INTEGRAL CLOSURE AND GENERALIZED TRANSFORMS IN GRADED DOMAINS

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In this article we consider the integral closure of integral domains by using the generalized transform and valuation rings. The first section establishes the basic theory in a general setting while the second deals with applications to graded rings, ending with a generalization of theorems due to Kuan and Seidenberg on integral closure in $Z^+$ graded rings. As in a number of recent articles, we investigate the idea that if a property holds in the graded case, and it holds for $R_\mathcal{S} = \{a/b \mid a, b \in R, b \text{ a homogeneous non-zero divisor}\}$, then the property holds for the ring.

The notation will be fairly standard: all rings are commutative with identity; for an integral domain $R$, $\overline{R}$ is the integral closure of $R$; valuation rings will often be written $(V, M)$ where $M$ is the maximal ideal; $V(I)$ denotes the variety of $I$; and $V(\mathcal{S}) = \bigcup_{I \in \mathcal{S}} V(I)$.

1. Integral closure and the generalized transform. Let $R$ be a commutative ring with identity and $K$ the total quotient ring of $R$. In [4] Arnold and Brewer defined the generalized transform of a ring $R$ at a multiplicatively closed set of ideals $\mathcal{S}$ as $x \in K \mid xI \subseteq R$ for some $I \in \mathcal{S}$ and used the notation $R_\mathcal{S}$. $R_\mathcal{S}$ is also called the $\mathcal{S}$-transform of $R$.

**Definition 1.1.** For an integral domain $R$, the normal locus of $R$ is the set of all prime ideals $p \in \text{Spec}(R)$ so that $R_p$ is integrally closed. The non-normal locus of $R$ is the set of prime ideals $q \in \text{Spec} R$ so that $R_q$ is not integrally closed.

We'll be using the following easy result.

**Proposition 1.2.** If $\mathcal{S}$ contains the non-normal locus and $R_\mathcal{S} = \bigcap_{p \notin V(\mathcal{S})} R_p$ then $R_\mathcal{S}$ is integrally closed.

The next definition will be mainly used in graded domains where the relation "$\mathcal{S}$-related" is an equivalence relation.

**Definition 1.3.** Let $\mathcal{S}$ be a multiplicatively closed set of ideals and $\mathcal{Q}$ the set of prime ideals in $V(\mathcal{S})$. We say that for two valuation rings
(\(V_1, M_1\)) and (\(V_2, M_2\)) \(V_1\) and \(V_2\) are \(\bar{s}\)-related (or \(\bar{p}\)-related) if there exists a valuation ring \((V, M)\) so that \(V_i \cap R_\bar{s} \supseteq V \cap R_\bar{s}\) and \(M_i \cap R \supseteq M \cap R\) for \(i = 1, 2\).

In general this will not be an equivalence relation. However, the valuation rings that are \(\bar{s}\)-related are downwardly directed in that \(V_1 \supset V_2\) if \(V_1 \cap R_\bar{s} \supseteq V_2 \cap R_\bar{s}\).

**Theorem 1.4.** Let \(R\) be an integral domain, \(\bar{\mathfrak{g}}\) the non-normal locus of \(R\), \(\mathfrak{s}\) the multiplicative set of ideals generated by products of primes in \(\bar{\mathfrak{g}}\), and assume that \(R_\mathfrak{s} = \bigcap_{p \in \mathfrak{g}(\mathfrak{s})} R_p\). Then \(R = R_\mathfrak{s} \cap (\bigcap V_\alpha) = \bigcap (R_\mathfrak{s} \cap V_\alpha)\) where the \(V_\alpha\)'s can be chosen to be minimal elements in the \(\bar{s}\)-related classes on valuation rings, if the minimal representatives exist.

**Proof.** With the assumptions as stated in the Theorem, \(R_\mathfrak{s} = \bigcap_{p \in \mathfrak{g}(\mathfrak{s})} R_p\) is integrally closed by Proposition 1.2 and so \(R \subseteq R_\mathfrak{s}\). For \(\{V_\beta\}\) the set of all valuation rings containing \(R\), \(R = \bigcap V_\beta = R_\mathfrak{s} \cap (\bigcap V_\beta) \subseteq R_\mathfrak{s} \cap (\bigcap V_\alpha)\) where the \(V_\alpha\)'s are minimal representatives. To show equality, let \(x \in R_\mathfrak{s} \cap (\bigcap V_\alpha)\) and let \((V, M)\) be a valuation ring with \(P = M \cap R\). If \(P\) is in the non-normal locus, there exists a valuation ring \((V', M')\) minimal (we are assuming that minimal representatives exist) in the \(\bar{s}\)-relation class containing \((V, M)\) and \(V \cap R_\mathfrak{s} \supseteq V' \cap R_\mathfrak{s}\). Hence \(x \in V \cap R_\mathfrak{s}\). On the other hand, if \((V, M)\) is from the normal locus then \(x \in R_\mathfrak{s} \subseteq R_p \subseteq V\) since \(p \in P\). In either case we have \(x \in R_\mathfrak{s} \cap (\bigcap V_\alpha)\) implies \(x \in R\). Thus \(R = R_\mathfrak{s} \cap (\bigcap V_\alpha)\).

2. Application to graded rings. In this section, \(R = \bigoplus_{\alpha \in \Gamma} R_\alpha\) will be an integral domain graded by an arbitrary torsionless grading monoid \(\Gamma\). By this we mean that \(R\) is an integral domain, \(\Gamma\) a commutative cancellative monoid, the quotient group \(\langle \Gamma \rangle\) generated by \(\Gamma\) is a torsion free ordered abelian group, and if \(r_\alpha \in R_\alpha\), \(r_\beta \in R_\beta\), \(r_\alpha \cdot r_\beta \in R_{\alpha + \beta}\). For such an \(R\) we let \(R_\mathfrak{s} = \{a/b \mid a, b \in R, b \neq 0\text{ homogeneous}\}\) and call it the homogeneous quotient ring of \(R\). We let \(\bar{s}\) be the set of all nonzero homogeneous or graded ideals (those generated by homogeneous elements).

**Proposition 2.1.** \(R_\mathfrak{s} = R_\bar{s}\).

**Proof.** If \(a/s \in R_\mathfrak{s}\) where \(a \in R\) and \(s \in \mathfrak{s}\), then \(a/s \cdot (s) \subseteq R\). Since \((s) \in \bar{s}\), \(a/s \in R_\bar{s}\). Conversely, if \(x \in R_\bar{s}\) then \(x I \subseteq R\) for some \(I \in \bar{s}\). Let \(x I \cap S\) then \(xI \subseteq R\) so \(x = xi/i \in R_\bar{s}\).
As in [6, 7, 9] one is able to define a graded valuation ring (or g-valuation ring) for \( \Gamma \) grading as well as \( Z \) or \( Z^+ \) grading. This is done by calling \( R = \bigoplus_{\alpha \in \Gamma} R_{\alpha} \) a \( \Gamma \)-graded valuation domain if for each homogeneous element \( x \in R_{\alpha}, x \) or \( 1/x \in R \). Equivalently if for each pair of homogeneous ideal \( I \) and \( J \) we have \( I \supseteq J \) or \( J \supseteq I \) (the homogeneous ideals are totally ordered under inclusion). Note that for a grading monoid \( \Gamma \) to admit a graded valuation domain \( g \in \langle \Gamma \rangle \) must imply that \( g \) or \(-g \) \( \in \Gamma \). Thus, when we speak of a \( \Gamma \)-graded valuation ring (or domain) we are assuming that the grading is done by the group \( \langle \Gamma \rangle \) or that \( \Gamma \) admits a \( \Gamma \)-graded valuation ring. We list three results that carry over from the \( Z \) or \( Z^+ \) grading to \( \Gamma \) grading. The proofs are identical to those given in [7, Lemma 1.6 through Proposition 1.9] with \( R_{\alpha} \) substituted for \( K[x, 1/x] \).

**Lemma 2.2.** Let \( D = \bigoplus_{\alpha \in \Gamma} D_{\alpha} \) be a \( \Gamma \) graded integral domain with quotient field \( L \) and let \( G \) be an ordered abelian group. If \( f: D \to G \) is defined so that the \( f|_{D_{\alpha}} = f_{\alpha} \) have the properties:

1. \( f_{\alpha}(d_{\alpha} + g_{\alpha}) \geq \inf \{f_{\alpha}(d_{\alpha}), f_{\alpha}(g_{\alpha})\} \) for \( d_{\alpha}, g_{\alpha} \in D_{\alpha} \);
2. \( f_{\alpha}(d_{\alpha}d_{\beta}) = f_{\alpha}(d_{\alpha}) + f_{\beta}(d_{\beta}) \) for \( d_{\alpha} \in D_{\alpha}, d_{\beta} \in D_{\beta} \); and
3. for \( r = \sum r_{\alpha}, r_{\alpha} \in D_{\alpha}, f(r) = \inf \{f_{\alpha}(r_{\alpha})\} \), then \( f \) can be extended to a valuation on \( L_{S} \).

**Theorem 2.3.** Let \( V^* \) be a \( \Gamma \) graded g-valuation ring with homogeneous quotient ring \( R_{\alpha} \). Then there exists a valuation ring \( V \) in the quotient field of \( V^* \) so that \( V \cap R_{\alpha} = V^* \).

In a manner similar to that found in [7], we can define a homogeneously defined valuation as a valuation that satisfies \( v(\sum r_{\alpha}) = \inf \{v(r_{\alpha})\} \) for \( r_{\alpha} \) homogeneous of degree \( \alpha \). The corresponding valuation ring \( V \) is called a homogeneously defined valuation ring [cf., 3, inf valuation]. We also have:

**Proposition 2.4.** Let \( V_{1} \) and \( V_{2} \) be homogeneously defined valuation rings so that \( V_{1} \cap R_{\alpha} = V_{2} \cap R_{\alpha} = V^* \). Then \( V_{1} = V_{2} \).

Note that we are able to set up an equivalence relation on the valuation rings in the quotient field of \( R_{\alpha} \). We do this by first letting \( V \) be a valuation ring. \( V \cap R_{\alpha} \) is then a ring which contains a unique largest graded valuation ring \( V^* \) defined from the valuation \( v \) of \( V \) restricted to the homogeneous components as in Lemma 2.2. Thus there is a canonical homogeneously defined valuation ring which we denote by \( V' \).
equivalence relation \( \sim_{R_S} \) is defined by \( V_1 \sim_{R_S} V_2 \) means \( V_1' = V_2' \). It is easy to check that this is an equivalence relation and that \( V \cap R_S \supseteq V' \cap R_S \). Thus the homogeneously defined valuation ring will be a minimal representative of the equivalence class, minimal meaning minimal with respect to the intersection in \( R_S \). We shall use these facts at a later time in this section.

**Definition 2.5.** An ideal \( I \) in a \( \Gamma \) graded ring \( R \) is called **totally non-homogeneous** if \( I \) fails to contain a non-zero homogeneous element.

**Proposition 2.6.** Let \( I \) be a totally non-homogeneous ideal, then there exists a totally non-homogeneous prime ideal \( J \supseteq I \).

**Proof.** Since \( I \cap S = \emptyset \) then \( I \) can be enlarged to an ideal \( J \) maximal with respect to \( J \cap S = \emptyset \). Any such \( J \) is prime.

**Remarks.** (1) If \( R \) is a \( Z \) or \( Z^+ \) graded domain, \( S = \{ \text{homogeneous non-zero elements in } R \} \), then the totally non-homogeneous primes of \( R \) are preserved in \( R_S \). \( R_S \) is of the form \( K[x, 1/x] \) for \( K \) a field and is hence of Krull dimension one. Thus if \( t \) is a non-zero non-homogeneous element of \( R \), then \( t \) is contained in a height one totally non-homogeneous prime.

(2) If \( t \) is an element of an integral domain \( R \) and each prime which contains \( t \) is of height \( \geq 2 \), then there fails to exist a non-trivial \( Z \) or \( Z^+ \) grading of \( R \) which makes \( t \) homogeneous. Equivalently, all \( Z \) and \( Z^+ \) gradings of \( R \) make \( t \) non-homogeneous.

The following material uses heavily the notation and ideas from \([5, 4]\) and we refer the reader to that for the necessary background.

Let \( P \) be the set of totally non-homogeneous prime ideals, \( \mathcal{S} \) the set of non-zero homogeneous ideals in \( R \), and \( V(\mathcal{S}) \) the graded prime ideals and those primes which contain graded primes. Using the notation in \([5]\), \( G(P) = \{ \text{ideals } A \text{ in } R \mid A \not\subset Q \forall Q \in P \} \).

**Lemma 2.7.** With the notation as above, \( G(P) = \{ \text{ideals } I \text{ of } R \mid I \supseteq \text{graded ideal} \} \).

**Proof.** It is clear that \( G(P) \) contains all graded non-zero ideals since if \( A \) is a graded ideal then no totally non-homogeneous prime may contain it. So let \( I \) be an ideal which does not contain any graded elements. By Proposition 2.6, \( I \) is contained in a totally non-graded prime. Thus \( I \in G(P) \) and we have equality.
Lemma 2.8. $R_{G(P)} = R_\bar{s}$.

Proof. From [5] we know that $G(P)$ is a multiplicatively closed set of ideals, and so we are comparing two generalized transforms. Let $x \in R_{G(P)}$, then $x \cdot I \subseteq R$ for some $I \in G(P)$. Let $I^*$ be the ideal generated by the homogeneous elements in $I$. $I^* \subseteq I$ so $x \cdot I^* \subseteq R$. This means that $x \in R_\bar{s}$ and we obtain $R_{G(P)} \subseteq R_\bar{s}$. Since $G(P) \supseteq \bar{s}$ we have $R_{G(P)} \supseteq R_\bar{s}$. Thus $R_\bar{s} = R_{G(P)}$.

Proposition 2.9. With $R$, $P$ and $\bar{s}$ as above, $R_\bar{s} = \bigcap_{p \in P} R_p$.

Proof. $R_\bar{s} = R_{G(P)}$ by Lemma 2.8 and $R_{G(P)} = \bigcap \{ R_q \mid q \in P \}$ by [5, Proposition 4.3].

We are now able to apply Theorem 1.4 to $\Gamma$-graded rings.

Theorem 2.10. If $R$ is a $\Gamma$ graded integral domain, then the integral closure of $R$ is the intersection of all $g$-valuation rings containing $R$.

Proof. Let $\bar{s}$ be the set of non-zero homogeneous ideals and $P$ the set of totally non-graded prime ideals, then $R_\bar{s} = R_\bar{s} = \bigcap_{p \in P} R_p$ by Propositions 2.1 and 2.9. $R_\bar{s}$ is integrally closed by [1, Propositions 2.1 and 3.2] and we apply Theorem 1.4 to obtain $R = \bigcap (R_\bar{s} \cap V_a)$ where the $V_a$'s are chosen to be minimal. The discussion following Proposition 2.4 shows that each $V_a$ is a homogeneously defined valuation ring and so each $R_\bar{s} \cap V_a$ is a graded $g$-valuation ring.

We conclude with a theorem that generalizes Theorem 1 of [10] and Lemma 1 of [11]:

Theorem 2.11. If $R$ is a $\Gamma$ graded domain then for each totally non-graded prime $P$, $R_p$ is integrally closed.

Proof. Let $P$ be a totally nonhomogeneous prime ideal. $P \cap S = \emptyset$ implies that $R_p = R_{SP}$, which is a localization of an integrally closed GCD domain and hence integrally closed.

Remark. The referee noted that $R_S$ is also completely integrally closed and that when $P$ is height one, $R_p$ will be a one dimensional GCD domain and hence completely integrally closed.
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REFERENCES


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John Kelly Beem and Phillip E. Parker, Klein-Gordon solvability and the geometry of geodesics ........................................ 1

David Borwein and Amnon Jakimovski, Transformations of certain sequences of random variables by generalized Hausdorff matrices ...... 15

Willy Brandal and Erol Barbut, Localizations of torsion theories ............. 27

John David Brillhart, Paul Erdős and Richard Patrick Morton, On sums of Rudin-Shapiro coefficients. II ........................................ 39

Martin Lloyd Brown, A note on tamely ramified extensions of rings ............ 71

Chang P’ao Ch’ên, A generalization of the Gleason-Kahane-Żelazko theorem .......................................................... 81

I. P. de Guzman, Annihilator alternative algebras .................................. 89

Ralph Jay De Laubenfels, Extensions of $d/dx$ that generate uniformly bounded semigroups ................................................. 95

Patrick Ronald Halpin, Some Poincaré series related to identities of $2 \times 2$ matrices .......................................................... 107

Fumio Hiai, Masanori Ohya and Makoto Tsukada, Sufficiency and relative entropy in $*$-algebras with applications in quantum systems ...... 117

Dean Robert Hickerson, Splittings of finite groups ................................. 141

Jon Lee Johnson, Integral closure and generalized transforms in graded domains .......................................................... 173

Maria Grazia Marinari, Francesco Odetti and Mario Raimondo, Affine curves over an algebraically nonclosed field .......................... 179

Douglas Shelby Meadows, Explicit PL self-knottings and the structure of PL homotopy complex projective spaces ......................... 189

Charles Kimbrough Megibben, III, Crawley’s problem on the unique $\omega$-elongation of $p$-groups is undecidable ............................. 205

Mary Elizabeth Schaps, Versal determinantal deformations ....................... 213

Stephen Scheinberg, Gauthier’s localization theorem on meromorphic uniform approximation ................................................. 223

Peter Frederick Stiller, On the uniformization of certain curves ............... 229

Ernest Lester Stitzinger, Engel’s theorem for a class of algebras ............. 245

Emery Thomas, On the zeta function for function fields over $F_p$ .............. 251