A GENERALIZATION OF THE GLEASON-KAHANE-ŻELAZKO THEOREM

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In this paper, we consider two classes of commutative Banach algebras, which include $C^\infty(T)$, $\text{Lip}_\beta(T)$, $BV(T)$, $L^1 \cap L^p(G)$, $A^p(G)$, $L^1 \cap C_0(G)$, $l^p$, $c_0$, and $C_0(S)$. We characterize ideals of finite codimension in these two classes of algebras and thereby partially answer a question suggested by C. R. Warner and R. Whitley.

In [5] and [9], A. M. Gleason, J. P. Kahane and W. Zelazko gave independently the following characterization of maximal ideals: Let $A$ be a commutative Banach algebra with unit element. Then a linear subspace $M$ of codimension 1 in $A$ is a maximal ideal in $A$ if and only if it consists of noninvertible elements, or equivalently, each element of $M$ belongs to some maximal ideal. This interesting result as first proved depended on the Hadamard Factorization Theorem.

This characterization of maximal ideals was extended in [15] and [16] to algebras without identity. In [16], C. R. Warner and R. Whitley also gave a characterization of ideals of finite codimension in $L^1(R)$ and $C[0,1]$. They showed: Let $A$ be any one of $L^1(R)$ and $C(S)$, where $S$ is a compact subset of $R$. If $M$ is a closed subspace of codimension $n$ in $A$ with the property that each element in $M$ belongs to at least $n$ regular maximal ideals, then $M$ is an ideal. In fact, $M$ is the intersection of $n$ regular maximal ideals. Also in [16], C. R. Warner and R. Whitley suggested the following question: For what locally compact abelian group $G$ does $L^1(G)$ have the property of $L^1(R)$ described above?

In this paper, we partially answer this question and generalize the work of C. R. Warner and R. Whitley. In this paper, two methods are introduced; One uses the Baire category theorem and the other generalizes the ideas of Theorems 2 and 4 in [16].

**Theorem 1.** Let $A$ be a commutative Banach algebra with a countable maximal ideal space $\mathfrak{M}$. If $M$ is a closed subspace of codimension $n$ in $A$ with the property that each element in $M$ belongs to at least $n$ regular maximal ideals, then $M$ is an ideal, which is the intersection of $n$ regular maximal ideals.
Proof. From the hypothesis, we know that \( M \subset \bigcup I_{s_1,s_2,\ldots,s_n} \) where \( I_{s_1,s_2,\ldots,s_n} \) denotes the space \( \{x \in A : x \text{ vanishes at } s_1, s_2, \ldots, s_n\} \) and the union is taken over all sets of distinct elements \( s_1, s_2, \ldots, s_n \) in \( \mathcal{M} \). Since \( \mathcal{M} \) is countable, the union is a countable union. By the Baire category theorem, \( M \subset I_{s_1,s_2,\ldots,s_n} \) for some set of distinct elements \( s_1, s_2, \ldots, s_n \) in \( \mathcal{M} \). If not, for any set of distinct elements \( s_1, s_2, \ldots, s_n \) in \( \mathcal{M} \), we have \( M \cap I_{s_1,s_2,\ldots,s_n} \neq M \). By the open mapping theorem, we find that \( M \cap I_{s_1,s_2,\ldots,s_n} \) is of first category in \( M \) and so the union \( \bigcup (M \cap I_{s_1,s_2,\ldots,s_n}) \) is of first category in \( M \). This implies that \( M \) is of first category in itself and contradicts the fact that \( M \) is a Banach space. Therefore \( M \subset I_{s_1,s_2,\ldots,s_n} \) for some set of distinct elements \( s_1, s_2, \ldots, s_n \) in \( \mathcal{M} \). Since \( M \) and \( I_{s_1,s_2,\ldots,s_n} \) are of codimension \( n \) in \( A \), \( M = I_{s_1,s_2,\ldots,s_n} \). We have completed the proof.

**Example 2.** Any of the following spaces has the property described in Theorem 1: \( C^n(T) \); \( \text{Lip}_\alpha(T) \), \( 0 < \alpha \leq 1 \); \( BV(T) \); \( L^p(G) \), \( 1 \leq p \leq \infty \), or \( A^p(G) \) or \( C(G) \), or any normed ideal in \( L^1(G) \), where \( G \) is a metrizable compact abelian group; \( l^p \), \( 1 \leq p < \infty \), and \( c_0 \) (cf. [1, 2, 4, 7, 8, 10, 11, 12, 14]).

**Remark 3.** The structure of a metrizable compact abelian group can be found in [12, Theorem 2.2.6]. It is well-known that the maximal ideal space of \( l^\infty \) coincides with the Stone-Čech compactification \( \beta Z^+ \), whose cardinal number is uncountable. (See [2, pp. 58] and [3, pp. 244].) Therefore Theorem 1 cannot be applied to this case. Theorem 1 answers the question suggested by C. R. Warner and R. Whitley for \( L^1(G) \) in the case \( G \) is compact and metrizable.

The following theorem extends the results presented in Theorem 1 to another kind of algebra while not hypothesizing that \( M \) be closed. (Compare this with Theorem 1 and [16, Theorems 2 and 4].) This theorem generalizes Theorems 2 and 4 in [16].

**Theorem 4.** Let \( A \) be a commutative Banach algebra with involution \( x \to x^* \) satisfying \( \hat{x}^* = \hat{x}^\sim \). Suppose that there is an element \( x_0 \) in \( A \), with \( \hat{x}_0 \) never zero, and that there is a one-to-one real-valued function \( \phi \) on the maximal ideal space \( \mathfrak{M} \) of \( A \) such that \( \hat{x}_0 \phi^j = \hat{x}_j \) for some \( x_j \) in \( A \) \((1 \leq j \leq n)\). If \( M \) is a subspace (not a priori closed) of codimension \( n \) in \( A \) with the property that each element in \( M \) belongs to at least \( n \) regular maximal ideals, then \( M \) is an ideal which is the intersection of \( n \) regular maximal ideals.
Proof. Without loss of generality, we may assume that \( \tilde{x}_0 \) is real-valued. Let \( \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{n-1} \) denote the cosets in the quotient space \( A/M \) corresponding to \( x_0, x_1, \ldots, x_{n-1} \). If \( \lambda_0 \tilde{x}_0 + \lambda_1 \tilde{x}_1 + \cdots + \lambda_{n-1} \tilde{x}_{n-1} = 0 \), then \( \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{n-1} x_{n-1} = 0 \) has \( n \) distinct solutions in \( s \). This implies that the polynomial \( \lambda_0 + \lambda_1 s + \cdots + \lambda_{n-1} s^{n-1} = 0 \) has \( n \) distinct zeros, which occurs only if all \( \lambda_j \)'s are zero. Hence \( \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{n-1} \) form a basis for \( A/M \).

There exist scalars \( \lambda_0, \ldots, \lambda_{n-1} \) such that \( x_n - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1} \) is in \( M \). Denote this element of \( M \) by \( m_0 \). We claim that \( m_0 \) is real-valued. By hypothesis and since \( m_0 \in M \), we find that the equation \( \lambda_0 + \lambda_1 \phi(s) + \cdots + \lambda_{n-1} \phi(s)^{n-1} = \phi(s)^n \) has \( n \) distinct solutions, say \( s_1, s_2, \ldots, s_n \).

We write down these relations as follows:

\[
\begin{align*}
\lambda_0 + \lambda_1 \phi(s_1) + \cdots + \lambda_{n-1} \phi(s_1)^{n-1} &= \phi(s_1)^n, \\
\lambda_0 + \lambda_1 \phi(s_2) + \cdots + \lambda_{n-1} \phi(s_2)^{n-1} &= \phi(s_2)^n, \\
&\vdots \\
\lambda_0 + \lambda_1 \phi(s_n) + \cdots + \lambda_{n-1} \phi(s_n)^{n-1} &= \phi(s_n)^n.
\end{align*}
\]

By hypothesis, we know that \( \phi(s_1), \phi(s_2), \ldots, \phi(s_n) \) are \( n \) distinct real numbers. By Cramer's rule, we find that \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) are all real and so \( m_0 \) is real-valued. As we saw above, \( m_0 \) vanishes exactly at \( s_1, s_2, \ldots, s_n \).

Let \( m \) be an element in \( M \) with \( m \) real-valued. We have \( m + im_0 \in M \) and so the equation \( \hat{m}(s) + i\hat{m}_0(s) = 0 \) has \( n \) distinct solutions in \( s \). This implies that \( \hat{m}(s_1) = \cdots = \hat{m}(s_n) = 0 \), because \( m_0 \) vanishes exactly at \( s_1, s_2, \ldots, s_n \).

Fix \( m \) in \( M \). There exist scalars \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) such that \( m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1} \) is in \( M \). We have \( m + m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1} \in M \) and so the equation \( 2\text{Re}\, \hat{m}(s) - \lambda_0 \hat{x}_0(s) - \cdots - \lambda_{n-1} \hat{x}_0(s) \phi(s)^{n-1} = 0 \) has \( n \) distinct solutions in \( s \). By Cramer's rule, we find that \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) are all real. On the other hand, we have \(-m + m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1} \in M \) and so the equation \(-2i\, \text{Im}\, \hat{m}(s) - \lambda_0 \hat{x}_0(s) - \cdots - \lambda_{n-1} \hat{x}_0(s) \phi(s)^{n-1} = 0 \) has \( n \) distinct solutions in \( s \). By Cramer's rule, we find that \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \) are all pure imaginary. Combining these two results we find that all \( \lambda_j \)'s are zero. This shows that \( m^* \) is in \( M \).

We know that

\[
m = 2^{-1}(m + m^*) + i\left[(2i)^{-1}(m - m^*)\right],
\]
where the Fourier-Gelfand transforms of \( m + m^* \) and \( (2i)^{-1}(m - m^*) \) are real-valued. From the results presented in the preceding two paragraphs, we find that \( \hat{m} \) vanishes at \( s_1, s_2, \ldots, s_n \) for every \( m \) in \( M \). This says that \( M \subset I_{1, s_2, \ldots, s_n} \) where \( I_{1, s_2, \ldots, s_n} \) denotes the space \( \{ x \in A : \hat{x} \text{ vanishes at } s_1, s_2, \ldots, s_n \} \). Since \( M \) and \( I_{1, s_2, \ldots, s_n} \) are of codimension \( n \) in \( A \), \( M = I_{1, s_2, \ldots, s_n} \). We have completed the proof.

**Example 5.** Any of the following spaces has the property described in Theorem 4: \( C^n(T) \), \( Lip_\alpha(T) \), \( 0 < \alpha \leq 1 \); \( BV(T) \), \( L^1 \cap L^p(G) \), \( 1 \leq p \leq \infty \), or \( A^p(G) \) or \( L^1 \cap C_0(G) \), or any normed ideal in \( L^1(G) \) which is invariant under involution, where \( G \) is either a metrizable compact abelian group or the direct product of the real line \( R \) and a metrizable compact abelian group; \( L^p \), \( 1 \leq p < \infty \), and \( C_0(S) \), where \( S \) is any closed subset of \( R \times Z^\infty \).

Example 5 follows immediately from the following lemma:

**Lemma 6.** The following two types of algebras have the property described in Theorem 4:

(i) Any normed ideal in \( L^1(G) \) which is invariant under involution, where \( G \) is a metrizable compact abelian group or the direct product of \( R \) and such a \( G \).

(ii) \( C_0(S) \), where \( S \) is any closed subset of \( R \times Z^\infty \).

**Proof.** Let \( A \) be a normed ideal in \( L^1(G) \) which is invariant under involution, where \( G \) is either a metrizable compact abelian group or the direct product of the real line \( R \) and a metrizable compact abelian group. From Theorems 2.2.2 and 2.2.6 in [12] we find that \( \Gamma \) is of the form \( \Gamma_1 \times \Gamma_2 \), where \( \Gamma_1 \) is \( \{0\} \) or \( R \) and \( \Gamma_2 \) is countable. Write \( \Gamma_2 \) as \( \{ \gamma_1, \gamma_2, \ldots \} \). Define a function \( \phi \) on \( \Gamma \) as follows:

\[
\phi(\gamma_m) = m \quad \text{if } \Gamma_1 = \{0\},
\]

\[
\phi(x, \gamma_m) = \frac{x}{(1 + 4\pi^2x^2)^{1/2}} + m \quad \text{if } \Gamma_1 = R,
\]

then \( \phi \) is a one-to-one real-valued function on \( \Gamma \).

Choose an integrable function \( h_0 \) on \( G \) with the following property:

\[
\hat{h}_0(\gamma_m) = e^{-m^2} \quad \text{if } \Gamma_1 = \{0\},
\]

\[
\hat{h}_0(x, \gamma_m) = e^{-(x^2 + m^2)} \quad \text{if } \Gamma_1 = R.
\]
It is well-known that $\Gamma$ is sigma-compact, say $\Gamma = \bigcup_{j=1}^{\infty} K_j$, where $K_j$ are compact subsets of $\Gamma$. There exists functions $g_j$ in $A$ such that $\hat{g}_j \geq 0$ on $\Gamma$ and $\hat{g}_j = 1$ on $K_j$. Define
\[
g_0 = \frac{\sum_{j=1}^{\infty} g_j}{\sum_{j=1}^{\infty} 1} \text{ and } f_0 = g_0 * h_0,
\]
then $f_0$ is in $A$ and $\hat{f}_0$ is never zero.

For the case $\Gamma_1 = \mathbb{R}$ we have
\[
\hat{f}_0(x, \gamma_m) = \hat{g}_0(x, \gamma_m) e^{-(x^2 + m^2)} \left[ \frac{x}{(1 + 4\pi^2 x^2)^{1/2}} + m \right]^j
\]
\[
\hat{g}_0(x, \gamma_m) = \hat{g}_0(x, \gamma_m) - \sum_{j=0}^{\infty} \frac{x^j}{j!} \hat{G}_1(x)^j m^{-j} - \sum_{j=0}^{\infty} \frac{x^j}{j!} \hat{G}_1(x)^j m^{-j}
\]
\[
\hat{G}_1(x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-\pi x^2/8} e^{-\delta x^2/4\pi} d\delta
\]
\[
\hat{H}_k(x) = e^{-x^2} x^k,
\]
\[
F_j = \sum_{k=0}^{j} \binom{j}{k} \left[ \frac{H_k * G_1 * \cdots * G_1}{\sum_{m=1}^{k \text{ terms}}} \right] \left( \sum_{m=1}^{\infty} e^{-m^2} m^{-j-k} \right),
\]
\[
f_j = g_0 * F_j.
\]

The definition of $G_1$ can be found in [13, pp. 132]. The existence of integrable functions $H_k$ on $\mathbb{R}$ is based on the fact that the function $e^{-x^2}$ is
rapidly decreasing. We have $G_1 \in L^1(R), H_k \in L^1(R)$ and the functions 
\[ \sum_{m=1}^{\infty} e^{-m^2 m^{1-k}} \gamma_m \]
are integrable. This implies that $F_j \in L^1(G)$ and so $f_j$ is in $A$. This result is also true for the case $\Gamma_1 = \{0\};$ with minor modifications the preceding proof applies.

It remains to show (ii). Let $S$ be any closed subset of the space $R \times Z^\infty$. From Theorem XI.6.5 in [3] we find that $S$ is locally compact. It is well-known that $R \times Z^\infty$ is the dual group of $R \times T^\omega$. (See [12, §2.2].) Take $G = R \times T^\omega$ and define $\phi$ and $h_0$ as above. Denote the restriction of $h_0$ on $S$ by $f_0$ and the restriction of $\phi$ on $S$ by itself, then $f_0 \in C_0(S), f_0$ is never zero, $\phi$ is one-to-one and real-valued and $f_0 \phi^j \in C_0(S)$ for all $j$. (Here we use the assumption that $S$ is closed.) We have completed the proof.

The problem of characterizing the ideals of finite codimension for $L^1(R^2)$ and $C(D), D$ the closed unit disk, raised in [16] remains open.

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References


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