known that $R \times Z^\infty$ is the dual group of $R \times T^\omega$. (See [**12**, §2.2].) Take $G = R \times T^\omega$ and define ϕ and h_0 as above. Denote the restriction of $\hat{h}_0$ on S by f_0 and the restriction of ϕ on S by itself, then $f_0 \in C_0(S)$, f_0 is never zero, ϕ is one-to-one and real-valued and $f_0\phi^j \in C_0(S)$ for all j. (Here we use the assumption that S is closed.) We have completed the proof.

The problem of characterizing the ideals of finite codimension for $L^1(R^2)$ and $C(D)$, D the closed unit disk, raised in **[16]** remains open.

Acknowledgement. I would like to thank the referee for his valuable suggestions.

References

[1] J. Cigler, *Normed ideals in $L^1(G)$*, Nederl. Akad. Wetensch. Indag. Math., **31** (1969), 273–282.

[2] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York and London, 1972.

[3] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.

[4] R. E. Edwards, *Fourier Series, a Modern Introduction*, 2 Vols. New York, N. Y.: Holt, Rinehart and Winston, Inc. 1967.

[5] A. M. Gleason, *A characterization of maximal ideals*, J. Analyse Math., **19** (1967), 171–172.

[6] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer-Verlag, Berlin, 1963.

[7] ______, *Abstract Harmonic Analysis*, Vol. II, Springer-Verlag, Berlin, 1970.

[8] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.

[9] J. P. Kahane and W. Zelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math., **29** (1968), 339–343.

[10] Y. Katznelson, *An Introduction to Harmonic Analysis*, New York, 1968.

[11] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford, 1968.

[12] W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

[13] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton, 1970.

[14] H. C. Wang, *Homogeneous Banach Algebras*, Lecture Notes in Pure and Appl. Mathematics, Dekker, New York, 1977.

[15] C. R. Warner and R. Whitley, *A characterization of regular maximal ideals*, Pacific J. Math., **30** (1969), 277–281.

[16] ______, *Ideals of finite codimension in* $C[0,1]$ *and* $L^1(R)$, Proc. Amer. Math. Soc., **76** (1979), 263–267.

Received September 10, 1981 and in revised form January 20, 1982.

INSTITUTE OF MATHEMATICS
NATIONAL TSING HUA UNIVERSITY
HSINCHU 300, TAIWAN
REPUBLIC OF CHINA

Pacific Journal of Mathematics
Vol. 107, No. 1 January, 1983