SUFFICIENCY AND RELATIVE ENTROPY IN ∗-ALGEBRAS
WITH APPLICATIONS IN QUANTUM SYSTEMS

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The sufficiency and weak sufficiency in *-algebras are discussed. Some properties are studied concerning the relative entropy and the sufficiency for invariant states and KMS states in $W^*$- and $C^*$-dynamical systems.

Introduction. The concept of sufficiency is very important in mathematical statistics. The abstract measure theoretic investigation of sufficient statistics was initiated by Halmos and Savage [13]. Kullback and Leibler [19] gave the characterization of sufficiency in terms of the information (i.e., the classical relative entropy). Umegaki [33, 34] studied the sufficiency and the relative entropy in the noncommutative case of semi-finite von Neumann algebras.

Araki [4, 5] extended the relative entropy to the case for normal positive linear functionals of general von Neumann algebras and showed its several properties. Furthermore Uhlmann [32] showed the general WYDL concavity using a quadratic interpolation theory and defined the relative entropy of positive linear functionals of arbitrary *-algebras.

In the previous paper [14], we discussed the sufficiency and the relative entropy in von Neumann algebras and gave the characterizations of invariant states and KMS states with respect to the modular automorphism group of a faithful normal state.

In this paper, we further develop the sufficiency and the relative entropy in *-algebras. In §1, we introduce besides the sufficiency another notion of weak sufficiency and establish the relation between them. In §2, we deal with the weak sufficiency of positive linear maps between *-algebras. In §3, we mention the Araki’s and Uhlmann’s relative entropies which are equal in the von Neumann algebra case. We further give a formula of relative entropy for states of $C^*$-algebras. In §4, we establish some properties of invariant states and KMS states in $W^*$-dynamical systems and $C^*$-dynamical systems through the relative entropy and the sufficiency. The theorems there improve or extend the results obtained in [14]. Finally we give an application to the Gibbs states of quantum lattice systems.
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The authors wish to express their gratitude to Professor H. Umegaki for his advice and encouragement.

1. Sufficiency and weak sufficiency of \(*\)-subalgebras. In this paper, we shall assume that all \(*\)-algebras, \(C^*\)-algebras and von Neumann algebras have the unity \(I\) and their \(*\)-subalgebras always contain \(I\). Let \(\mathcal{A}\) be a \(*\)-algebra and \(\mathcal{S}\) be the set of all states of \(\mathcal{A}\).

**DEFINITION 1.1.** A \(*\)-subalgebra \(\mathcal{B}\) of \(\mathcal{A}\) is said to be **sufficient** for \(\mathcal{S}\) if there exists a projection \(\varepsilon\) of \(\mathcal{A}\) onto \(\mathcal{B}\) such that

(i) \(\varepsilon(A^*) = \varepsilon(A)^*\) for all \(A \in \mathcal{A}\),

(ii) \(\varepsilon(A)^*\varepsilon(A) \leq \varepsilon(A^*A)\) for all \(A \in \mathcal{A}\),

(iii) \(\varepsilon(B_1AB_2) = B_1\varepsilon(A)B_2\) for all \(A \in \mathcal{A}\) and \(B_1, B_2 \in \mathcal{B}\),

(iv) \(\varphi = \varphi \circ \varepsilon\) for all \(\varphi \in \mathcal{S}\).

We here call a projection \(\varepsilon\) of \(\mathcal{A}\) onto \(\mathcal{B}\) satisfying (i)–(iii) a **conditional expectation** of \(\mathcal{A}\) onto \(\mathcal{B}\). If \(\mathcal{A}\) is a \(C^*\)-algebra and \(\mathcal{B}\) is a \(C^*\)-subalgebra, then a conditional expectation of \(\mathcal{A}\) onto \(\mathcal{B}\) is nothing but a norm one projection of \(\mathcal{A}\) onto \(\mathcal{B}\) (cf. [31]).

We first give some examples of sufficiency in von Neumann algebras. Let \(\mathcal{H}\) be a von Neumann algebra and \(\mathcal{S}\) be the set of all normal states of \(\mathcal{H}\). The definition in [14] of sufficiency of a von Neumann subalgebra for \(\mathcal{S}\) is somewhat different from Definition 1.1. However these are equivalent if \(\mathcal{S}\) contains a faithful normal state (this is the case dealt in [14]).

**EXAMPLE 1.2.** Let \(\varphi \in \mathcal{S}\) be faithful and \(\sigma_\varphi\) be its modular automorphism group (cf. [28]). We showed in [14] that the centralizer of \(Z_\varphi\) of \(\varphi\) is sufficient for the set of all \(\sigma_\varphi\)-invariant states in \(\mathcal{S}\) and the center \(\mathcal{Z} = \mathcal{H} \cap \mathcal{H}^*\) is sufficient for the set of all states in \(\mathcal{S}\) satisfying the KMS condition with respect to \(\sigma_\varphi\) (at \(\beta = 1\)).

**EXAMPLE 1.3.** Assume that \(\mathcal{H}\) is semi-finite with a faithful normal semi-finite trace \(\tau\) of \(\mathcal{H}\). For each \(\varphi \in \mathcal{S}\), there exists a unique positive self-adjoint operator \(\rho_\varphi = d\varphi/d\tau\) such that \(\varphi(A) = \tau(\rho_\varphi A)\) for all \(A \in \mathcal{H}\). For any set \(\mathcal{S} \subset \mathcal{S}\), the von Neumann subalgebra \(\mathcal{M}\) generated by \(\{d\varphi/d\tau: \varphi \in \mathcal{S}\}\) is proved to be sufficient for \(\mathcal{S}\) (see [16, p. 72]).

**EXAMPLE 1.4.** Let \(\{\mathcal{H}, G, \alpha\}\) be a \(W^*\)-dynamical system where \(g \mapsto \alpha_g\) is a representation of a group \(G\) in \(\text{Aut}(\mathcal{H})\). Let \(\mathcal{H}^\alpha\) be the fixed point subalgebra of \(\alpha\) and \(\mathcal{S}_\alpha\) be the set of all \(\alpha\)-invariant states in \(\mathcal{S}\). Then the
result of Kovács and Szűcs [18] asserts that if \( \mathcal{M} \) is \( G \)-finite, i.e., \( \varphi(A^*A) = 0 \) for all \( \varphi \in \mathcal{E} \), implies \( A = 0 \), then \( \mathcal{M}^\alpha \) is sufficient for \( \mathcal{E} \).

For \( \ast \)-subalgebras \( \mathcal{B} \) of \( \mathcal{E} \), the existence of a conditional expectation of \( \mathcal{E} \) onto \( \mathcal{B} \) is usually a rather strict condition. In the sequel, we introduce another weak notion of sufficiency by using cyclic representations of \( \mathcal{E} \). Unbounded \( \ast \)-representations of \( \ast \)-algebras were studied in [23]. A \( \ast \)-representation \( \pi \) of \( \mathcal{E} \) on a Hilbert space \( \mathcal{H} \) is a map of \( \mathcal{E} \) into linear operators all defined on a common dense domain \( D(\pi) \subset \mathcal{H} \) which satisfies \( \pi(I) = I \) and

\[
(\pi(\alpha A + \beta B) \Phi = \alpha \pi(A) \Phi + \beta \pi(B) \Phi \quad \text{for all} \quad A, B \in \mathcal{E}, \alpha, \beta \in \mathbb{C} \quad \text{and} \quad \Phi \in D(\pi),
\]

\[\pi(A)D(\pi) \subset D(\pi) \quad \text{for all} \quad A \in \mathcal{E} \quad \text{and} \quad \pi(A)\pi(B) \Phi = \pi(AB) \Phi \quad \text{for all} \quad A, B \in \mathcal{E} \quad \text{and} \quad \Phi \in D(\pi),\]

\[\langle \Phi, \pi(A)\Psi \rangle = \langle \pi(A^*) \Phi, \Psi \rangle \quad \text{for all} \quad \Phi, \Psi \in D(\pi), \text{ i.e.,} \quad \pi(A^*) \subset \pi(A)^* \quad \text{for all} \quad A \in \mathcal{E}.
\]

The unbounded commutant \( \pi(\mathcal{E})^c \) of \( \pi(\mathcal{E}) \) consists of all linear operators \( T: D(\pi) \to \mathcal{H} \) such that

\[\langle \Phi, T\pi(A)\Psi \rangle = \langle \pi(A^*) \Phi, T\Psi \rangle, \quad A \in \mathcal{E}, \Phi, \Psi \in D(\pi).
\]

The commutant \( \pi(\mathcal{E})' \) of \( \pi(\mathcal{E}) \) is the set of all bounded operators \( T \) on \( \mathcal{H} \) such that \( T \upharpoonright D(\pi) \in \pi(\mathcal{E})^c \). For each positive linear functional \( \varphi \) of \( \mathcal{E} \), the GNS construction gives rise to a cyclic representation \( \{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\} \) of \( \mathcal{E} \) induced by \( \varphi \) which is unique up to unitary equivalence, that is, \( \pi_\varphi \) is a \( \ast \)-representation of \( \mathcal{E} \) on a Hilbert space \( \mathcal{H}_\varphi \) with \( \Omega_\varphi \in D(\pi_\varphi) \) such that

\[D(\pi_\varphi) = \pi_\varphi(\mathcal{E}) \Omega_\varphi, \quad \mathcal{H}_\varphi = \pi_\varphi(\mathcal{E}) \overline{\Omega_\varphi},
\]

\[\varphi(A) = \langle \Omega_\varphi, \pi_\varphi(A) \Omega_\varphi \rangle, \quad A \in \mathcal{E}.
\]

If for every \( A \in \mathcal{E} \) there exists a \( c > 0 \) with \( A^*A \leq cI \) (particularly if \( \mathcal{E} \) is a \( C^* \)-algebra), then \( \pi_\varphi \) becomes a bounded \( \ast \)-representation of \( \mathcal{E} \) on \( \mathcal{H}_\varphi \). We shall use in this paper the following three conditions of absolute continuity.

1. A positive linear functional \( \psi \) is absolutely continuous with respect to \( \varphi \) (we write \( \psi \ll \varphi \)) if \( \varphi(A^*A) = 0 \) implies \( \psi(A^*A) = 0 \).

2. A linear functional \( \psi \) is strongly absolutely continuous with respect to \( \varphi \) (we write \( \psi \prec \varphi \)) if for each sequence \( \{A_n\} \) in \( \mathcal{E} \), \( \varphi(A_n^*A_n) \to 0 \) implies \( \psi(BA_n) \to 0 \) for all \( B \in \mathcal{E} \).

3. A positive linear functional \( \psi \) is dominated by \( \varphi \) if \( \psi \leq c\varphi \) for some \( c > 0 \).

Note that for any positive \( \psi \), (3) implies (2) and (2) implies (1). If \( \psi \) is a linear functional of \( \mathcal{E} \) with \( \psi \ll \varphi \), then by [11, Theorem 1] there exists a
unique $T \in \pi_\varphi(\mathcal{B})^c$ (we denote by $T = d\varphi/d\varphi$) such that

$$\psi(A) = \langle T\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle, \quad A \in \mathcal{B}.$$  

Then $\psi$ is positive if and only if $T$ is positive, and moreover $\psi$ is dominated by $\varphi$ if and only if $T$ is bounded so that $T \in \pi_\varphi(\mathcal{B})'$. For each $\ast$-subalgebra $\mathcal{B}$ of $\mathcal{C}$, let $\mathcal{B}_\varphi = \pi_\varphi(\mathcal{B})\Omega_\varphi$ and $\overline{\mathcal{B}}_\varphi = \pi_\varphi(\mathcal{B})\overline{\Omega}_\varphi$. For every $A \in \mathcal{B}$, we define a vector $P_\varphi(A \mid \mathcal{B})$ in $\overline{\mathcal{B}}_\varphi$ by

$$P_\varphi(A \mid \mathcal{B}) = P_{\overline{\mathcal{B}}_\varphi}(\pi_\varphi(A)\Omega_\varphi)$$

where $P_{\overline{\mathcal{B}}_\varphi}$ is the orthogonal projection onto $\overline{\mathcal{B}}_\varphi$.

**Definition 1.5.** A $\ast$-subalgebra $\mathcal{B}$ of $\mathcal{C}$ is said to be **weakly sufficient** for $S \subset \mathcal{S}$ if for each $A \in \mathcal{B}$ there exists a sequence $\{B_n\}$ in $\mathcal{B}$ such that

$$P_\varphi(A \mid \mathcal{B}) = s\text{-lim} \pi_\varphi(B_n)\Omega_\varphi, \quad \varphi \in S.$$

**Theorem 1.6.** Assume that there is a finite subset $\{\varphi_1, \ldots, \varphi_k\}$ of $S$ such that every $\varphi \in S$ is dominated by $\rho = \sum_{i=1}^k \varphi_i$. Then a $\ast$-subalgebra $\mathcal{B}$ of $\mathcal{C}$ is weakly sufficient for $S$ if and only if $(d\varphi/d\rho)(\mathcal{B}_\rho) \subset \overline{\mathcal{B}}_\rho$ for every $\varphi \in S$.

**Proof.** Suppose that $\mathcal{B}$ is weakly sufficient for $S$. For each $A \in \mathcal{B}$, there exists a sequence $\{B_n\}$ in $\mathcal{B}$ such that

$$P_\varphi(A \mid \mathcal{B}) = s\text{-lim} \pi_\varphi(B_n)\Omega_\varphi, \quad \varphi \in S.$$  

Since $\{\pi_\varphi(B_n)\Omega_\rho\}$ is Cauchy, it follows that $\Psi = s\text{-lim} \pi_\rho(B_n)\Omega_\rho$ exists in $\overline{\mathcal{B}}_\rho$. If $B \in \mathcal{B}$, then we have

$$\|\pi_\rho(A)\Omega_\rho - \pi_\rho(B)\Omega_\rho\|^2 = \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - \pi_{\varphi_i}(B)\Omega_{\varphi_i}\|^2$$

$$\geq \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - P_{\varphi_i}(A \mid \mathcal{B})\|^2$$

$$= \lim \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - \pi_{\varphi_i}(B_n)\Omega_{\varphi_i}\|^2$$

$$= \lim \|\pi_\rho(A)\Omega_\rho - \pi_\rho(B_n)\Omega_\rho\|^2$$

$$= \|\pi_\rho(A)\Omega_\rho - \Psi\|^2,$$

so that $P_\rho(A \mid \mathcal{B}) = \Psi = s\text{-lim} \pi_\rho(B_n)\Omega_\rho$. For each $\varphi \in S$, let $T = d\varphi/d\rho$ and $\hat{T} = d(\varphi \uparrow \mathcal{B})/d(\rho \uparrow \mathcal{B})$ where the cyclic representation of $\mathcal{B}$ induced
by $\rho \uparrow \mathcal{B}$ is given by $\{\overline{B}_\rho, \pi_\rho \uparrow \mathcal{B}, \Omega_\rho\}$. Then for every $B \in \mathcal{B}$ we have
\[
\left\langle T\pi_\rho(B)\Omega_\rho, \pi_\rho(A)\Omega_\rho \right\rangle = \varphi(B^*A) = \left\langle \pi_\varphi(B)\Omega_\varphi, P_\varphi(A \mid \mathcal{B}) \right\rangle
= \lim \varphi(B^n) = \left\langle \hat{T}\pi_\rho(B)\Omega_\rho, P_\rho(A \mid \mathcal{B}) \right\rangle
= \left\langle \hat{T}\pi_\rho(B)\Omega_\rho, \pi_\rho(A)\Omega_\rho \right\rangle.
\]
Since this holds for each $A \in \mathcal{B}$, we obtain
\[
T\pi_\rho(B)\Omega_\rho = \hat{T}\pi_\rho(B)\Omega_\rho \in \overline{\mathcal{B}_\rho}, \quad B \in \mathcal{B},
\]
and hence $T\mathcal{B}_\rho \subseteq \overline{\mathcal{B}_\rho}$.

Conversely suppose that $(d\varphi/d\rho)\mathcal{B}_\rho \subseteq \overline{\mathcal{B}_\rho}$ for all $\varphi \in \mathcal{S}$. Let $A \in \mathcal{A}$ and take a sequence $\{B_n\}$ in $\mathcal{B}$ such that $P_\rho(A \mid \mathcal{B}) = s-lim \pi_\rho(B_n)\Omega_\rho$. For each $\varphi \in \mathcal{S}$, since $\varphi$ is dominated by $\rho$, it follows that $\{\pi_\varphi(B_n)\Omega_\varphi\}$ is Cauchy, so that $\Phi = s-lim \pi_\varphi(B_n)\Omega_\varphi$ exists in $\overline{\mathcal{B}_\varphi}$. If $B \in \mathcal{B}$, then we have
\[
\left\langle \pi_\varphi(B)\Omega_\varphi, P_\varphi(A \mid \mathcal{B}) \right\rangle = \varphi(B^*A) = \left\langle \left( d\varphi/d\rho \right) \pi_\rho(B)\Omega_\rho, P_\rho(A \mid \mathcal{B}) \right\rangle
= \lim \varphi(B^n) = \left\langle \pi_\varphi(B)\Omega_\varphi, \Phi \right\rangle,
\]
and hence $P_\varphi(A \mid \mathcal{B}) = \Phi = s-lim \pi_\varphi(B_n)\Omega_\varphi$. Thus $\mathcal{B}$ is weakly sufficient for $\mathcal{S}$.

\textbf{Remark.} Theorem 1.6 is considered as the noncommutative extension of Halmos-Savage's theorem [13]. For the proof of "only if" part of Theorem 1.6, we need only $\varphi \prec \rho$ for every $\varphi \in \mathcal{S}$. If $\pi_\rho$ is a bounded $*$-representation (particularly if $\mathcal{A}$ is a $C^*$-algebra), we see that $(d\varphi/d\rho)\mathcal{B}_\rho \subseteq \overline{\mathcal{B}_\rho}$ is equivalent to $(d\varphi/d\rho)\Omega_\rho \subseteq \overline{\mathcal{B}_\rho}$ since $T\pi_\rho(A)\Omega_\rho = \pi_\rho(A)T\Omega_\rho$ for all $A \in \mathcal{A}$ and $T \in \pi_\rho(\mathcal{A})^*$.

In the following theorem, we state the elementary facts of weak sufficiency which are immediately seen from the definition and Theorem 1.6.

\textbf{Theorem 1.7.} (1) If a $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is weakly sufficient for $\{\varphi, \psi\}$ and $\varphi = \psi$ on $\mathcal{B}$, then $\varphi = \psi$ on $\mathcal{A}$.

When the assumption in Theorem 1.6 is satisfied, then:
(2) If a $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is weakly sufficient for $\mathcal{S}$, then $\mathcal{B}$ is weakly sufficient for the convex hull of $\mathcal{S}$.

(3) If a $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is weakly sufficient for $\mathcal{S}$ and a $*$-subalgebra $\mathcal{C}$ of $\mathcal{B}$ is weakly sufficient for $\{\varphi \uparrow \mathcal{B} : \varphi \in \mathcal{S}\}$, then $\mathcal{C}$ is weakly sufficient for $\mathcal{S}$.

(4) If a $*$-subalgebra $\mathcal{B}$ of a $C^*$-algebra $\mathcal{A}$ is weakly sufficient for $\mathcal{S}$, then any $*$-subalgebra $\mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$ is weakly sufficient for $\mathcal{S}$. 

THEOREM 1.8. (1) If a *-subalgebra $\mathcal{B}$ of $\mathcal{G}$ is sufficient for $S$, then $\mathcal{B}$ is weakly sufficient for $S$.

(2) Assume that there is a $\varphi \in S$ such that $\psi < \varphi$ for all $\psi \in S$. Then a *-subalgebra $\mathcal{B}$ of $\mathcal{G}$ is sufficient for $S$ if and only if $\mathcal{B}$ is weakly sufficient for $S$ and there exists a conditional expectation $\varepsilon_\varphi$ of $\mathcal{G}$ onto $\mathcal{B}$ with $\varphi = \varphi \circ \varepsilon_\varphi$.

Proof. (1) Let $\varepsilon$ be a conditional expectation of $\mathcal{G}$ onto $\mathcal{B}$ with $\varphi = \varphi \circ \varepsilon$ for all $\varphi \in S$. If $A \in \mathcal{G}$, $B \in \mathcal{B}$ and $\varphi \in S$, then we have

$$
\langle P_\varphi(A \mid \mathcal{B}), \pi_\varphi(B)\Omega_\varphi \rangle = \varphi(A^*B) = \varphi(\varepsilon(A)^*B)
$$

and hence $P_\varphi(A \mid \mathcal{B}) = \pi_\varphi(\varepsilon(A))\Omega_\varphi$. Thus $\mathcal{B}$ is weakly sufficient for $S$.

(2) Suppose that $\mathcal{B}$ is weakly sufficient for $S$ and there exists a conditional expectation $\varepsilon_\varphi$ of $\mathcal{G}$ onto $\mathcal{B}$ with $\varphi = \varphi \circ \varepsilon_\varphi$. We show that $\psi = \psi \circ \varepsilon_\varphi$ for all $\psi \in S$. For each $\psi \in S$, since $(d\psi/d\varphi)\Omega_\varphi \in \mathcal{B}_\varphi$ by Theorem 1.6 (Remark), we can choose $(B_n)$ in $\mathcal{B}$ such that

$$(d\psi/d\varphi)\Omega_\varphi = s\text{-lim } \pi_\varphi(B_n)\Omega_\varphi.$$ 

Then $\psi = \psi \circ \varepsilon_\varphi$ follows from

$$
\psi(\varphi(A)) = \langle (d\psi/d\varphi)\Omega_\varphi, \pi_\varphi(\varphi(A))\Omega_\varphi \rangle 
= \lim \langle \pi_\varphi(B_n)\Omega_\varphi, \pi_\varphi(\varepsilon_\varphi(A))\Omega_\varphi \rangle = \lim \varphi(\varepsilon_\varphi(A))
= \lim \varphi(B_n^*A) = \langle (d\psi/d\varphi)\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle = \psi(A), \quad A \in \mathcal{G}. \quad \square
$$

EXAMPLE 1.9. We recall the usual concept of sufficiency in the classical probability theory (cf. [7, 13]). Let $(X, \mathcal{F})$ be a measurable space and $S$ be a set of probability measures on $\mathcal{F}$. A $\sigma$-subalgebra $\mathcal{G}$ of $\mathcal{F}$ is sufficient for $S$ if and only if for each $A \in \mathcal{F}$ there exists a $\mathcal{G}$-measurable function $g$ such that $g = E_\mu(1_A \mid \mathcal{G})$ a.e. $[\mu]$ for every $\mu \in S$, where $E_\mu(1_A \mid \mathcal{G})$ denotes the conditional expectation of the characteristic function $1_A$ of $A$ with respect to $\mu$ and $\mathcal{G}$. Let $\mathcal{G}$ (resp. $\mathcal{B}$) be the set of all complex-valued $\mathcal{F}$ (resp. $\mathcal{G}$)-measurable simple functions. Under the pointwise operations, $\mathcal{G}$ becomes a *-algebra and $\mathcal{B}$ is a *-subalgebra of $\mathcal{G}$. Each $\mu \in S$ is naturally regarded as a state of $\mathcal{G}$. The cyclic representation $(\mathcal{K}_\mu, \pi_\mu, \Omega_\mu)$ is given as follows: $\mathcal{K}_\mu = L^2(X, \mathcal{G}, \mu)$, $\pi_\mu(f)$ is the multiplication operator by $f \in \mathcal{G}$, and $\Omega_\mu = 1$. Moreover $\mathcal{B}_\mu = L^2(X, \mathcal{G}, \mu)$ and $P_\mu(f \mid \mathcal{B}) = E_\mu(f \mid \mathcal{G})$. Then it is easy to see that if $S$ is dominated, i.e., there is a measure $m$ on $\mathcal{F}$ with $\mu \ll m$ for all $\mu \in S$, then $\mathcal{G}$ is sufficient for $S$ if and only if $\mathcal{B}$ is weakly sufficient for $S$. 
EXAMPLE 1.10. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\Omega$ with $\|\Omega\| = 1$, and $\varphi$ be a faithful normal state given by $\varphi(A) = \langle \Omega, A\Omega \rangle$. For each von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$, let $S$ be the set of all states $\psi$ defined by $\psi(A) = \langle T\Omega, A\Omega \rangle$ with $T \in \mathcal{M}'$, $T\Omega \in \mathcal{M}\Omega$ and $\|T^{1/2}\Omega\| = 1$. Then it follows from Theorem 1.6 that $\mathcal{M}$ is weakly sufficient for $S$. Furthermore Theorem 1.8 shows that $\mathcal{M}$ is sufficient for $S$ if and only if there exists a conditional expectation $\varepsilon_{\varphi}$ of $\mathcal{M}$ onto $\mathcal{N}$ with $\varphi = \varphi \circ \varepsilon_{\varphi}$, which is if and only if $\mathcal{M}$ is invariant under the modular automorphism group $\sigma_{\varphi}$ (cf. [29]).

2. Weak sufficiency of positive linear maps. In this section, let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\gamma: \mathcal{B} \to \mathcal{A}$ be a linear map such that $\gamma(I) = I$, $\gamma(B^*) = \gamma(B)^*$ and $\gamma(B)^*\gamma(B) \leq \gamma(B^*B)$ for all $B \in \mathcal{B}$. We also assume that for every $B \in \mathcal{B}$ there is a $c > 0$ with $B^*B \leq cI$, which is satisfied if $\mathcal{B}$ is a C*-algebra. Let $S_\mathcal{A}$ and $S_\mathcal{B}$ be the sets of all states of $\mathcal{A}$ and $\mathcal{B}$. Then it is immediate that $\varphi \in S_\mathcal{A}$ implies $\varphi \circ \gamma \in S_\mathcal{B}$. For each $\varphi \in S_\mathcal{A}$ and $A \in \mathcal{A}$, define a linear functional $\varphi_A$ of $\mathcal{A}$ by $\varphi_A(A) = \varphi(A^*A_1)$. Then we have $\varphi_A \circ \gamma < \varphi \circ \gamma$ since

$$|\varphi_A \circ \gamma(BB_1)| = |\varphi(A^*\gamma(BB_1))|$$

$$\leq \varphi(A^*A)^{1/2} \varphi(\gamma(BB_1)^*\gamma(BB_1))^{1/2}$$

$$\leq \varphi(A^*A)^{1/2} \varphi(\gamma(B^*_1B^*BB_1))^{1/2}$$

$$\leq \varphi(A^*A)^{1/2} c^{1/2}(\varphi \circ \gamma)(B^*_1B_1)^{1/2}$$

for every $B, B_1 \in \mathcal{B}$ where $B^*B \leq cI$. Therefore $d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \in \pi_{\varphi \circ \gamma}(\mathcal{B})^c$ is defined.

Definition 2.1. We call $\gamma$ to be weakly sufficient for $S$ if for each $A \in \mathcal{A}$ there exists a sequence $\{B_n\}$ in $\mathcal{B}$ such that

$$[d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_{\varphi \circ \gamma} = s\text{-lim} \pi_{\varphi \circ \gamma}(B_n)\Omega_{\varphi \circ \gamma}, \quad \varphi \in S.$$

Definition 2.1 is compatible with Definition 1.5. Indeed we have

Theorem 2.2. Let $\gamma: \mathcal{B} \to \mathcal{A}$ be a $*$-homomorphism. Then $\gamma$ is weakly sufficient for $S \subset S_\mathcal{A}$ if and only if the $*$-subalgebra $\gamma(\mathcal{B})$ of $\mathcal{A}$ is weakly sufficient for $S$.

Proof. If $(\mathcal{K}_\varphi, \pi_\varphi, \Omega_\varphi)$ is the cyclic representation of $\mathcal{A}$ induced by $\varphi \in S_\mathcal{A}$, then the cyclic representation of $\mathcal{B}$ induced by $\varphi \circ \gamma$ is obtained
by \{(γ®)φ, ττφ © γ5 Ωφ\}. Now it suffices to show that
\[
[d(φ_A o γ)/d(φ o γ)]Ωφ = P_φ (A | γ®), \quad φ ∈ S_δ, A ∈ ©.
\]
This follows from
\[
\langle [d(φ_A o γ)/d(φ o γ)]Ωφ, π_φ(γB)Ωφ \rangle = (φ_A o γ)(B)
\]
\[
= φ(A^*(γB)) = \langle π_φ(A)Ωφ, π_φ(γB)Ωφ \rangle
\]
\[
= \langle P_φ (A | γ®), π_φ(γB)Ωφ \rangle, \quad B ∈ ©. \quad \square
\]

We assume further that © is abelian and γ: © → © is completely positive, i.e.,
\[
\sum_{i,j=1}^n A_i^*γ(B_i^*B_j)A_j \geq 0
\]
for every \(A_1, \ldots, A_n ∈ ©\) and \(B_1, \ldots, B_n ∈ ©\). Note (see [30, IV. 3]) that when © and © are C*-algebras, any completely positive map γ: © → © with γ(I) = I satisfies automatically γ(B)*γ(B) ≤ γ(B*B) for all \(B ∈ ©\), and any positive linear map γ: © → © is completely positive if either © or © is abelian. Let © ⊗ © be the *-algebraic tensor product of © and ©. For each φ ∈ S_δ, we can define the compound state φ ⊗ γ of © ⊗ © by
\[
(φ ⊗ γ)(A ⊗ B) = (φ_A^* o γ)(B) = φ(A(γB)), \quad A ∈ ©, B ∈ ©,
\]
since
\[
(φ ⊗ γ)\left(\left( \sum_{i=1}^n A_i ⊗ B_i \right)^* \left( \sum_{i=1}^n A_i ⊗ B_i \right) \right) = φ\left( \sum_{i,j=1}^n A_i^*γ(B_i^*B_j)A_j \right) \geq 0.
\]
Identifying © and © with *-subalgebras © ⊗ I and I ⊗ © of © ⊗ ©, we then have

**Theorem 2.3.** (1) © is sufficient for \{φ ⊗ γ: φ ∈ S_δ\}.
(2) γ is weakly sufficient for \(S ⊂ S_δ\) if and only if © is weakly sufficient for \{φ ⊗ γ: φ ∈ S\}.

**Proof.** (1) Define ε: © ⊗ © → © by ε(A ⊗ B) = Aγ(B), A ∈ ©, B ∈ ©. Since γ is completely positive and © is abelian, it follows that ε is a conditional expectation of © ⊗ © onto ©. Hence (1) is seen from (φ ⊗ γ) o ε = φ ⊗ γ for all φ ∈ S_δ.

(2) For φ ∈ S_δ, let \{φ_φ, π_φ, Ω_φ\} be the cyclic representation of © ⊗ © induced by φ = φ ⊗ γ. Since φ_φ ⊗ © = φ o γ, the cyclic representation of © induced by φ o γ is given by \{©_φ, π_φ ⊗ ©, Ω_φ\}. Let A ∈ ©,
$B \in \mathfrak{B}$ and $T = d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \in \pi_{\tilde{\varphi}}(\mathfrak{B})^c$. It follows that

$$P_{\tilde{\varphi}}(A \otimes B \mid \mathfrak{B}) = \left[ \frac{d(\tilde{\varphi}_A \otimes \lambda_{B \mid \mathfrak{B}})}{d(\tilde{\varphi} \mid \mathfrak{B})} \right] \Omega_{\tilde{\varphi}} = \left[ \frac{d(\varphi_A \circ \gamma)_B}{d(\varphi \circ \gamma)} \right] \Omega_{\tilde{\varphi}} = T\pi_{\tilde{\varphi}}(B)\Omega_{\tilde{\varphi}},$$

where the first equality is a special case of the equation in the proof of Theorem 2.2 and the last equality is seen from

$$\left[ \frac{d(\varphi_A \circ \gamma)_B}{d(\varphi \circ \gamma)} \right] \Omega_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}(B_i)\Omega_{\tilde{\varphi}}$$

$$= (\varphi_A \circ \gamma)(B^*B_i) = \left( T\pi_{\tilde{\varphi}}(B)\Omega_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}(B_i)\Omega_{\tilde{\varphi}} \right), \quad B_i \in \mathfrak{B}.$$

The "if" part of (2) is now immediate by taking $B = I$. Conversely if $\gamma$ is weakly sufficient for $S$, then there exists a sequence $\{B_n\}$ in $\mathfrak{B}$ such that

$$\Omega_{\tilde{\varphi}} = s-lim \pi_{\tilde{\varphi}}(B_n)\Omega_{\tilde{\varphi}}, \quad \varphi \in S.$$

Since $\pi_{\tilde{\varphi}}(B)$ is bounded from $B^*B \leq cI$, we have

$$P_{\tilde{\varphi}}(A \otimes B \mid \mathfrak{B}) = \pi_{\tilde{\varphi}}(B)T\Omega_{\tilde{\varphi}} = s-lim \pi_{\tilde{\varphi}}(BB_n)\Omega_{\tilde{\varphi}}, \quad \varphi \in S.$$

Hence $\mathfrak{B}$ is weakly sufficient for $\{\varphi \otimes \gamma: \varphi \in S\}$.

**Example 2.4.** Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two measurable spaces and $\nu$ be a channel distribution from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$, i.e., $\nu$ is a real-valued function on $X \times \mathcal{G}$ such that for every $x \in X$, $\nu(x, \cdot)$ is a probability measure on $\mathcal{G}$ and for every $B \in \mathcal{G}$, $\nu(\cdot, B)$ is $\mathcal{F}$-measurable on $X$. Let $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ be the abelian $C^*$-algebras of bounded complex-valued measurable functions on $X$ and $Y$. Define a positive linear map $\gamma: \mathfrak{B}(Y) \to \mathfrak{B}(X)$ by

$$(\gamma g)(x) = \int_Y g(y)\nu(x, dy), \quad x \in X, g \in \mathfrak{B}(Y).$$

Let $S$ be a set of probability measures on $\mathcal{F}$. For each $\mu \in S$, $\mu \otimes \gamma$ is given by

$$(\mu \otimes \gamma)(f \otimes g) = \int_{X \times Y} f \otimes g \, d(\mu \otimes \nu), \quad f \in \mathfrak{B}(X), g \in \mathfrak{B}(Y),$$

where $\mu \otimes \nu$ is the probability measure on $\mathcal{F} \otimes \mathcal{G}$ defined by $(\mu \otimes \nu) \times (A \times B) = \int_A \nu(x, B) \, d\mu$. Then we see in connection with Theorem 2.3(2) that $\gamma$ is weakly sufficient for $S$ if and only if the $\sigma$-subalgebra $X \times \mathcal{G} = \{X \times B: B \in \mathcal{G}\}$ of $\mathcal{F} \otimes \mathcal{G}$ is sufficient in the classical sense for $\{\mu \otimes \nu: \mu \in S\}$.  

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Example 2.5. Let \( R \) be a von Neumann algebra. An \( \mathcal{R} \)-valued PO-measure \( M \) on a measurable space \((X, \mathcal{F})\) is a map \( M: \mathcal{F} \to \mathcal{R} \) such that \( M(F) \geq 0 \) for all \( F \in \mathcal{F} \) and \( \sum_{n=1}^{\infty} M(F_n) = I \) (\( \sigma \)-weakly) for every countable measurable partition \( \{F_n\} \) of \( X \). Let \( \mathcal{B}(X) \) be the abelian \( C^* \)-algebra of bounded measurable functions on \( X \). We define a positive linear map \( \gamma: \mathcal{B}(X) \to \mathcal{R} \) with \( \gamma(1) = I \) by

\[
\varphi(\gamma(f)) = \int_X f d(\varphi \circ M), \quad f \in \mathcal{B}(X), \varphi \in \mathcal{R}_*.
\]

For each \( \varphi \in \mathcal{S} \), the cyclic representation \( \{ \mathcal{H}_{\varphi \circ \gamma}, \pi_{\varphi \circ \gamma}, \Omega_{\varphi \circ \gamma} \} \) of \( \mathcal{B}(X) \) induced by \( \varphi \circ \gamma \) is given as follows \( \mathcal{H}_{\varphi \circ \gamma} = L^2(X, \varphi \circ M), \pi_{\varphi \circ \gamma}(f) \) is the multiplication operator by \( f \), and \( \Omega_{\varphi \circ \gamma} = 1 \). For \( A \in \mathcal{R} \), \( d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \) is identical to the Radon-Nikodym derivative \( d(\varphi_A \circ M)/d(\varphi \circ M) \) which is in \( L^2(X, \varphi \circ M) \). Now assume that \( \mathcal{R} \) is \( \sigma \)-finite, so that \( \mathcal{R} \) has a faithful normal state. Then it is proved that \( \gamma \) is weakly sufficient for \( S \subset \mathcal{S} \) if and only if for every \( A \in \mathcal{R} \) there exists a measurable function \( f \) on \( X \) satisfying

\[
d(\varphi_A \circ M)/d(\varphi \circ M) = f \quad \text{a.e.} \quad \varphi \circ M, \quad \varphi \in S.
\]

Further assume that \( M \) is pure, i.e., \( M \) is a spectral measure. Then \( \gamma \) is a \(*\)-homomorphism and \( \gamma(\mathcal{B}(X)) \) is equal to the subalgebra \( \mathcal{M} = \{ M(F): F \in \mathcal{F} \}'' \). Hence Theorem 2.2 shows that \( \gamma \) is weakly sufficient for \( S \) if and only if \( \mathcal{M} \) is weakly sufficient for \( S \).

Example 2.6. Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( C(S) \) be the abelian \( C^* \)-algebra of continuous functions on \( S \). Define a positive linear map \( \gamma: \mathcal{A} \to C(S) \) with \( \gamma(1) = 1 \) by \( (\gamma A)(\omega) = \omega(A), \quad A \in \mathcal{A}, \omega \in S \). For each \( \rho \in S \) and each abelian von Neumann subalgebra \( \mathcal{B} \) of \( \pi_{\rho}(\mathcal{A})' \), we take the \( \mathcal{B} \)-orthogonal measure \( \lambda \) of \( \rho \) (cf. [30, p. 241]). Now assume that \( \mathcal{A} \) is separable and \( \mathcal{B} \subset \mathcal{B}_\rho = \pi_{\rho}(\mathcal{A})'' \cap \pi_{\rho}(\mathcal{A})', \) i.e., \( \lambda \) is a subcentral measure of \( \rho \), and let \( S \) be the set of all Borel probability measures \( \mu \) on \( S \) with \( \mu \ll \lambda \). Then \( \gamma \) is weakly sufficient for \( S \). This is proved as follows. There is a \(*\)-isomorphism \( \theta \) of \( L^\infty(S, \lambda) \) onto \( \mathcal{B} \) such that

\[
\langle \Omega_\rho, \theta(f)\pi_\rho(A)\Omega_\rho \rangle = \int_S f(\omega)\omega(A) \, d\lambda(\omega), \quad A \in \mathcal{A}, f \in L^\infty(S, \lambda).
\]

For each \( \mu \in S \) and \( f \in C(S) \), taking \( g_{\mu n} = \min((d\mu/d\lambda)^{1/2}, n) \) we obtain
(μ \circ γ)(A) = \int_S \omega(A) \, d\mu(\omega) = \lim \langle \theta(g_{\mu_n})\Omega_\rho, \pi_\rho(A)\theta(g_{\mu_n})\Omega_\rho \rangle,

(\mu_f \circ γ)(A) = \int_S \tilde{\omega}(A) \, d\mu(\omega)

= \lim \langle \theta(f)\theta(g_{\mu_n})\Omega_\rho, \pi_\rho(A)\theta(g_{\mu_n})\Omega_\rho \rangle.

Since \{g_{\mu_n}\} is Cauchy in L^2(\mathcal{S}, \lambda), it follows that \Phi_\mu = s-lim \theta(g_{\mu_n})\Omega_\rho exists and

(μ \circ γ)(A) = \langle \Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle,

(μ_f \circ γ)(A) = \langle \theta(f)\Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle, \quad A \in \mathcal{A}.

Hence the cyclic representation of \mathcal{A} induced by μ \circ γ is given by

\{\pi_\rho(\mathcal{A})\Phi_\mu, \pi_\rho(\cdot)\pi_\rho(\mathcal{A})\Phi_\mu, \pi_\rho(\Phi_\mu)\}

and we have

\[ d(μ_f \circ γ)/d(μ \circ γ) \Phi_\mu = \theta(f)\Phi_\mu. \]

Since \mathcal{A} is separable, there exists a sequence \{A_n\} in \mathcal{A} such that \pi_\rho(A_n) \to \theta(f) (strongly), and hence

\[ d(μ_f \circ γ)/d(μ \circ γ) \Phi_\mu = s-lim \pi_\rho(A_n)\Phi_\mu, \quad \mu \in S. \]

This shows that γ is weakly sufficient for S.

A linear map γ: \mathcal{B} \to \mathcal{A} considered here describes more or less a quantum communication channel with the input space \mathcal{A} and the output space \mathcal{B} (cf. [15, 21]). Examples 2.4–2.6 provide classical-classical, quantum-classical and classical-quantum channels. Roughly speaking, the physical meaning of weak sufficiency of γ is that the indirect measurement through γ gives as much information (measured by the relative entropy) as the direct measurement of observables in \mathcal{A} given a set S of input states (see §§3, 4).

3. Relative entropy of states of *-algebras. We begin with the definitions of Araki's relative entropy and Uhlmann's relative entropy.

(I) Araki's relative entropy. Let (\mathcal{M}, \mathcal{N}, J, \mathcal{D}) be a standard form of a von Neumann algebra \mathcal{M} (cf. [2, 12]). Araki [4, 5] defined the relative entropy of normal positive linear functionals \varphi and \psi of \mathcal{M} as follows.
There exist unique vector representatives $\Phi$ and $\Psi$ in $\mathcal{F}$ such that $\varphi(A) = \langle \Phi, A\Phi \rangle$ and $\psi(A) = \langle \Psi, A\Psi \rangle$ for all $A \in \mathcal{A}$. The operator $S_{\psi, \varphi}$ with the domain

$$D(S_{\psi, \varphi}) = \mathcal{R} \Phi + (I - s^{\mathcal{R}}(\Phi))$$

is defined by

$$S_{\psi, \varphi}(A\Phi + \Omega) = s^{\mathcal{R}}(\Phi)A^*\Psi, \quad A \in \mathcal{A}, \ s^{\mathcal{R}}(\Phi)\Omega = 0,$$

where $s^{\mathcal{R}}(\Phi)$ denotes the $\mathcal{R}$-support of $\Phi$. Then $S_{\psi, \varphi}$ is a closable conjugate-linear operator and the relative modular operator $\Delta_{\psi, \varphi}$ is defined by $\Delta_{\psi, \varphi} = (S_{\psi, \varphi})^*S_{\psi, \varphi}$. Let $\Delta_{\psi, \varphi} = \int_{0}^{\infty} \lambda \, d\varepsilon_{\psi, \varphi}(\lambda)$ be the spectral decomposition of $\Delta_{\psi, \varphi}$. The Araki's relative entropy $S(\psi \mid \varphi)$ is now given by

$$S(\psi \mid \varphi) = \begin{cases} \int_{0}^{\infty} \log \lambda \, d\langle \Psi, e_{\psi, \varphi}(\lambda)\Psi \rangle & \text{if } \psi \ll \varphi, \\ +\infty & \text{otherwise}. \end{cases}$$

Note that the relative entropy $S(\psi \mid \varphi)$ is independent of the choice of a standard form of $\mathcal{R}$ which is unique up to unitary equivalence. We used in [14] the notation $S(\psi \mid \psi)$ instead of $S(\psi \mid \varphi)$.

(II) Uhlmann's relative entropy. Let $\mathcal{E}$ be a complex linear space. Given two seminorms $p$ and $q$ on $\mathcal{E}$, the quadratical mean $QM(p, q)$ is defined by

$$QM(p, q)(x) = \sup_{\alpha \in H} \alpha(x, x)^{1/2}, \quad x \in \mathcal{E},$$

where $H$ is the set of all positive hermitian forms $\alpha$ on $\mathcal{E}$ satisfying $|\alpha(x, y)| \leq p(x)q(y)$ for all $x, y \in \mathcal{E}$. A function $t \mapsto p_t$ on $[0, 1]$ whose values are seminorms on $\mathcal{E}$ is called a quadratical interpolation from $p$ to $q$ if for every $x \in \mathcal{E}$ the function $t \mapsto p_t(x)$ is continuous and if the following properties hold:

$$p_t = QM(p_{t_1}, p_{t_2}), \quad t = (t_1 + t_2)/2, \ t_1, t_2 \in [0, 1],$$

$$p_{1/2} = QM(p, q),$$

$$p_{t/2} = QM(p, p_t), \quad t \in [0, 1],$$

$$p_{(1+t)/2} = QM(q, p_t), \quad t \in [0, 1].$$

Uhlmann [32] showed that for each positive hermitian forms $\alpha$ and $\beta$ there exists a unique function $t \mapsto QF_t(\alpha, \beta)$ on $[0, 1]$ with values in the set of
positive hermitian forms on $\mathcal{L}$ such that the function $p_t$ given by $p_t(x) = QF_t(\alpha, \beta)(x, x)^{1/2}$ is the quadratical interpolation from $\alpha(x, x)^{1/2}$ to $\beta(x, x)^{1/2}$, and defined the relative entropy functional $S(\alpha; \beta)(x)$ of $\alpha$ and $\beta$ by

$$S(\alpha; \beta)(x) = -\liminf_{t \to +0} \frac{1}{t} \{ QF_t(\alpha, \beta)(x, x) - \alpha(x, x) \}, \quad x \in \mathcal{L}.$$ 

Now let $\mathcal{A}$ be a $*$-algebra, and $\varphi$ and $\psi$ be positive linear functionals of $\mathcal{A}$. The Uhlmann's relative entropy $S(\psi | \varphi)$ is defined by

$$S(\psi | \varphi) = S(\psi^R; \varphi^L)(I),$$

where $\varphi^L$ and $\psi^R$ are the positive hermitian forms given by $\varphi^L(A, B) = \varphi(A^*B)$ and $\psi^R(A, B) = \psi(BA^*)$.

For each normal positive linear functionals $\varphi$ and $\psi$ of a von Neumann algebra $\mathcal{M}$, the Uhlmann's relative entropy is equal to the Araki's relative entropy. We here contain the proof for completeness.

Let $\mathcal{K}$ be the domain of $(I + \Delta_{\psi, \phi})^{1/2}$, which becomes a Hilbert space with an inner product:

$$\langle \Omega_1, \Omega_2 \rangle = \langle (I + \Delta_{\psi, \phi})^{1/2} \Omega_1, (I + \Delta_{\psi, \phi})^{1/2} \Omega_2 \rangle, \quad \Omega_1, \Omega_2 \in \mathcal{K}.$$ 

The operators $(I + \Delta_{\psi, \phi})^{-1}$ and $\Delta_{\psi, \phi}(I + \Delta_{\psi, \phi})^{-1}$ are positive bounded linear operators on $\mathcal{K}$. Define positive hermitian forms $\alpha$ and $\beta$ on $\mathcal{K}$ by

$$\alpha(\Omega_1, \Omega_2) = \langle \Omega_1, (I + \Delta_{\psi, \phi})^{-1} \Omega_2 \rangle,$$

$$\beta(\Omega_1, \Omega_2) = \langle \Omega_1, (I + \Delta_{\psi, \phi})^{-1} \Omega_2 \rangle.$$ 

We then have (cf. [24], [32, Example 4])

$$QF_t(\alpha, \beta)(\Omega, \Omega) = \langle \Omega, [\Delta_{\psi, \phi}(I + \Delta_{\psi, \phi})^{-1}]^{1-t} [(I + \Delta_{\psi, \phi})^{-1}]^t \Omega \rangle$$

$$= \langle \Omega, (\Delta_{\psi, \phi})^{1-t} \Omega \rangle, \quad t \in (0, 1), \Omega \in \mathcal{K}.$$ 

Since $\mathcal{R} \Phi \subset \mathcal{K}$ and

$$\psi^R(A, B) = \langle A\Phi, \Delta_{\psi, \phi}(I + \Delta_{\psi, \phi})^{-1} B\Phi \rangle,$$

$$\varphi^L(A, B) = \langle A\Phi, (I + \Delta_{\psi, \phi})^{-1} B\Phi \rangle,$$

it is easy to check that

$$QF_t(\psi^R, \varphi^L)(A, A) = QF_t(\alpha, \beta)(A\Phi, A\Phi), \quad A \in \mathcal{R}.$$ 

Take the spectral decomposition $\Delta_{\psi, \phi} = \int_0^\infty \lambda d\nu_{\psi, \phi}(\lambda)$. If $\psi \preceq \varphi$, then $(\Delta_{\psi, \phi})^{1/2} \Phi = JS_{\psi, \phi} \Phi = J\Psi = \Psi$ and hence $\psi^R(I, I) = \|\Delta_{\psi, \phi}\|_2^2$. 

We have

\[ S(\psi^R; \varphi^L)(I) = -\liminf_{t \to +0} \int_0^\infty \frac{\lambda^{-t} - 1}{t} d\langle \Phi, e_{\psi, \varphi}(\lambda)\Phi \rangle \]

because the function \((\lambda^{-t} - 1)/t\) converges decreasingly to \(-\log \lambda\) as \(t \to +0\). If \(\psi \ll \varphi\) does not hold, then \(\psi^R(I, I) < \| (\Delta_{\psi, \varphi})^{1/2} \Phi \|_2^2\) and hence \(S(\psi^R; \varphi^L)(I) = +\infty\). Thus the Uhlmann's relative entropy is equal to the Araki's one.

**Lemma 3.1.** Let \(\mathfrak{A}\) be a C*-algebra and \(\pi\) be a nondegenerate representation of \(\mathfrak{A}\) on a Hilbert space. If \(\varphi\) and \(\psi\) are positive linear functionals of \(\mathfrak{A}\) having the normal extensions \(\tilde{\varphi}\) and \(\tilde{\psi}\) to \(\pi(\mathfrak{A})''\) such that \(\varphi(A) = \tilde{\varphi}(\pi(A))\) and \(\psi(A) = \tilde{\psi}(\pi(A))\), then \(S(\psi | \varphi) = S(\tilde{\psi} | \tilde{\varphi})\).

**Proof.** According to the Uhlmann's definition of relative entropy, it suffices to show that

\[ QF_t(\psi^R, \varphi^L)(A, A) = QF_t(\tilde{\psi}^R, \tilde{\varphi}^L)(\pi(A), \pi(A)), \quad t \in [0, 1], A \in \mathfrak{A}. \]

Let \(\Gamma\) be the set of \(t \in [0, 1]\) for which the above equation holds for every \(A \in \mathfrak{A}\). Let \(H\) be the set of all positive hermitian forms \(\alpha\) on \(\mathfrak{A}\) satisfying

\[ |\alpha(A_1, A_2)| \leq \psi^R(A_1, A_1)^{1/2} \varphi^L(A_2, A_2)^{1/2}, \quad A_1, A_2 \in \mathfrak{A}, \]

and \(\tilde{H}\) be the set of all positive hermitian forms \(\tilde{\alpha}\) on \(\pi(\mathfrak{A})''\) satisfying

\[ |\tilde{\alpha}(Q_1, Q_2)| \leq \tilde{\psi}^R(Q_1, Q_1)^{1/2} \tilde{\varphi}^L(Q_2, Q_2)^{1/2}, \quad Q_1, Q_2 \in \pi(\mathfrak{A})''. \]

If \(\tilde{\alpha} \in \tilde{H}\), then the form \(\alpha\) on \(\mathfrak{A}\) defined by \(\alpha(A_1, A_2) = \tilde{\alpha}(\pi(A_1), \pi(A_2))\) is in \(H\). Conversely if \(\alpha \in H\), then there exists a positive hermitian form \(\tilde{\alpha}\) on \(\pi(\mathfrak{A})\) such that \(\alpha(A_1, A_2) = \tilde{\alpha}(\pi(A_1), \pi(A_2))\) and hence

\[ |\tilde{\alpha}(\pi(A_1), \pi(A_2))| \leq \psi^R(\pi(A_1), \pi(A_2))^{1/2} \varphi^L(\pi(A_2), \pi(A_2))^{1/2}, \quad A_1, A_2 \in \mathfrak{A}. \]
By the Kaplansky density theorem, $\hat{a}$ can be uniquely extended to a positive hermitian form $\tilde{a}$ on $\pi(\hat{A})''$ which is in $\hat{H}$. Therefore

$$QF_{1/2}(\varphi^R, \varphi^L)(A, A) = \sup_{\alpha} \alpha(A, A)$$

$$= \sup_{\tilde{a}} \tilde{a}(\pi(A), \pi(A))$$

$$= QF_{1/2}(\tilde{\varphi}^R, \tilde{\varphi}^L)(\pi(A), \pi(A)), \quad A \in \mathfrak{A}.$$  

This implies $1/2 \in \Gamma$. Noting that

$$QF_t(\varphi^R, \varphi^L)(A, A) \leq \varphi^R(A, A)^{1-t} \varphi^L(A, A)^t, \quad t \in [0, 1], A \in \mathfrak{A},$$

we can see by the similar arguments that $t \in \Gamma$ implies $t/2 \in \Gamma$ and $(1 + t)/2 \in \Gamma$, and that $t_1, t_2 \in \Gamma$ implies $(t_1 + t_2)/2 \in \Gamma$. Since $\Gamma$ is closed, we deduce that $\Gamma = [0, 1]$. \hfill \Box

In the above lemma, we can take as $\pi$ the cyclic representation induced by $\varphi + \psi$ or the universal representation of $\mathfrak{A}$.

We here remark that the relative entropy defined in (I) and (II) contains the usual relative entropies in the classical and quantum systems. Let $(X, \mathcal{F})$ be a measurable space, and $\mu$ and $\nu$ be probability measures on $\mathcal{F}$. Take a measure $m$ on $\mathcal{F}$ with $\mu, \nu \ll m$. Then $\mu$ and $\nu$ are naturally regarded as normal states of the abelian von Neumann algebra $\mathcal{N} = L^{\infty}(X, m)$ acting on $\mathcal{K} = L^2(X, m)$. Then the relative entropy $S(\nu | \mu)$ is equal to the classical relative entropy $I(\nu | \mu)$ (known as the Kullback-Leibler information):

$$I(\nu | \mu) = \begin{cases} \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \, d\mu & \text{if } \nu \ll \mu, \\ + \infty & \text{otherwise.} \end{cases}$$

Indeed, $\Phi = (d\mu/dm)^{1/2}$ and $\Psi = (d\nu/dm)^{1/2}$ are vector representatives for $\mu$ and $\nu$, and $\Delta_{\Psi, \Phi}$ is the multiplication operator by $1_{\text{supp} \Phi}(\Psi/\Phi)^2$ where $1_{\text{supp} \Phi}$ is the characteristic function of the support of $\Phi$. If $\nu \ll \mu$, then we have

$$S(\nu | \mu) = \int \Psi^2 1_{\text{supp} \Phi} \log(\Psi/\Phi)^2 \, dm$$

$$= \int \frac{d\nu}{dm} \left( \log \frac{d\nu}{dm} - \log \frac{d\mu}{dm} \right) \, dm = I(\nu | \mu).$$
Next let $\varphi$ and $\psi$ be normal states of the full von Neumann algebra $\mathcal{B} = \mathcal{B}(\mathcal{K})$ on a Hilbert space $\mathcal{K}$. Then $\varphi$ and $\psi$ are given by $\varphi(A) = \text{Tr}(\rho_\varphi A)$ and $\psi(A) = \text{Tr}(\rho_\psi A)$ with positive trace class operators $\rho_\varphi$ and $\rho_\psi$ on $\mathcal{K}$, and we obtain

$$S(\psi \mid \varphi) = \text{Tr}(\rho_\psi \log \rho_\psi - \rho_\psi \log \rho_\varphi).$$

The relative entropy $S(\psi \mid \varphi)$ has several basic properties such as joint convexity, monotonicity, lower semicontinuity, etc. (cf. [4, 5, 32]). The monotonicity is stated as follows (cf. [32, Proposition 18]). Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras and $\gamma$: $\mathcal{B} \to \mathcal{A}$ be a linear map such that $\gamma(I) = I$, $\gamma(B^*) = \gamma(B)^*$ and $\gamma(B)^* \gamma(B) \leq \gamma(B^* B)$ for all $B \in \mathcal{B}$. If $\varphi$ and $\psi$ are positive linear functionals on $\mathcal{B}$, then

$$S(\psi \circ \gamma \mid \varphi \circ \gamma) \leq S(\psi \mid \varphi).$$

This monotonicity is applied to positive linear maps such as in Examples 2.4–2.6. Particularly if $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$, then we have $S_\mathcal{B}(\psi \mid \varphi) \leq S(\psi \mid \varphi)$ where $S_\mathcal{B}(\psi \mid \varphi)$ denotes the relative entropy of the restrictions $\varphi \upharpoonright \mathcal{B}$ and $\psi \upharpoonright \mathcal{B}$.

In connection with Example 2.6, it is proved that the relative entropy of states of a $C^*$-algebra is equal to that of their decomposition measures in some cases.

**Theorem 3.2.** Let $\mathcal{A}$ be a $C^*$-algebra and $\mu$, $\nu$ be regular Borel probability measures on $\mathcal{S}$ with barycenters $q$, $p \in \mathcal{S}$. If there is a subcentral measure $\lambda$ on $\mathcal{S}$ such that $\mu$, $\nu \ll \lambda$, then $S(\psi \mid \varphi) = I(\nu \mid \mu)$.

**Proof.** Let $\lambda$ be the $\mathcal{B}$-orthogonal measure of $\rho \in \mathcal{S}$ with an abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{B}_\rho = \pi_\rho(\mathcal{A})'' \cap \pi_\rho(\mathcal{A})'$, and $\theta$ be the $*$-isomorphism of $L^\infty(\mathcal{S}, \lambda)$ onto $\mathcal{B}$ such that

$$\langle \Omega_\rho, \theta(f) \pi_\rho(A) \Omega_\rho \rangle = \int \omega(f) \omega(A) \, d\lambda(\omega), \quad A \in \mathcal{A}, f \in L^\infty(\mathcal{S}, \lambda).$$

As is seen in Example 2.6, there exists a $\Phi_\mu \in \mathcal{K}_\rho$ such that $\varphi(A) = \langle \Phi_\mu, \pi_\rho(A) \Phi_\mu \rangle$ for all $A \in \mathcal{A}$. Hence $\varphi$ has the normal extension $\tilde{\varphi}$ to $\pi_\rho(\mathcal{A})''$ and it is easily checked that

$$\tilde{\varphi}(\theta(f)) = \int \omega(f) \, d\mu, \quad f \in L^\infty(\mathcal{S}, \lambda).$$

Analogously $\psi$ has the normal extension $\tilde{\psi}$ to $\pi_\rho(\mathcal{A})''$ satisfying

$$\tilde{\psi}(\theta(f)) = \int \omega(f) \, d\nu, \quad f \in L^\infty(\mathcal{S}, \lambda).$$
Using Lemma 3.1, we have
\[ S(\psi | \varphi) = S(\bar{\psi} | \bar{\varphi}) \geq S_{\mathcal{B}}(\bar{\psi} | \bar{\varphi}) = I(\nu | \mu). \]

The inverse inequality always holds by the monotonicity. \( \square \)

**COROLLARY 3.3.** (1) Let \( \varphi, \psi \in \mathcal{S} \) which satisfy the KMS condition with respect to a strongly continuous one-parameter automorphism group \( \alpha_t \) of \( \mathfrak{A} \). If \( \mu \) and \( \nu \) are the central measures of \( \varphi \) and \( \psi \), then \( S(\psi | \varphi) = I(\nu | \mu) \).

(2) Let \( (\mathfrak{A}, G, \alpha) \) be a C*-dynamical system such that \( \alpha_G \) is a large group of automorphisms of \( \mathfrak{A} \), and \( \varphi, \psi \in \mathcal{S} \) be \( \alpha \)-invariant. If \( \mu \) and \( \nu \) are the ergodic decomposition measures of \( \varphi \) and \( \psi \), then \( S(\psi | \varphi) = I(\nu | \mu) \).

**Proof.** (1) Let \( K \) be the set of all states satisfying the KMS condition with respect to \( \alpha_t \). Then \( K \) is a Choquet simplex and the central measure of \( \rho \in K \) is identical to the unique maximal measure on \( K \) representing \( \rho \) (cf. [8, p. 121]). Hence it follows that \( \lambda = (\mu + \nu)/2 \) is the central measure of \( \rho = (\varphi + \psi)/2 \), so that Theorem 3.2 gives the desired equality.

(2) First note that the set \( \mathcal{S}_\alpha \) of all \( \alpha \)-invariant states becomes a Choquet simplex, because the condition of large group implies the \( G \)-abelianness (cf. [10]). Hence \( \lambda = (\mu + \nu)/2 \) is the ergodic decomposition measure of \( \rho = (\varphi + \psi)/2 \). It follows (cf. [26, Theorem 3.6], [27, Theorem 3.1]) that \( \lambda \) is the \( \mathcal{B} \)-orthogonal measure of \( \rho \) with \( \mathcal{B} = (\pi_\rho(\mathfrak{A}) \cup U_\rho(G))' = \mathfrak{A}_\rho \cap U_\rho(G)' \) where \( g \mapsto U_\rho(g) \) is the unitary representation of \( G \) on \( \mathfrak{H}_\rho \) such that \( \pi_\rho(\alpha_g(A)) = U_\rho(g) \pi_\rho(A) U_\rho(g)^* \) and \( U_\rho(g) \Omega_\rho = \Omega_\rho \). Thus we have the desired equality. \( \square \)

**4. Relative entropy, sufficiency and KMS condition.** In this section, we establish some relations between the relative entropy, the sufficiency and the KMS condition in \( W^* \)-dynamical systems and \( C^* \)-dynamical systems. The following theorem is obvious from Definition 1.1 and the monotonicity of relative entropy.

**THEOREM 4.1.** If a \( * \)-subalgebra \( \mathcal{B} \) of \( \mathfrak{A} \) is sufficient for \( \{ \varphi, \psi \} \) in \( \mathcal{S} \), then \( S(\psi | \varphi) = S_\mathcal{B}(\psi | \varphi) \).

**THEOREM 4.2.** Let \( \mathfrak{H} \) be a von Neumann algebra and \( \mathcal{S} \) be the set of all normal states of \( \mathfrak{H} \).

(1) Let \( (\mathfrak{H}, G, \alpha) \) be a \( W^* \)-dynamical system. If \( \varphi, \psi \in \mathcal{S} \) are \( \alpha \)-invariant, then \( S(\psi | \varphi) = S_{\mathfrak{H}^\alpha}(\psi | \varphi) \) where \( \mathfrak{H}^\alpha \) is the fixed point subalgebra of \( \alpha \).
Let $\alpha_t$ be a strongly continuous one-parameter automorphism group of $\mathcal{R}$. If $\varphi, \psi \in \mathcal{S}$ satisfy the KMS condition with respect to $\alpha_t$, then $S(\psi | \varphi) = S_\beta(\psi | \varphi)$ where $\beta = \mathcal{R} \cap \mathcal{R}'$.

**Proof.** (1) Let $s(\varphi)$ and $s(\psi)$ be the support projections of $\varphi$ and $\psi$, which are in $\mathcal{R}^a$ from the $\alpha$-invariance of $\varphi$ and $\psi$. Since $S(\psi | \varphi) = S_{e\mathcal{R}}(\psi | \varphi) = +\infty$ if $s(\psi) \leq s(\varphi)$ does not hold, we assume that $s(\psi) \leq s(\varphi)$. Letting $e = s(\varphi)$, we can define a $W^*$-dynamical system $(\hat{\mathcal{R}}, G, \hat{\alpha})$ by $\hat{\mathcal{R}} = e\mathcal{R} e$ and $\hat{\alpha}_t = \alpha_t \mid \hat{\mathcal{R}}$. Then $\hat{\varphi} = \varphi \mid \hat{\mathcal{R}}$ and $\hat{\psi} = \psi \mid \hat{\mathcal{R}}$ are $\hat{\alpha}$-invariant. Since $\hat{\varphi}$ is faithful, it follows (see Example 1.4) that $\hat{\mathcal{R}}^{\hat{\alpha}} = e\mathcal{R}^a e$ is sufficient for $\{\varphi, \psi\}$. Hence we have $S(\hat{\psi} | \hat{\varphi}) = S_{e\mathcal{R}}(\hat{\psi} | \hat{\varphi})$ by Theorem 4.1. It now suffices to show the equations:

$$S(\psi | \varphi) = S(\hat{\psi} | \hat{\varphi}) \quad \text{and} \quad S_{e\mathcal{R}}(\psi | \varphi) = S_{e\mathcal{R}}(\hat{\psi} | \hat{\varphi}).$$

Define a linear map $\gamma: \mathcal{R} \to \hat{\mathcal{R}}$ by $\gamma(A) = eAe$. Then we have $\gamma(I) = e$, and $\gamma(A^*) = \gamma(A)^*$ and $\gamma(A)^* \gamma(A) \leq \gamma(A^* A) \gamma(A) = \gamma(A^* A)$ for all $A \in \mathcal{R}$. Since $\varphi = \hat{\varphi} \circ \gamma$ and $\psi = \hat{\psi} \circ \gamma$, the monotonicity gives $S(\psi | \varphi) \leq S(\hat{\psi} | \hat{\varphi})$. Next define a linear map $\hat{\gamma}: \hat{\mathcal{R}} \to \mathcal{R}$ by $\hat{\gamma}(B) = B + \hat{\varphi}(B)(I - e)$. Then we have $\hat{\gamma}(e) = I$, $\hat{\gamma}(B^*) = \hat{\gamma}(B)^*$ and $\hat{\gamma}(B) \hat{\gamma}(B) = B^* B + |\hat{\varphi}(B)|^2 (I - e)$.

Since $\hat{\varphi} = \varphi \circ \hat{\gamma}$ and $\hat{\psi} = \psi \circ \hat{\gamma}$, the monotonicity again gives $S(\hat{\psi} | \hat{\varphi}) \leq S(\psi | \varphi)$. We hence obtain the first equation and analogously the second equation.

(2) By the KMS condition, the support projections $s(\varphi)$ and $s(\psi)$ are in $\beta$ (cf. [22, Lemma 5.1]). Letting $s(\psi) \leq s(\varphi) = e$, we define $\hat{\mathcal{R}} = e\mathcal{R} e$ and $\hat{\alpha}_t = \alpha_t \mid \hat{\mathcal{R}}$. Then $\hat{\varphi} = \varphi \mid \hat{\mathcal{R}}$ and $\hat{\psi} = \psi \mid \hat{\mathcal{R}}$ satisfy the KMS condition with respect to $\hat{\alpha}_t$. Since $\hat{\varphi}$ is faithful and hence $\hat{\alpha}_t = \alpha_t^\varphi$ the modular automorphism group of $\varphi$, it follows (see Example 1.2) that $\beta = \beta e$ is sufficient for $\{\varphi, \psi\}$. As in the proof of (1), we thus have

$$S(\psi | \varphi) = S(\hat{\psi} | \hat{\varphi}) = S_{e\mathcal{R}}(\hat{\psi} | \hat{\varphi}) = S_{\beta}(\psi | \varphi).$$

**Theorem 4.3.** Let $\alpha_t$ be a strongly continuous one-parameter automorphism group of $\mathcal{R}$ and $\varphi, \psi \in \mathcal{S}$. Assume that $\varphi$ satisfies the KMS condition with respect to $\alpha_t$.

(1) If $S(\psi | \varphi) = S_{\mathcal{R}}(\psi | \varphi) < +\infty$, then $\psi$ is $\alpha_t$-invariant.

(2) If $S(\psi | \varphi) = S_{\beta}(\psi | \varphi) < +\infty$, then $\psi$ satisfies the KMS condition with respect to $\alpha_t$. 

\qed
Proof. By the assumptions, we have $s(\varphi) \in J \cap \mathcal{M}^\alpha$ and $s(\psi) \leq s(\varphi)$. As is seen from the proof of Theorem 4.2, we may suppose that $\varphi$ is faithful, so that $\alpha_t = \sigma_t^{\varphi}$ the modular automorphism group and $\mathcal{M}^\alpha = Z_\varphi$ the centralizer of $\varphi$ (cf. [28, Lemma 15.8]). In [14, Corollaries 4.2 and 4.3], we proved (1) and (2) for the case when also $\psi$ is faithful. Now let $\psi$ be not faithful and $p = s(\psi)$.

(1) We first show that $p \in Z_\varphi$. Let $\hat{\mathcal{H}} = (\mathcal{M}^{\psi} \cup \{p\})'$, $\hat{\varphi} = \varphi \uparrow \hat{\mathcal{H}}$ and $\hat{\psi} = \psi \uparrow \hat{\mathcal{H}}$. Then $\hat{\varphi}$ is a trace of $\hat{\mathcal{H}}$ and we have $S(\hat{\psi} | \hat{\varphi}) = S_{Z_\psi}(\hat{\psi} | \hat{\varphi}) < +\infty$ by the assumption. Let $\varepsilon$ be the conditional expectation of $\mathcal{H}$ onto $\mathcal{M}^\psi$ with $\hat{\varphi} \circ \varepsilon = \hat{\varphi}$. Define $\hat{\psi}' = \hat{\psi} \circ \varepsilon$, $\hat{\psi}_t = (1 - t)\hat{\psi} + t\hat{\varphi}$ and $\hat{\psi}'_t = \hat{\psi}_t \circ \varepsilon = (1 - t)\hat{\psi}' + t\hat{\varphi}$ for $0 < t < 1$. Since $\hat{\psi}_t$ is faithful, it follows by [14, Theorem 3.3] that

\[
\|\hat{\psi}'_t - \hat{\psi}_t\| \leq \left\{2 \left( S(\hat{\psi}_t | \hat{\varphi}) - S_{Z_\psi}(\hat{\psi}_t | \hat{\varphi}) \right) \right\}^{1/2}, \quad 0 < t < 1.
\]

Since $\hat{\varphi}$ is a trace, there exists a positive self-adjoint operator $h$ affiliated with $\hat{\mathcal{H}}$ such that $\hat{\psi}(A) = \hat{\varphi}(hA)$ for all $A \in \hat{\mathcal{H}}$. Take the spectral decomposition $h = \int_0^\infty \lambda \, de(\lambda)$. Noting that $\Delta\hat{\psi}, \hat{\psi}_t = h$ and $\Delta\hat{\psi}, \hat{\psi}_t = (1 - t)h + tI$ where $\hat{\Phi}, \hat{\Psi}$, and $\hat{\psi}_t$ are vector representatives of $\hat{\varphi}$, $\psi$ and $\psi_t$ in the standard form of $\hat{\mathcal{H}}$, we have

\[
S(\hat{\psi} | \hat{\varphi}) = \int_0^\infty \lambda \log \lambda \, d\hat{\varphi}(e(\lambda)),
\]

\[
S(\hat{\psi}_t | \hat{\varphi}) = \int_0^\infty [(1 - t)\lambda + t] \log[(1 - t)\lambda + t] \, d\hat{\varphi}(e(\lambda)).
\]

Since

\[
-\frac{1}{e} \leq [(1 - t)\lambda + t] \log[(1 - t)\lambda + t] \leq (1 - t)\lambda \log \lambda,
\]

it follows from the Lebesgue’s convergence theorem that

\[
S(\hat{\psi} | \hat{\varphi}) = \lim_{t \to +0} S(\hat{\psi}_t | \hat{\varphi}),
\]

and analogously

\[
S_{Z_\psi}(\hat{\psi} | \hat{\varphi}) = \lim_{t \to +0} S_{Z_\psi}(\hat{\psi}_t | \hat{\varphi}).
\]

By letting $t \to +0$ in (*), we obtain $\hat{\psi}' = \hat{\psi}$, which implies that $Z_\psi$ is sufficient for $\{\hat{\varphi}, \hat{\psi}\}$. Then it is easy to see that $h$ is affiliated with $Z_\psi$, so that $p = s(h) \in Z_\psi$. Now define a faithful state $\tilde{\psi} = c\psi + (1 - c)\varphi$ where
$c = \varphi(p) < 1$ and $\bar{\varphi} = (1 - c)^{-1}\varphi_{1-p}$. Since $s(\psi) \perp s(\bar{\varphi})$, by [5, Theorem 3.6] we have
\[
S(\bar{\psi} | \varphi) = cS(\psi | \varphi) + (1 - c)S(\bar{\varphi} | \varphi)
+ c \log c + (1 - c) \log(1 - c),
\]
\[
S_{Z_\varphi}(\bar{\psi} | \varphi) = cS_{Z_\varphi}(\psi | \varphi) + (1 - c)S_{Z_\varphi}(\bar{\varphi} | \varphi)
+ c \log c + (1 - c) \log(1 - c).
\]

Since $\bar{\varphi}$ is $\sigma_\varphi$-invariant and $\bar{\varphi} \leq (1 - c)^{-1}\varphi$, it follows from Theorem 4.2 (1) that $S(\bar{\psi} | \varphi) = S_{Z_\varphi}(\bar{\varphi} | \varphi) < +\infty$, and hence $S(\bar{\psi} | \varphi) = S_{Z_\varphi}(\bar{\psi} | \varphi) < +\infty$. This implies by [14, Corollary 4.2] that $\bar{\psi}$ is $\sigma_\varphi$-invariant. Thus $\psi$ is $\sigma_\varphi$-invariant.

(2) Substituting $\beta$ for $Z_\varphi$ in the proof of (1), we can show that $p \in \beta$ and a faithful state $\bar{\psi}$ defined as above satisfies the KMS condition with respect to $\sigma_\varphi$, and thus $\psi$ satisfies the same condition.

Let $\mathfrak{A}$ be a $C^*$-algebra and $\alpha_t$ be a strongly continuous one-parameter automorphism group of $\mathfrak{A}$. Let $\varphi \in \mathcal{S}$ and $\{\mathcal{K}_\varphi, \pi_\varphi, \Omega_\varphi\}$ be the cyclic representation of $\mathfrak{A}$ induced by $\varphi$. Suppose that $\varphi$ satisfies the KMS condition with respect to $\alpha_t$. Since $\varphi$ is $\alpha_t$-invariant, there is a strongly continuous one-parameter unitary group $U_\varphi(t)$ on $\mathcal{K}_\varphi$ such that $U_\varphi(t)\Omega_\varphi = \Omega_\varphi$ and
\[
\pi_\varphi(\alpha_t(A)) = U_\varphi(t)\pi_\varphi(A)U_\varphi(t)^*, \quad t \in \mathbb{R}, A \in \mathfrak{A}.
\]
The normal extensions $\bar{\psi}$ and $\bar{\alpha}_t$ of $\varphi$ and $\alpha_t$ to $\pi_\varphi(\mathfrak{A})''$ are given by
\[
\bar{\psi}(Q) = \langle \Omega_\varphi, Q\Omega_\varphi \rangle, \quad Q \in \pi_\varphi(\mathfrak{A})'',
\]
\[
\bar{\alpha}_t(Q) = U_\varphi(t)QU_\varphi(t)^*, \quad t \in \mathbb{R}, Q \in \pi_\varphi(\mathfrak{A})''.
\]
and it is known (cf. [1, Lemma 2.4]) that $\bar{\psi}$ satisfies the KMS condition with respect to $\bar{\alpha}_t$, i.e., $\bar{\alpha}_t = \sigma_\bar{\varphi}$ the modular automorphism group of $\bar{\psi}$. Then we have

**Theorem 4.4.** Let $\mathfrak{A}$, $\alpha_t$, and $\varphi$ be as above. For each $\psi \in \mathcal{S}$ with $\psi < \varphi$, let $\bar{\psi}$ be the normal extension of $\psi$ to $\pi_\varphi(\mathfrak{A})''$. Then the following conditions are equivalent:

(i) $\psi$ satisfies the KMS condition with respect to $\alpha_t$;
(ii) $\beta_\varphi = \pi_\varphi(\mathfrak{A})'' \cap \pi_\varphi(\mathfrak{A})'$ is sufficient for $\{\bar{\psi}, \bar{\varphi}\}$;
(iii) $\beta_\varphi$ is weakly sufficient for $\{\bar{\psi}, \bar{\varphi}\}$;
(iv) $(d\psi/d\varphi)\Omega_\varphi \in \beta_\varphi\Omega_\varphi$. 

(v) \((D\tilde{\phi}: D(\tilde{\phi} + \tilde{\psi}))_t \in \mathcal{Z}_\varphi\) for all \(t \in \mathbb{R}\) where \((D\tilde{\phi}: D(\tilde{\phi} + \tilde{\psi}))_t\) is the Connes Radon-Nikodym derivative (cf. [9]);
(vi) \(S(\psi|\varphi) = S_{\mathcal{Z}_\varphi}(\tilde{\psi}|\tilde{\phi})\).

**Proof.** Note that \(\tilde{\psi}\) is given by

\[
\tilde{\psi}(Q) = \left( (d\psi/d\varphi) \Omega_{\varphi}, Q \Omega_{\varphi} \right), \quad Q \in \pi_{\varphi}(\mathcal{A})'',
\]
and hence \(\tilde{\psi} < \tilde{\varphi}\). Since there exists a conditional expectation \(e_{\tilde{\varphi}}\) of \(\pi_{\varphi}(\mathcal{A})''\) onto \(\mathcal{Z}_\varphi\) with \(\tilde{\varphi} = \tilde{\varphi} \circ e_{\tilde{\varphi}}\), the equivalence of (ii), (iii) and (iv) follows from Theorem 1.6 (Remark) and the proof of Theorem 1.8. Because the KMS condition of \(\psi\) with respect to \(\alpha_t\) and the same of \(\tilde{\psi}\) with respect to \(\tilde{\alpha}_t\) are equivalent, it follows from [14, Theorem 2.3] that (i) and (ii) are equivalent. Since \((D\tilde{\psi}: D(\tilde{\psi} + \tilde{\psi})) = (D(\tilde{\psi} + \tilde{\psi}): D\tilde{\varphi})^*\), we see by [14, Lemma 2.1] the equivalence of (ii) and (v). Finally the equivalence of (i) and (vi) follows from Theorems 4.2 (2) and 4.3 (2) and Lemma 3.1 if we prove \(S_{\mathcal{Z}_\varphi}(\tilde{\psi}|\tilde{\varphi}) < +\infty\). There exists a positive self-adjoint operator \(h\) affiliated with \(\mathcal{Z}_\varphi\) such that \(\tilde{\psi}(Q) = \tilde{\varphi}(hQ)\) for all \(Q \in \mathcal{Z}_\varphi\). Take the spectral decomposition \(h = \int_0^\infty \lambda \, de(\lambda)\). Then the condition \(\tilde{\psi} < \tilde{\varphi}\) gives rise to \(\tilde{\varphi}(h^2) = \int_0^\infty \lambda^2 \, d\tilde{\varphi}(e(\lambda)) < +\infty\). Hence we have

\[
S_{\mathcal{Z}_\varphi}(\tilde{\psi}|\tilde{\varphi}) = \int_0^\infty \lambda \log \lambda \, d\tilde{\varphi}(e(\lambda)) \\
\leq \int_0^\infty \lambda^2 \, d\tilde{\varphi}(e(\lambda)) < +\infty. \quad \square
\]

**Remark.** Assuming only that \(\psi\) has the normal extension \(\tilde{\psi}\) to \(\pi_{\varphi}(\mathcal{A})''\) (which is necessarily a vector state), we obtain the equivalence of the conditions (i), (ii) and (v) in Theorem 4.4, which imply (vi) and are implied by the equality (vi) with a finite value. For the case of \(\psi\) being dominated by \(\varphi\), the condition (iv) can be replaced by \(d\psi/d\varphi \in \mathcal{Z}_\varphi\) (see e.g. [17, p. 104]). Also for the \(\alpha_t\)-invariance of \(\psi \in \mathcal{S}\) with \(\psi < \varphi\), we can obtain the similar equivalent conditions by substituting \(Z_{\tilde{\varphi}} = \pi_{\varphi}(\mathcal{A})'' \cap U_{\varphi}(\mathcal{R})'\) for \(\mathcal{Z}_\varphi\) in the above conditions (ii)–(vi).

Theorem 4.4 finds an application in quantum lattice systems. Let \(L\) be a countable set and \(\mathcal{K}_0\) a finite-dimensional Hilbert space. For each nonempty finite set \(\Lambda \subset L\), let \(\mathcal{K}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{K}_x\) with \(\mathcal{K}_x = \mathcal{K}_0\) and \(\mathcal{C}_\Lambda = \mathcal{B}(\mathcal{K}_\Lambda)\). Then the quantum lattice system on \(L\) is described by the quasi-local C*-algebra \(\mathcal{A} = \bigcup_{\Lambda \subset L} \mathcal{C}_\Lambda\). An interaction \(\Phi\) is defined as a function from finite subsets \(\Lambda \subset L\) into the self-adjoint elements of \(\mathcal{A}\) such that \(\Phi(\Lambda) \in \mathcal{C}_\Lambda\). Let \(\varphi\) be a state of \(\mathcal{A}\) satisfying the Gibbs condition with
respect to $\Phi$ (see [8] for the definition). Now assume that $\Phi$ satisfies
\[
\sum_{n=0}^{\infty} e^{rn} \left( \sup_{x \in L} \sum_{|\Lambda| = n+1} \|\Phi(\Lambda)\| \right) < +\infty
\]
for some $r > 0$. Then the strongly continuous one-parameter automorphism group $\alpha^\Phi_t$ of $\varnothing$ can be given by
\[
\alpha^\Phi_t(A) = \lim_{\Lambda \to L} e^{itH_\Phi(\Lambda)} A e^{-itH_\Phi(\Lambda)}, \quad A \in \varnothing, \ t \in \mathbb{R},
\]
where $H_\Phi(\Lambda) = \sum_{X \subseteq \Lambda} \Phi(X)$. It is known (cf. [8, p. 268]) that $\psi \in \mathcal{S}$ satisfies the Gibbs condition with respect to $\Phi$ if and only if $\psi$ satisfies the KMS condition with respect to $\alpha^\Phi_t$. Then we have

**Corollary 4.5.** Let $\varnothing$, $\Phi$, $\alpha_t = \alpha^\Phi_t$ and $\varphi$ be as above, and let $\psi \in \mathcal{S}$ with $\psi < \varphi$. Then the Gibbs condition for $\psi$ with respect to $\Phi$ is equivalent to each of the conditions (i)–(vi) in Theorem 4.4, and these conditions imply the following:

(i) for each $\Lambda \subseteq L$, $\varnothing_{\Lambda^c} = \bigcup_{X \subseteq \Lambda^c} \varnothing_X$ is weakly sufficient for $\{\varphi, \psi\}$;

(ii) for each $\Lambda \subseteq L$, $S(\psi | \varphi) = S_{\varnothing_{\Lambda^c}}(\psi | \varphi)$.

Further if $\bigcup_{\Lambda \subseteq L} \pi_\varphi(\varnothing_\Lambda) \Omega_\varphi$ is a core for the modular operator $\Delta_{\Omega_\varphi}$ associated with $\Omega_\varphi$, the condition (vi) conversely implies the Gibbs condition for $\psi$ with respect to $\Phi$.

The main part of the corollary is immediate from Theorem 4.4 and the fact that $\mathcal{B}_\varphi$ is identical to $\bigcap_{\Lambda \subseteq L} (\varnothing_{\Lambda^c})''$ the algebra of observables at infinity. The last part follows by [6, Lemma 3].

We finally give some notes on the translationally invariant case of $L = \mathbb{Z}^d$. Let $\tau$ be the automorphism group of translations on $\mathbb{Z}^d$. Let $\Phi$ be a $\tau$-invariant interaction satisfying $\sum_{\Lambda \ni 0} e^{r|\Lambda|} \|\Phi(\Lambda)\| < +\infty$ for some $r > 0$. A $\tau$-invariant state $\varphi$ is said to be equilibrium with respect to $\Phi$ if the following variational equality holds (see [17, 25]):

\[
P(\Phi) = s(\varphi) - \varphi(A_\varphi)
\]
where $s(\varphi)$ is the mean entropy of $\varphi$ and

\[
P(\Phi) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log \tau_{1\Lambda}(e^{-H_\Phi(\Lambda)}), \quad A_\varphi = \sum_{\Lambda \ni 0} |\Lambda|^{-1} \Phi(\Lambda).
\]

Then the equilibrium condition with respect to $\Phi$, the Gibbs condition with respect to $\Phi$ and the KMS condition with respect to $\alpha^\Phi_t$ are all equivalent for $\tau$-invariant states of $\varnothing$ (cf. [3, 8, 20]). Let $\varphi, \psi \in \mathcal{S}$ be
τ-invariant. Since \( \{ \emptyset, \mathbb{Z}^d, \tau \} \) is asymptotically abelian, we obtain \( S(\psi \mid \varphi) = I(v \mid \mu) \) by Corollary 3.3 (2) where \( \mu \) and \( v \) are the ergodic decomposition measures of \( \varphi \) and \( \psi \). If \( \varphi \) is equilibrium and \( \psi < \varphi \) (or more weakly \( \psi \) has the normal extension to \( \pi_\varphi(\mathbb{E})' \)), then it can be proved that \( v \ll \mu \), so that \( \psi \) is automatically equilibrium because \( \mu \) is supported on the set of equilibrium states.

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Received July 6, 1981 and in revised form November 11, 1981.

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