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**INTEGRAL CLOSURE AND GENERALIZED TRANSFORMS IN
GRADED DOMAINS**

JON LEE JOHNSON

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In this article we consider the integral closure of integral domains by using the generalized transform and valuation rings. The first section establishes the basic theory in a general setting while the second deals with applications to graded rings, ending with a generalization of theorems due to Kuan and Seidenberg on integral closure in Z^+ graded rings. As in a number of recent articles, we investigate the idea that if a property holds in the graded case, and it holds for $R_s = \{a/b \mid a, b \in R, b \text{ a homogeneous non-zero divisor}\}$, then the property holds for the ring.

The notation will be fairly standard: all rings are commutative with identity; for an integral domain R , \bar{R} is the integral closure of R ; valuation rings will often be written (V, M) where M is the maximal ideal; $V(I)$ denotes the variety of I ; and $V(\mathfrak{s})$ is $\bigcup_{I \in \mathfrak{s}} V(I)$.

1. Integral closure and the generalized transform. Let R be a commutative ring with identity and K the total quotient ring of R . In [4] Arnold and Brewer defined the *generalized transform* of a ring R at a multiplicatively closed set of ideals \mathfrak{s} as $\{x \in K \mid xI \subseteq R \text{ for some } I \in \mathfrak{s}\}$ and used the notation $R_{\mathfrak{s}}$. $R_{\mathfrak{s}}$ is also called the \mathfrak{s} -transform of R .

DEFINITION 1.1. For an integral domain R , the *normal locus* of R is the set of all prime ideals $p \in \text{Spec}(R)$ so that R_p is integrally closed. The *non-normal locus* of R is the set of prime ideals $q \in \text{Spec } R$ so that R_q is not integrally closed.

We'll be using the following easy result.

PROPOSITION 1.2. *If \mathfrak{s} contains the non-normal locus and $R_{\mathfrak{s}} = \bigcap_{p \notin V(\mathfrak{s})} R_p$ then $R_{\mathfrak{s}}$ is integrally closed.*

The next definition will be mainly used in graded domains where the relation “ \mathfrak{s} -related” is an equivalence relation.

DEFINITION 1.3. Let \mathfrak{s} be a multiplicatively closed set of ideals and \mathcal{P} the set of prime ideals in $V(\mathfrak{s})$. We say that for two valuation rings

(V_1, M_1) and (V_2, M_2) V_1 and V_2 are \mathfrak{s} -related (or \mathfrak{P} -related) if there exists a valuation ring (V, M) so that $V_i \cap R_{\mathfrak{s}} \supseteq V \cap R_{\mathfrak{s}}$ and $M_i \cap R \supseteq M \cap R$ for $i = 1, 2$.

In general this will not be an equivalence relation. However, the valuation rings that are \mathfrak{s} -related are downwardly directed in that $V_1 > V_2$ if $V_1 \cap R_{\mathfrak{s}} \supset V_2 \cap R_{\mathfrak{s}}$.

THEOREM 1.4. *Let R be an integral domain, \mathfrak{P} the non-normal locus of R , \mathfrak{s} the multiplicative set of ideals generated by products of primes in \mathfrak{P} , and assume that $R_{\mathfrak{s}} = \bigcap_{p \notin \mathfrak{P}(\mathfrak{s})} R_p$. Then $\bar{R} = R_{\mathfrak{s}} \cap (\bigcap V_{\alpha}) = \bigcap (R_{\mathfrak{s}} \cap V_{\alpha})$ where the V_{α} 's can be chosen to be minimal elements in the \mathfrak{s} -related classes on valuation rings, if the minimal representatives exist.*

Proof. With the assumptions as stated in the Theorem, $R_{\mathfrak{s}} = \bigcap_{p \notin \mathfrak{P}(\mathfrak{s})} R_p$ is integrally closed by Proposition 1.2 and so $\bar{R} \subseteq R_{\mathfrak{s}}$. For $\{V_{\beta}\}$ the set of all valuation rings containing R , $\bar{R} = \bigcap V_{\beta} = R_{\mathfrak{s}} \cap (\bigcap V_{\beta}) \subseteq R_{\mathfrak{s}} \cap (\bigcap V_{\alpha})$ where the V_{α} 's are minimal representatives. To show equality, let $x \in R_{\mathfrak{s}} \cap (\bigcap V_{\alpha})$ and let (V, M) be a valuation ring with $P = M \cap R$. If P is in the non-normal locus, there exists a valuation ring (V', M') minimal (we are assuming that minimal representatives exist) in the \mathfrak{s} -relation class containing (V, M) and $V \cap R_{\mathfrak{s}} \supseteq V' \cap R_{\mathfrak{s}}$. Hence $x \in V \cap R_{\mathfrak{s}}$. On the other hand, if (V, M) is from the normal locus then $x \in R_{\mathfrak{s}} \subseteq R_p \subseteq V$ since $p \in P$. In either case we have $x \in R_{\mathfrak{s}} \cap (\bigcap V_{\alpha})$ implies $x \in \bar{R}$. Thus $\bar{R} = R_{\mathfrak{s}} \cap (\bigcap V_{\alpha})$.

2. Application to graded rings. In this section, $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ will be an integral domain graded by an arbitrary torsionless grading monoid Γ . By this we mean that R is an integral domain, Γ a commutative cancellative monoid, the quotient group $\langle \Gamma \rangle$ generated by Γ is a torsion free ordered abelian group, and if $r_{\alpha} \in R_{\alpha}$, $r_{\beta} \in R_{\beta}$, $r_{\alpha} \cdot r_{\beta} \in R_{\alpha+\beta}$. For such an R we let $R_S = \{a/b \mid a, b \in R, b \neq 0 \text{ homogeneous}\}$ and call it the homogeneous quotient ring of R . We let \mathfrak{s} be the set of all nonzero homogeneous or graded ideals (those generated by homogeneous elements).

PROPOSITION 2.1. $R_{\mathfrak{s}} = R_S$.

Proof. If $a/s \in R_S$ where $a \in R$ and $s \in S$, then $a/s \cdot (s) \subseteq R$. Since $(s) \in \mathfrak{s}$, $a/s \in R_{\mathfrak{s}}$. Conversely, if $x \in R_{\mathfrak{s}}$ then $xI \subseteq R$ for some $I \in \mathfrak{s}$. Let $i \in I \cap S$ then $xi \in R$ so $x = xi/i \in R_S$.

As in [6, 7, 9] one is able to define a graded valuation ring (or g -valuation ring) for Γ grading as well as Z or Z^+ grading. This is done by calling $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ a Γ -graded valuation domain if for each homogeneous element $x \in R_S$, x or $1/x \in R$. Equivalently if for each pair of homogeneous ideal I and J we have $I \supseteq J$ or $J \supseteq I$ (the homogeneous ideals are totally ordered under inclusion). Note that for a grading monoid Γ to admit a graded valuation domain $g \in \langle \Gamma \rangle$ must imply that g or $-g \in \Gamma$. Thus, when we speak of a Γ -graded valuation ring (or domain) we are assuming that the grading is done by the group $\langle \Gamma \rangle$ or that Γ admits a Γ -graded valuation ring. We list three results that carry over from the Z or Z^+ grading to Γ grading. The proofs are identical to those given in [7, Lemma 1.6 through Proposition 1.9] with R_S substituted for $K[x, 1/x]$.

LEMMA 2.2. *Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$ be a Γ graded integral domain with quotient field L and let G be an ordered abelian group. If $f: D \rightarrow G$ is defined so that the $f|_{D_\alpha} = f_\alpha$ have the properties:*

- (1) $f_\alpha(d_\alpha + g_\alpha) \geq \inf\{f_\alpha(d_\alpha), f_\alpha(g_\alpha)\}$ for $d_\alpha, g_\alpha \in D_\alpha$;
- (2) $f_\alpha(d_\alpha d_\beta) = f_\alpha(d_\alpha) + f_\beta(d_\beta)$ for $d_\alpha \in D_\alpha, d_\beta \in D_\beta$; and
- (3) for $r = \sum r_\alpha, r_\alpha \in D_\alpha, f(r) = \inf\{f_\alpha(r_\alpha)\}$, then f can be extended to a valuation on L_S .

THEOREM 2.3. *Let V^* be a Γ graded g -valuation ring with homogeneous quotient ring R_S . Then there exists a valuation ring V in the quotient field of V^* so that $V \cap R_S = V^*$.*

In a manner similar to that found in [7], we can define a *homogeneously defined valuation* as a valuation that satisfies $v(\sum r_\alpha) = \inf\{v(r_\alpha)\}$ for r_α homogeneous of degree α . The corresponding valuation ring V is called a *homogeneously defined valuation ring* [cf., 3, inf valuation].

We also have:

PROPOSITION 2.4. *Let V_1 and V_2 be homogeneously defined valuation rings so that $V_1 \cap R_S = V_2 \cap R_S = V^*$. Then $V_1 = V_2$.*

Note that we are able to set up an equivalence relation on the valuation rings in the quotient field of R_S . We do this by first letting V be a valuation ring. $V \cap R_S$ is then a ring which contains a unique largest graded valuation ring V^* defined from the valuation v of V restricted to the homogeneous components as in Lemma 2.2. Thus there is a canonical homogeneously defined valuation ring which we denote by V' . The

equivalence relation \sim_{R_S} is defined by $V_1 \sim_{R_S} V_2$ means $V'_1 = V'_2$. It is easy to check that this is an equivalence relation and that $V \cap R_S \supseteq V' \cap R_S$. Thus the homogeneously defined valuation ring will be a minimal representative of the equivalence class, minimal meaning minimal with respect to the intersection in R_S . We shall use these facts at a later time in this section.

DEFINITION 2.5. An ideal I in a Γ graded ring R is called *totally non-homogeneous* if I fails to contain a non-zero homogeneous element.

PROPOSITION 2.6. *Let I be a totally non-homogeneous ideal, then there exists a totally non-homogeneous prime ideal $J \supseteq I$.*

Proof. Since $I \cap S = \emptyset$ then I can be enlarged to an ideal J maximal with respect to $J \cap S = \emptyset$. Any such J is prime.

REMARKS. (1) If R is a Z or Z^+ graded domain, $S = \{\text{homogeneous non-zero elements in } R\}$, then the totally non-homogeneous primes of R are preserved in R_S . R_S is of the form $K[x, 1/x]$ for K a field and is hence of Krull dimension one. Thus if t is a non-zero non-homogeneous element of R , then t is contained in a height one totally non-homogeneous prime.

(2) If t is an element of an integral domain R and each prime which contains t is of height ≥ 2 , then there fails to exist a non-trivial Z or Z^+ grading of R which makes t homogeneous. Equivalently, all Z and Z^+ gradings of R make t non-homogeneous.

The following material uses heavily the notation and ideas from [5, 4] and we refer the reader to that for the necessary background.

Let P be the set of totally non-homogeneous prime ideals, \mathfrak{s} the set of non-zero homogeneous ideals in R , and $V(\mathfrak{s})$ the graded prime ideals and those primes which contain graded primes. Using the notation in [5], $G(P) = \{\text{ideals } A \text{ in } R \mid A \not\subseteq Q \forall Q \in P\}$.

LEMMA 2.7. *With the notation as above, $G(P) = \{\text{ideals } I \text{ of } R \mid I \supseteq \text{graded ideal}\}$.*

Proof. It is clear that $G(P)$ contains all graded non-zero ideals since if A is a graded ideal then no totally non-homogeneous prime may contain it. So let I be an ideal which does not contain any graded elements. By Proposition 2.6, I is contained in a totally non-graded prime. Thus $I \in G(P)$ and we have equality.

LEMMA 2.8. $R_{G(P)} = R_{\mathfrak{s}}$.

Proof. From [5] we know that $G(P)$ is a multiplicatively closed set of ideals, and so we are comparing two generalized transforms. Let $x \in R_{G(P)}$, then $x \cdot I \subseteq R$ for some $I \in G(P)$. Let I^* be the ideal generated by the homogeneous elements in I . $I^* \subseteq I$ so $x \cdot I^* \subseteq R$. This means that $x \in R_{\mathfrak{s}}$ and we obtain $R_{G(P)} \subseteq R_{\mathfrak{s}}$. Since $G(P) \supseteq \mathfrak{s}$ we have $R_{G(P)} \supseteq R_{\mathfrak{s}}$. Thus $R_{\mathfrak{s}} = R_{G(P)}$.

PROPOSITION 2.9. *With R, P and \mathfrak{s} as above, $R_{\mathfrak{s}} = \bigcap_{p \in P} R_p$.*

Proof. $R_{\mathfrak{s}} = R_{G(P)}$ by Lemma 2.8 and $R_{G(P)} = \bigcap \{R_q \mid q \in P\}$ by [5, Proposition 4.3].

We are now able to apply Theorem 1.4 to Γ -graded rings.

THEOREM 2.10. *If R is a Γ graded integral domain, then the integral closure of R is the intersection of all g -valuation rings containing R .*

Proof. Let \mathfrak{s} be the set of non-zero homogeneous ideals and P the set of totally non-graded prime ideals, then $R_S = R_{\mathfrak{s}} = \bigcap_{p \in P} R_p$ by Propositions 2.1 and 2.9. R_S is integrally closed by [1, Propositions 2.1 and 3.2] and we apply Theorem 1.4 to obtain $\overline{R} = \bigcap (R_{\mathfrak{s}} \cap V_{\alpha} 0)$ where the V_{α} 's are chosen to be minimal. The discussion following Proposition 2.4 shows that each V_{α} is a homogeneously defined valuation ring and so each $R_{\mathfrak{s}} \cap V_{\alpha}$ is a graded g -valuation ring.

We conclude with a theorem that generalizes Theorem 1 of [10] and Lemma 1 of [11]:

THEOREM 2.11. *If R is a Γ graded domain then for each totally non-graded prime P , R_P is integrally closed.*

Proof. Let P be a totally nonhomogeneous prime ideal. $P \cap S = \emptyset$ implies that $R_P = R_{SP_S}$, which is a localization of an integrally closed GCD domain and hence integrally closed.

REMARK. The referee noted that R_S is also completely integrally closed and that when P is height one, R_P will be a one dimensional GCD domain and hence completely integrally closed.

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Pacific Journal of Mathematics

Vol. 107, No. 1

January, 1983

John Kelly Beem and Phillip E. Parker , Klein-Gordon solvability and the geometry of geodesics	1
David Borwein and Annon Jakimovski , Transformations of certain sequences of random variables by generalized Hausdorff matrices	15
Willy Brandal and Erol Barbut , Localizations of torsion theories	27
John David Brillhart, Paul Erdős and Richard Patrick Morton , On sums of Rudin-Shapiro coefficients. II	39
Martin Lloyd Brown , A note on tamely ramified extensions of rings	71
Chang P'ao Ch'ên , A generalization of the Gleason-Kahane-Zelazko theorem	81
I. P. de Guzman , Annihilator alternative algebras	89
Ralph Jay De Laubenfels , Extensions of d/dx that generate uniformly bounded semigroups	95
Patrick Ronald Halpin , Some Poincaré series related to identities of 2×2 matrices	107
Fumio Hiai, Masanori Ohya and Makoto Tsukada , Sufficiency and relative entropy in $*$ -algebras with applications in quantum systems	117
Dean Robert Hickerson , Splittings of finite groups	141
Jon Lee Johnson , Integral closure and generalized transforms in graded domains	173
Maria Grazia Marinari, Francesco Odetti and Mario Raimondo , Affine curves over an algebraically nonclosed field	179
Douglas Shelby Meadows , Explicit PL self-knottings and the structure of PL homotopy complex projective spaces	189
Charles Kimbrough Megibben, III , Crawley's problem on the unique ω -elongation of p -groups is undecidable	205
Mary Elizabeth Schaps , Versal determinantal deformations	213
Stephen Scheinberg , Gauthier's localization theorem on meromorphic uniform approximation	223
Peter Frederick Stiller , On the uniformization of certain curves	229
Ernest Lester Stitzinger , Engel's theorem for a class of algebras	245
Emery Thomas , On the zeta function for function fields over F_p	251