

# Pacific Journal of Mathematics

**SOME POINCARÉ SERIES RELATED TO IDENTITIES OF  $2 \times 2$   
MATRICES**

PATRICK RONALD HALPIN

## SOME POINCARÉ SERIES RELATED TO IDENTITIES OF $2 \times 2$ MATRICES

PATRICK HALPIN

A partial solution to a problem of Procesi has recently been given by Formanek, Halpin, Li by determining the Poincaré series of the ideal of two variable identities of  $M_2(k)$ . Two related results are obtained in this article.

A weak identity of  $M_n(k)$  is a polynomial which vanishes identically on  $sl_n$ , the subspace of  $M_n(k)$  of matrices of trace zero. We show that the Poincaré series of the ideal of two variable weak identities of  $M_2(k)$  is rational. In addition it is shown that the ideal of identities of upper triangular  $2 \times 2$  matrices in an arbitrary finite number of variables has a rational Poincaré series. As an application we are able to determine this ideal precisely.

**Introduction.** Let  $S = K \langle x_1, \dots, x_n \rangle$  be the free associative algebra over  $k$  where  $k$  is any field of characteristic zero.  $S$  is naturally graded by giving  $x_1$  degree  $(1, 0, \dots, 0)$ ,  $x_2$  degree  $(0, 1, \dots, 0)$ , etc. Denote by  $S_{(i_1, \dots, i_n)}$  the subspace of  $S$  generated by monomials of degree  $(i_1, \dots, i_n)$ . If  $A$  is a homogeneously generated ideal of  $S$  then we associate a series to  $A$ , called the Poincaré series of  $A$ , via

$$P(A) = \sum_{i_1, \dots, i_n \geq 0} a(i_1, \dots, i_n) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}$$

where  $a(i_1, \dots, i_n) = \dim_k(A \cap S_{(i_1, \dots, i_n)})$ . In [1] Formanek, Halpin, Li showed that the Poincaré series of the ideal of two variables identities of  $M_2(k)$  is a rational function in  $s_1$  and  $s_2$ . In this article we obtain two related results.

A weak identity of  $M_n(k)$  is a polynomial which vanishes upon substitution of elements of  $sl_n(k)$ , where  $sl_n(k)$  denotes the subspace of  $M_n(k)$  of matrices of trace zero. The notion of a weak identity was introduced by Razmyslov [2] in connection with the study of central polynomials. Let  $T_2^W(x_1, x_2)$  denote the ideal of  $k \langle x_1, x_2 \rangle$  of weak identities of  $M_2(k)$ . In Section 1 we determine  $P(T_2^W(x_1, x_2))$  and find that it is again a rational function in  $s_1$  and  $s_2$ .

In §2 we consider the identities of the subalgebra of  $M_2(k)$  consisting of upper triangular matrices. By restricting to upper triangular matrices we are able to obtain results more complete than those obtained in [1]. We

calculate the Poincaré series of the ideal of identities of upper triangular  $2 \times 2$  matrices in an arbitrary finite number of variables. As an application the ideal of identities of upper triangular  $2 \times 2$  matrices is determined explicitly.

**1. Weak identities of  $M_2(k)$ .** Let  $T_2^W(x_1, x_2)$  denote the collection of two variable weak identities of  $M_2(k)$  where  $k$  is a field of characteristic zero. It is easy to see that  $T_2^W(x_1, x_2)$  is an ideal of  $k\langle x_1, x_2 \rangle$ , although it is not a  $T$ -ideal in the usual sense. As in the case of the identities of  $M_n(k)$ , the ideal of weak identities  $M_n(k)$  is homogeneously generated. The goal of this section is to determine  $P(T_2^W(x_1, x_2))$ .

Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{22} & -X_{11} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & -Y_{11} \end{pmatrix}$$

be  $2 \times 2$  generic matrices of trace zero. The  $x_{ij}, y_{ij}$  are commuting indeterminates. Define  $R = k[X, Y]$  as the algebra generated over  $k$  by  $X$  and  $Y$ .  $R$  may be graded by assigning  $X$  degree  $(1, 0)$  and  $Y$  degree  $(0, 1)$ . Let  $A = k[x_{ij}, y_{ij}]$  be the commutative polynomial ring generated over  $k$  by the six indeterminates  $x_{ij}, y_{ij}$ .  $A$  may be graded by assigning each  $x_{ij}$  degree  $(1, 0)$  and each  $y_{ij}$  degree  $(0, 1)$ .

The following lemma, which is analogous to a well known result on identities of  $M_n(k)$ , is clear.

LEMMA 1. *The sequence*

$$0 \rightarrow T_2^W(x_1, x_2) \rightarrow k\langle x_1, x_2 \rangle \xrightarrow{\pi} k[X, Y] \rightarrow 0,$$

where  $\pi(x_1) = X$  and  $\pi(x_2) = Y$ , is an exact sequence of graded  $k$ -modules.

By  $D, T$  we denote determinant, trace respectively. We define

$$\begin{aligned} B &= k[D(X), D(Y), T(XY)] \\ &= k[x_{11}^2 + x_{12}x_{21}, y_{11}^2 + y_{12}y_{21}, x_{12}y_{21} + x_{21}y_{12} + 2x_{11}y_{11}] \end{aligned}$$

$B$  inherits a grading as a homogeneously generated submodule of  $A$ .

LEMMA 2.  *$B$  is a commutative polynomial ring over  $k$  in  $D(X), D(Y), T(XY)$ .*

*Proof.* This is easily seen by specializing  $x_{12} = x_{21} = 0$ .

The proof of the following lemma is routine and is therefore omitted.

LEMMA 3.  $I, X, Y, XY$  are linearly independent over  $A$  and so are linearly independent over  $B$ .

THEOREM 4.  $R = BI \oplus BX \oplus BY \oplus BXY$ , a direct sum of  $k$ -spaces.

*Proof.* The following relations are easily verified and show that  $BI \oplus BX \oplus BY \oplus BYX \subseteq R$ :

$$\begin{aligned} X^2 &= -D(X)I, \\ Y^2 &= -D(Y)I, \\ XY + YX &= T(XY)I. \end{aligned}$$

For the other inclusion note that  $B$  is the ring generated by  $D(X), D(Y), T(XY)$ . Therefore the three relations above show that  $BI \oplus BX \oplus BY \oplus BXY$  is a ring containing  $X, Y$  and hence  $R \subseteq BI \oplus BX \oplus BY \oplus BXY$ .

The following easy lemma, used in [1], will be used extensively in the article.

LEMMA 5. Let  $M$  and  $N$  be homogeneous  $k$ -submodules of  $M_2(k[x_{ij}, y_{ij}])$ .

(1) If  $M \oplus N$  is a direct sum then  $P(M \oplus N) = P(M) + P(N)$ .

(2) If  $U \in M_2(k[x_{ij}, y_{ij}])$  is a homogeneous nonzero divisor of degree  $(p, q)$  then  $P(MU) = s_1^p s_2^q P(M)$ .

THEOREM 6. We have

$$(1) \quad P(R) = \frac{1}{(1 - s_1)(1 - s_2)(1 - s_1 s_2)}$$

and

$$(2) \quad P(T_2^w(x_1, x_2)) = \frac{s_1 s_2 (s_1 + s_2 - s_1 s_2)}{(1 - s_1)(1 - s_2)(1 - s_2 s_2)(1 - s_1 - n s_2)}.$$

*Proof.* By Lemma 2  $B$  is a commutative polynomial ring in  $D(X), D(Y), T(XY)$  of degrees  $(2, 0), (0, 2), (1, 1)$  respectively. Therefore

$$\begin{aligned} P(B) &= P(k[D(X), D(Y), T(XY)]) \\ &= (1 + s_1^2 + s_1^4 + \dots)(1 + s_2^2 + s_2^4 + \dots)(1 + s_1 s_2 + s_1^2 s_2^2 + \dots) \\ &= \frac{1}{(1 - s_1^2)(1 - s_2^2)(1 - s_1 s_2)}. \end{aligned}$$

Therefore

$$\begin{aligned} P(R) &= P(BI \oplus BX \oplus BY \oplus BXY) \\ &= P(B) + P(BX) + P(BY) + P(BXY) = (1 + s_1)(1 + s_2)P(B) \\ &= \frac{1}{(1 - s_1)(1 - s_2)(1 - s_1s_2)}. \end{aligned}$$

For (2) we note that by the exact sequence of Lemma 1

$$\begin{aligned} P(T_2^W(x_1, x_2)) &= P(k\langle x_1, x_2 \rangle) - P(R) \\ &= \frac{1}{1 - s_1 - s_2} - \frac{1}{(1 - s_1)(1 - s_2)(1 - s_1s_2)} \\ &= \frac{s_1s_2(s_1 + s_2 - s_1s_2)}{(1 - s_1)(1 - s_2)(1 - s_1s_2)(1 - s_1 - s_2)}. \end{aligned}$$

**2. Upper triangular matrices.** The object of study in this section is the ideal of identities of upper triangular  $2 \times 2$  matrices.

We first establish the notation that will be used in this section. Let  $A = k[x_{ij}^{(k)}; 1 \leq i \leq j \leq 2, 1 \leq k \leq n]$  be the commutative polynomial ring generated over  $k$  by the  $3n$  variables  $x_{ij}^{(k)}$ . By  $T_2^U(x_1, \dots, x_n)$  we mean the ideal of identities of upper triangular  $2 \times 2$  matrices in  $x_1, \dots, x_n$  with coefficients in  $k$ . Now let  $X_1, \dots, X_n$  be upper triangular  $2 \times 2$  generic matrices where

$$X_i = \begin{pmatrix} x_{11}^{(i)} & x_{12}^{(i)} \\ 0 & x_{22}^{(i)} \end{pmatrix}.$$

$R = k[X_1, \dots, X_n]$  denotes the algebra generated over  $k$  by  $X_1, \dots, X_n$ .

We begin with a version of the well known diagonalization technique.

**LEMMA 7.**  $R = k[X_1, X_2, \dots, X_n]$  is isomorphic (as  $k$ -algebras) to  $k[X, X_2, \dots, X_n]$  where

$$X = \begin{pmatrix} x_{11}^{(1)} & 0 \\ 0 & x_{22}^{(1)} \end{pmatrix}.$$

*Proof.* The matrix  $X_1$  is diagonalizable by some matrix  $T$  which may be taken upper triangular. Then

$$R \cong T^{-1}RT = k[X, T^{-1}X_2T, \dots, T^{-1}X_nT] \cong k[X, X_2, \dots, X_n].$$

In view of Lemma 7 from now on we will take  $R = k[X_1, \dots, X_n]$  where  $X_1 = X$ .

We grade  $k\langle x_1, \dots, x_n \rangle$  as in the previous section. Similarly  $A = k[x_{ij}^{(k)}; 1 \leq i \leq j \leq 2, 1 \leq k \leq n]$  and  $B = k[x_{ii}^{(k)}; i = 1, 2, 1 \leq k \leq n]$  are graded by giving each  $x_{ij}^{(1)}$  degree  $(1, 0, \dots, 0)$ , each  $x_{ij}^{(2)}$  degree  $(0, 1, \dots, 0)$ , etc. Also  $R$  is graded by assigning  $X_1$  degree  $(1, 0, \dots, 0)$ ,  $X_2$  degree  $(0, 1, \dots, 0)$ , etc.

With these gradings we state an obvious lemma which is analogous to Lemma 1.

LEMMA 8. *The sequence below, with the obvious maps, is an exact sequence of graded  $k$ -modules:*

$$0 \rightarrow T_2^U(x_1, \dots, x_n) \rightarrow k\langle x_1, \dots, x_n \rangle \rightarrow R \rightarrow 0.$$

The main theorem of this section is the evaluation of  $P(T_2^U(x_1, \dots, x_n))$  which will be proved by induction on  $n$ . In order to start the induction at  $n = 2$  we first calculate  $P(R_0)$  where  $R_0 = k[X_1, X_2]$ .

LEMMA 9. *The commutator ideal  $[R_0, R_0]$  equals*

$$k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}] \cdot [X_1, X_2].$$

*Proof.*  $[R_0, R_0]$  is the ideal of  $R_0$  generated by

$$[X_1, X_2] = \begin{pmatrix} 0 & (x_{11}^{(1)} - x_{22}^{(1)})x_{12}^{(2)} \\ 0 & 0 \end{pmatrix}.$$

Now notice that

$$X_i[X_1, X_2] = x_{11}^{(i)}[X_1, X_2]$$

and

$$[X_1, X_2]X_i = x_{22}^{(i)}[X_1, X_2].$$

Therefore

$$[R_0, R_0] \subseteq k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}] \cdot [X_1, X_2].$$

For the reverse inclusion if  $(x_{11}^{(1)})^a(x_{22}^{(1)})^b(x_{11}^{(2)})^c(x_{22}^{(2)})^d$  is any monomial in  $k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}]$  then one sees easily that

$$\begin{aligned} & (x_{11}^{(1)})^a(x_{22}^{(1)})^b(x_{11}^{(2)})^c(x_{22}^{(2)})^d[X_1, X_2] \\ &= X_1^a X_2^c [X_1, X_2] X_1^b X_2^d \in [R_0, R_0]. \end{aligned}$$

LEMMA 10.

$$P([R_0, R_0]) = \frac{s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}.$$

*Proof.* Since  $x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}$  have degrees  $(1, 0), (1, 0), (0, 1), (0, 1)$  respectively, we have

$$\begin{aligned} P([R_0, R_0]) &= P(k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}] \cdot [X_1, X_2]) \\ &= s_1 s_2 P(k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}]) \\ &= s_1 s_2 (1 + s_1 + s_1^2 + \cdots)^2 (1 + s_2 + s_2^2 + \cdots)^2 \\ &= \frac{s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}. \end{aligned}$$

LEMMA 11.

$$P(R_0) = \frac{1 - s_1 - s_2 + 2s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}.$$

*Proof.* Since  $R_0/[R_0, R_0] \cong k[x_1, x_2]$ , a commutative polynomial ring, it follows that as  $k$ -spaces

$$R_0 \cong_k [R_0, R_0] \oplus_k k[x_1, x_2].$$

Therefore

$$\begin{aligned} P(R_0) &= P([R_0, R_0]) + P(k[x_1, x_2]) \\ &= \frac{s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2} + \frac{1}{(1 - s_1)(1 - s_2)} = \frac{1 - s_1 - s_2 + 2s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}. \end{aligned}$$

In order to calculate  $P(T_2^U(x_1, \dots, x_n))$  it suffices to calculate  $P(k[X_1, \dots, X_n])$ . We proceed by induction on  $n$ , having established the case  $n = 2$ . The following lemma will be used to execute the inductive step.

LEMMA 12. *The ideal  $[X_1, R]$  of  $R$  equals  $[X_1, X_2]B \oplus_k [X_1, X_3]B \oplus_k \cdots \oplus_k [X_1, X_n]B$ .*

*Proof.* The ideal  $[X_1, R]$  is the ideal of  $R$  generated by

$$\begin{aligned} [X_1, X_2] &= \begin{pmatrix} 0 & (x_{11}^{(1)} - x_{22}^{(1)})x_{12}^{(2)} \\ 0 & 0 \end{pmatrix} \\ &\vdots \\ [X_1, X_n] &= \begin{pmatrix} 0 & (x_{11}^{(1)} - x_{22}^{(1)})x_{12}^{(n)} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Notice that

$$X_i[X_1, X_j] = x_{11}^{(i)}[X_1, X_j]$$

and

$$[X_1, X_j]X_i = x_{22}^{(i)}[X_1, X_j].$$

The lemma now follows easily as in Lemma 9. Of course the sum above is direct since the  $x_{12}^{(k)}$ ,  $1 \leq k \leq n$ , are distinct indeterminates.

As an immediate consequence of Lemma 12 we may compute  $P([X_1, R])$ .

LEMMA 13.

$$P([X_1, R]) = \frac{s_1s_2 + s_1s_3 + \cdots + s_1s_n}{(1-s_1)^2(1-s_2)^2 \cdots (1-s_n)^2}.$$

THEOREM 14.

$$P(R) = \frac{(2(1-s_1) \cdots (1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2(1-s_2)^2 \cdots (1-s_n)^2}.$$

*Proof.* We induct on  $n$ . The case  $n = 2$  is Lemma 11 so we assume  $n \geq 3$  and that the theorem is true for  $n - 1$  variables.

$R$  has the following decomposition as a  $k$ -space:

$$R \cong_k R/[X_1, R] \oplus_k [X_1, R] \cong_k \bigoplus_{i=0}^{\infty} X_i^i k[X_2, \dots, X_n] \oplus_k [X_1, R].$$

Therefore,

$$\begin{aligned} P(R) &= P\left(\bigoplus_{i=0}^{\infty} X_i^i k[X_2, \dots, X_n]\right) + P([X_1, R]) \\ &= (1 + s_1 + s_1^2 + \cdots)P(k[X_2, \dots, X_n]) + P([X_1, R]). \end{aligned}$$

By the inductive hypothesis  $P(k[X_2, \dots, X_n])$  equals

$$\frac{(2(1-s_2) \cdots (1-s_n)) + (s_2 + \cdots + s_n) - 1}{(1-s_2)^2(1-s_3)^2 \cdots (1-s_n)^2},$$

and by Lemma 13  $P([X_1, R])$  equals

$$\frac{s_1s_2 + \cdots + s_1s_n}{(1-s_1)^2 \cdots (1-s_n)^2}.$$



Thus

$$\begin{aligned} P(R) &= \frac{(2(1-s_2) \cdots (1-s_n)) + (s_2 + \cdots + s_n) - 1}{(1-s_1)(1-s_2)^2(1-s_3)^2 \cdots (1-s_n)^2} \\ &\quad + \frac{s_1s_2 + \cdots + s_1s_n}{(1-s_1)^2 \cdots (1-s_n)^2} \\ &= \frac{(2(1-s_1) \cdots (1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2(1-s_2)^2 \cdots (1-s_n)^2}. \end{aligned}$$

We now prove the main result of this section.

**THEOREM 15.**

$$P(T_2^U(x_1, \dots, x_n)) = \frac{((1-s_1) \cdots (1-s_n) - (1-s_1 - \cdots - s_n))^2}{(1-s_1 - \cdots - s_n)(1-s_1)^2 \cdots (1-s_n)^2}.$$

*Proof.* By the exact sequence of Lemma 8 we have

$$\begin{aligned} P(T_2^U(x_1, \dots, x_n)) &= P(k\langle x_1, \dots, x_n \rangle) - P(k[X_1, \dots, X_n]) \\ &= \frac{1}{1-s_1 - \cdots - s_n} - \frac{2((1-s_1) \cdots (1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2 \cdots (1-s_n)^2} \\ &= \frac{((1-s_1) \cdots (1-s_n) - (1-s_1 - \cdots - s_n))^2}{(1-s_1 - \cdots - s_n)(1-s_1)^2 \cdots (1-s_n)^2}. \end{aligned}$$

As an application of Theorem 15 we now give a precise description of  $T_2^U(x_1, \dots, x_n)$ . Let  $T_1(x_1, \dots, x_n)$  denote the commutator ideal of  $k\langle x_1, \dots, x_n \rangle$ . In other words,  $T_1(x_1, \dots, x_n)$  is the ideal of  $k\langle x_1, \dots, x_n \rangle$  such that the sequence

$$0 \rightarrow T_1(x_1, \dots, x_n) \rightarrow k\langle x_1, \dots, x_n \rangle \rightarrow k[x_1, \dots, x_n] \rightarrow 0$$

is exact. It follows that

$$\begin{aligned} P(T_1(x_1, \dots, x_n)) &= \frac{1}{1-s_1 - \cdots - s_n} - \frac{1}{(1-s_1) \cdots (1-s_n)} \\ &= \frac{(1-s_1) \cdots (1-s_n) - (1-s_1 - \cdots - s_n)}{(1-s_1 - \cdots - s_n)(1-s_1) \cdots (1-s_n)}. \end{aligned}$$

We will show that  $T_2^U(x_1, \dots, x_n) = (T_1(x_1, \dots, x_n))^2$ . To show one inclusion is very easy. It then suffices to show that both members have the same Poincaré series. To calculate the Poincaré series of  $(T_1(x_1, \dots, x_n))^2$  we need to make use of a combinatorial lemma, due to Formanek. We sketch a proof of the lemma.

LEMMA 16. (*Formanek*) Let  $I$  and  $J$  be homogeneously generated ideals of  $k\langle x_1, \dots, x_n \rangle$ . Then  $P(IJ) = P(I)P(J)(1 - s_1 - \dots - s_n)$ .

*Proof.* One first shows, using only elementary arguments, that  $I$  and  $J$  are free as left ideals on homogeneous generators. Let  $\alpha(i_1, \dots, i_n)$  equal the number of free generators of  $I$  considered as a left ideal of degree  $(i_1, \dots, i_n)$ . Define

$$G(I) = \sum_{i_1, \dots, i_n \geq 0} \alpha(i_1, \dots, i_n) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}.$$

Similarly define  $G(J)$  and  $G(IJ)$ . Then  $G(IJ) = G(I)G(J)$  and  $P(I) = G(I)/(1 - s_1 - \dots - s_n)$ . The lemma follows.

THEOREM 17.  $T_2^U(x_1, \dots, x_n) = (T_1(x_1, \dots, x_n))^2$ .

*Proof.* We first show that  $(T_1(x_1, \dots, x_n))^2 \subseteq T_2^U(x_1, \dots, x_n)$ . Any element of  $(T_1(x_1, \dots, x_n))^2$  is a sum of terms of the form  $r_1[x_i, x_j]r_2[x_k, x_l]r_3$  where  $1 \leq i, j, k, l \leq n$  and  $r_1, r_2, r_3 \in k\langle x_1, \dots, x_n \rangle$ . The commutator of two upper triangular matrices is strictly upper triangular. Therefore each term of the form above is an identity for  $R$  since any finite product of upper triangular  $2 \times 2$  matrices where at least two of the factors are strictly upper triangular is zero. Therefore  $(T_1(x_1, \dots, x_n))^2 \subseteq T_2^U(x_1, \dots, x_n)$ .

As mentioned above it now suffices to show that  $(T_1(x_1, \dots, x_n))^2$  and  $T_2^U(x_1, \dots, x_n)$  have the same Poincaré series. By Lemma 16

$$\begin{aligned} P\left(\left(T_1(x_1, \dots, x_n)\right)^2\right) &= (1 - s_1 - \dots - s_n)(P(T_1(x_1, \dots, x_n)))^2 \\ &= (1 - s_1 - \dots - s_n) \left( \frac{(1 - s_1) \cdots (1 - s_n) - (1 - s_1 - \dots - s_n)}{(1 - s_1 - \dots - s_n)(1 - s_1) \cdots (1 - s_n)} \right)^2 \\ &= P(T_2^U(x_1, \dots, x_n)). \end{aligned}$$

#### REFERENCES

1. E. Formanek, P. Halpin, W.-C. Li, *The Poincaré series of the ring of  $2 \times 2$  generic matrices*, J. Algebra, **69** (1981), 105–112.
2. Y. Razmyslov, *On a problem of Kaplansky*, Izv. A.N.S.S.S.R. Sec. Math., **37** (1973), 479–496 (Russian). Translation: Math. USSR-Izv., Vol. 7 (1973), No. 3.

Received August 24, 1981. This article is a portion of the author's Ph.D. thesis at The Pennsylvania State University.



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DONALD BABBITT (Managing Editor)

University of California

Los Angeles, CA 90024

HUGO ROSSI

University of Utah

Salt Lake City, UT 84112

C. C. MOORE and ARTHUR OGUS

University of California

Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics

University of Southern California

Los Angeles, CA 90089-1113

R. FINN and H. SAMELSON

Stanford University

Stanford, CA 94305

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

(1906–1982)

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF HAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

# Pacific Journal of Mathematics

Vol. 107, No. 1

January, 1983

<b>John Kelly Beem and Phillip E. Parker</b> , Klein-Gordon solvability and the geometry of geodesics .....	1
<b>David Borwein and Annon Jakimovski</b> , Transformations of certain sequences of random variables by generalized Hausdorff matrices .....	15
<b>Willy Brandal and Erol Barbut</b> , Localizations of torsion theories .....	27
<b>John David Brillhart, Paul Erdős and Richard Patrick Morton</b> , On sums of Rudin-Shapiro coefficients. II .....	39
<b>Martin Lloyd Brown</b> , A note on tamely ramified extensions of rings .....	71
<b>Chang P'ao Ch'ên</b> , A generalization of the Gleason-Kahane-Żelazko theorem .....	81
<b>I. P. de Guzman</b> , Annihilator alternative algebras .....	89
<b>Ralph Jay De Laubenfels</b> , Extensions of $d/dx$ that generate uniformly bounded semigroups .....	95
<b>Patrick Ronald Halpin</b> , Some Poincaré series related to identities of $2 \times 2$ matrices .....	107
<b>Fumio Hiai, Masanori Ohya and Makoto Tsukada</b> , Sufficiency and relative entropy in $*$ -algebras with applications in quantum systems .....	117
<b>Dean Robert Hickerson</b> , Splittings of finite groups .....	141
<b>Jon Lee Johnson</b> , Integral closure and generalized transforms in graded domains .....	173
<b>Maria Grazia Marinari, Francesco Odetti and Mario Raimondo</b> , Affine curves over an algebraically nonclosed field .....	179
<b>Douglas Shelby Meadows</b> , Explicit PL self-knottings and the structure of PL homotopy complex projective spaces .....	189
<b>Charles Kimbrough Megibben, III</b> , Crawley's problem on the unique $\omega$ -elongation of $p$ -groups is undecidable .....	205
<b>Mary Elizabeth Schaps</b> , Versal determinantal deformations .....	213
<b>Stephen Scheinberg</b> , Gauthier's localization theorem on meromorphic uniform approximation .....	223
<b>Peter Frederick Stiller</b> , On the uniformization of certain curves .....	229
<b>Ernest Lester Stitzinger</b> , Engel's theorem for a class of algebras .....	245
<b>Emery Thomas</b> , On the zeta function for function fields over $F_p$ .....	251