EXPLICIT PL SELF-KNOTTINGS AND THE STRUCTURE OF PL HOMOTOPY COMPLEX PROJECTIVE SPACES

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We show that certain piecewise-linear homotopy complex projective
spaces may be described as a union of smooth manifolds glued along
their common boundaries. These boundaries are sphere bundles and the
 glueing homeomorphisms are piecewise-linear self-knottings on these
bundles. Furthermore, we describe these self-knottings very explicitly
and obtain information on the groups of concordance classes of such
maps.

A piecewise linear homotopy complex projective space $\mathbb{C}P^n$ is a
compact PL manifold $M^{2n}$ homotopy equivalent to $CP^n$. In [22] Sullivan
gave a complete enumeration of the set of PL isomorphism classes of these
manifolds as a consequence of his Characteristic Variety theorem and his
have shown that these manifolds, the index 8 Milnor manifolds, and the
differentiable generators of the oriented smooth bordism ring provide a
complete generating set for the torsion-free part of the oriented PL
bordism ring. Hence a study of the geometric structure of these exotic
projective spaces $\mathbb{C}P^n$, especially with regard to their smooth singularities,
may extend our understanding of the PL bordism ring. This paper begins
such a study in which we obtain a geometric decomposition of $\mathbb{C}P^n$, into
(for many cases) a union of smooth manifolds glued together by PL
self-knottings on certain sphere bundles. We also obtain information on
groups of concordance classes of PL self-knottings from these decomposi-
tions and a number of fairly explicitly constructed examples of self-knot-
tings. I would like to thank by thesis advisor R. J. Milgram for many
helpful discussions.

I. Sullivan’s classification of PL homotopy $\mathbb{C}P^n$ proceeds as follows:
Given a homotopy equivalence $h: \mathbb{C}P^n \to CP^n$ make $h$ transverse regular
to $CP^j \subset \mathbb{C}P^n$, the standard inclusion. The restriction of $h$ to the trans-
verse inverse image $h^{-1}(CP^j) = N^{2j} \subset \mathbb{C}P^n$ is a degree one normal map
with simply connected surgery obstruction

\[ \sigma_j \in P_{2j} = \begin{cases} Z, & j \text{ even} \\ Z/2Z, & j \text{ odd} \end{cases} \]

For \( j = 2, \ldots, n - 1 \) these obstruction invariants yield a complete enumeration—i.e. the set of PL isomorphism classes of \( \tilde{\mathbb{C}}P^n \) is set-isomorphic to the product \( Z \times Z_2 \times Z \times \cdots \times P_{2(n-1)} \) with \( n - 2 \) factors.

We will use the following notation to specify elements with this classification:

\[ \tilde{\mathbb{C}}P^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \]

will denote the PL homotopy \( \tilde{\mathbb{C}}P^n \) with invariants \( \sigma_j \in P_{2j} \) in Sullivan’s enumeration.

We recall that a PL homeomorphism \( f: M \to M \) is a “self-knotting” and \( M \) is said to be “self knotted” if \( f \) is homotopic but not PL isotopic to the identity. Also, PL homeomorphisms \( f, g: M \to M \) are “PL concordant” (pseudo-isotopic) if we have a PL homeomorphism \( F: M \times I \to M \times I \) with \( F(m, 0) = (f(m), 0) \) and \( F(m, 1) = (g(m), 1) \) for \( m \in M \). We then define:

\[ SK(M) = \text{“the group (under composition of maps) of PL concordance classes of PL self-knottings of } M.\]

Unless otherwise noted \( \mathbb{C}P^j \subset \mathbb{C}P^n \) means the standard embedding of \( \mathbb{C}P^j \) onto the first \( (j + 1) \) homogeneous coordinates of \( \mathbb{C}P^n \) or a smooth ambient isotope of this embedding. In this context we define:

\[ v_N(\mathbb{C}P^j) = \text{“the smooth tubular disc bundle neighborhood of the embedding } \mathbb{C}P^j \subset \mathbb{C}P^n.\]

Our results are as follows:

**Theorem A.** For \( n \geq 4 \) and \( \sigma_2 \equiv 0 \) (2) every \( \tilde{\mathbb{C}}P^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \) is PL homeomorphic to the identification space

\[ \left[ \tilde{\mathbb{C}}P^n - v_n(\mathbb{C}P^1) \right] \cup_{\varphi_{n-1}} \left[ v_n(\mathbb{C}P^1) \right] \]

where \( \tilde{\mathbb{C}}P^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-2}, 0) \) and the identification is over a PL homeomorphism

\[ \varphi_{n-1}: \partial v_n(\mathbb{C}P^1) \to \partial v_n(\mathbb{C}P^1). \]

We prove Theorem A in Part II by a careful description of Sullivan’s classification and an easy \( h \)-cobordism argument.
An immediate consequence of Theorem A is the decomposition of 
\( CP^{n+1} \leftrightarrow (0, \ldots, 0, \sigma_n) \) into
\[
CP^{n+1} = [CP^{n+1} - \nu(CP^1)] \cup_{\varphi_0} \nu(CP^1).
\]

**Theorem B.** For every \( n \geq 4 \) and non-zero \( \tau \in P_{2n} \) there is a PL self-knotting
\[
\varphi_\tau: \partial \nu_{n+1}(CP^1) \to \partial \nu_{n+1}(CP^1)
\]
which will suffice for the glueing homeomorphism in Theorem A.

We establish this theorem by an explicit construction of \( \varphi_\tau \) in Part III.

**II.** Here we prove Theorem A by beginning with a construction which shows how to obtain 
\( CP^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \) from 
\( CP^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}) \) for \( n \geq 4 \):

Let \( h: \tilde{CP}^n \to CP^n \) be a homotopy equivalence, and let \( M^{2n} \) be the compact \((n-1)\)-connected Milnor or Kervaire manifold of Index \( 8\sigma_n \) or Kervaire-Arf invariant \( \sigma_n \) as the case may be [4]. Let \( r: M^{2n} \to S^{2n} \) be a degree one map. Then \( h\# r: \tilde{CP}^n\# M^{2n} \to CP^n\# S^{2n} = CP^n \) is a degree one normal map with 1-connected surgery obstruction \( \sigma_n \). We define \( \hat{H} \) as the \( D^2 \) bundle over \( \tilde{CP}^n\# M^{2n} \) induced by \( h\# r \) from \( \hat{H} \), the disc bundle associated to the complex line bundle over \( CP^n \). Let \( \hat{h}: \hat{H} \to H \) be the bundle mapping. We note that the map \( h\# r \) is \((n-1)\)-connected with homological kernel \( K_n = \pi_n(\tilde{M}^{2n}_0) \) where \( \tilde{M}^{2n}_0 = M^{2n} - D^{2n} \). The bundle \( \hat{H} \) is trivial over \( \tilde{M}^{2n}_0 \) since \( \tilde{M}^{2n}_0 = (h\# r)^{-1}(\text{point}) \). In \( \tilde{M}^{2n}_0 \times D^2 \) we can represent \( \pi_n(\tilde{M}^{2n}_0) \) by disjointly embedded spheres \( S^n \cong \tilde{M}^{2n}_0 \times S^1 \) with trivial normal bundles. This follows by general position and the fact that the normal bundles of the generating spheres \( S^n \subset \tilde{M}^{2n}_0 \) are the stably trivial tangent disc bundles \( \tau(S^n) \). We now attach a solid handle \( D^{n+1} \times D^{n+1} \) along \( S^n \times D^{n+1} \subset \tilde{M}^{2n}_0 \times S^1 \) for each generator of \( \pi_n(\tilde{M}^{2n}_0) \) and extend the map \( \hat{h} \) across these bundles. This is possible since the embedded spheres are in the homotopy kernel of \( \hat{h} \). Call the resulting PL manifold \( \hat{H} \) and the extended map \( \hat{h}: \hat{H} \to H \). In the process of extending \( \hat{h} \) across the handles, we may guarantee that \( \hat{h} \) is a map of pairs \( (\hat{H}, \partial) \to (H, \partial) \). We observe, then, the:

**Proposition.** \( \hat{h}: (\hat{H}, \partial) \to (H, \partial) \) is a homotopy equivalence of pairs.
This follows directly from the construction as \( \widetilde{H} \) deformation retracts onto \( \widetilde{CP}^n \# M^{2n} \cup \{e^n_o\} \) where the \( n \)-cells \( e^n_o \) are attached so as to kill the entire homology kernel of \( (h \# r) \). Hence \( \tilde{h}: \widetilde{H} \to H \) is a homology isomorphism, and as \( \tilde{H} \) is 1-connected we have by Whitehead's theorem that it is a homotopy equivalence. The restriction of \( \tilde{h} \) to the boundary is likewise a homology isomorphism as the boundaries, \( D^{n+1} \times S^n_e \), of the solid handles are precisely the surgeries needed to cobord \( \tilde{h}: \partial \widetilde{H} \to \partial H \) to a homotopy equivalence.

In particular as \( n \geq 3 \) we note that the boundary manifold, \( \partial \tilde{H} \), is a PL \((2n+1)\)-sphere by the Poincaré conjecture. Thus, we attach \( D^{2n+2} \) to \( \tilde{H} \) as the PL cone on \( \partial \tilde{H} \) and define:

\[
\widetilde{CP}^{n+1} = \tilde{H} \cup \{\partial \tilde{H}\} \quad \text{and} \quad h: \widetilde{CP}^{n+1} \to CP^{n+1} = H \cup \{\partial H\}
\]

by radial extension of \( \tilde{h} \) into \( c(\partial \tilde{H}) \).

Observe that \( h \) has “built-in” transverse inverse image \( \widetilde{CP}^n \# M^{2n} = h^{-1}(CP^n) \) with surgery obstruction \( \sigma_n \). Hence, this \( \widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \) is the space we require.

Now, given \( \widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}) \) let us consider a bit more closely the suspension and generalized suspension constructions described above. First, assume the homotopy equivalence

\[
h: \widetilde{CP}^n \to CP^n
\]

is the identity map on a disc \( D^{2n} \subset \widetilde{CP}^n \). Let \( \widetilde{CP}_0^n = \widetilde{CP}^n - D^n, M_0^{2n} = M^{2n} - D^{2n} \) and observe that \( \widetilde{CP}^n \# M^{2n} = \widetilde{CP}_0^n \cup M_0^{2n} \). Now, let \( \widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0) \) be the suspension\(^1\) of \( \widetilde{CP}^n \) with homotopy equivalence

\[
\tilde{h}: \widetilde{CP}^{n+1} \to CP^{n+1}
\]

and \( \widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \) be the general suspension of \( \widetilde{CP}^n \) with homotopy equivalence

\[
h: \widetilde{CP}^{n+1} \to CP^{n+1}.
\]

Let \( D^{2n} \subset CP^n \) be the image \( h(D^{2n}) \) and let \( CP^1 = S^2 \subset CP^{n+1} \) be represented as \( D^2 \cup c(\partial D^2) \) in \( CP^{n+1} = H \cup c(\partial H) \) with \( D^2 \) the fiber in \( H \) over the center of the disc \( D^{2n} \). Then \( \nu_{n+1}(CP^1) \subset CP^{n+1} \) may be represented as the set \( D^2 \times D^{2n} \cup c(\partial H) \), a \( D^{2n} \) bundle over the sphere \( S^2 = D^2 \cup c(\partial D^2) \).

Now let \( \tilde{V} = \tilde{h}^{-1}(\nu_{n+1}(CP^1)) \) and \( \tilde{V} = \tilde{h}^{-1}(\nu_{n+1}(CP^1)) \) in \( \widetilde{CP}^{n+1} \) and \( \widetilde{CP}^{n+1} \) respectively. We observe directly from the constructions that

\(^{1}\) We say \( \widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, 0) \) in the “suspension” of \( \widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \) as it is precisely the Thom complex of the line bundle induced over \( \widetilde{CP}^n \).
\( \mathcal{C}P^{n+1} - \tilde{V} \) and \( \mathcal{C}B^{n+1} - \hat{V} \) are precisely the same spaces. To prove Theorem A we must show that \( \tilde{V} \) and \( \hat{V} \) are PL homeomorphic to \( v_{n+1}(\mathbb{C}P^1) \).

**Lemma 1.** \( \tilde{V} \cong v_{n+1}(\mathbb{C}P^1) \) if \( \sigma_2 \) is even.

We observe this from PL block bundle theory as follows: by construction \( \tilde{V} \) is the union of two discs \( D_*^2 \times D^{2n} \) and \( c(\partial \tilde{H}) = D^{2n+2} \) along \( S_*^1 \times D^{2n} \). Hence \( \tilde{V} \) is trivially a block bundle regular neighborhood of \( \mathbb{C}P^1 = D_*^2 \cup c(D_*^2) \). Assume the obstruction \( \sigma_2 \) is even. Then as noted by Sullivan ([23] p. 43) the splitting obstruction of the homotopy equivalence

\[
\tilde{h}: \mathcal{C}B^{n+1} \to \mathbb{C}P^{n+1}
\]

along \( \mathbb{C}P^1 \) vanishes as it is the mod 2 reduction of \( \sigma_2 \). Hence, by a homotopic deformation we may conclude that the transverse inverse image of \( \mathbb{C}P^1 \) by \( \tilde{h} \) is \( \mathbb{C}P^1 \subset \mathcal{C}P^{n+1} \). Moreover, as any two homotopic PL embeddings of \( \mathbb{C}P^1 \subset \mathcal{C}P^{n+1} \) are ambiently PL isotopic (for \( n \geq 2 \) by Cor. 5.9 p. 65 [21]), we see by appeal to the uniqueness of normal block bundles (regular neighborhoods) [20] that \( \tilde{V} \) is block bundle isomorphic to the bundle induced from \( v_{n+1}(\mathbb{C}P^1) \) by \( \tilde{h} \). Conversely, the same argument on the homotopy inverse of \( \tilde{h} \) implies \( v_{n+1}(\mathbb{C}P^1) \) is block bundle induced from \( \hat{V} \). As we are in the stable block and vector bundle range and \( \pi_2 B_{PL} = \pi_2 B_0 = Z_2 \) we can conclude that \( \mathcal{C} \) and \( v(\mathbb{C}P^1) \) are block bundle isomorphic; hence PL homeomorphic.

**Lemma 2.** \( \hat{V} \cong S^2 \) (homotopy equivalent).

**Proof.** By construction \( \hat{V} = D^2 \times M_0^{2n} \cup X \cup c(\partial H) \) where \( X \) represents the solid handles we attached along \( S^1 \times M_0^{2n} \) to kill the homology kernel of \( \hat{h} \). The manifold \( D^2 \times M_0^{2n} \cup X \) is simply-connected with simply connected boundary and the homology of a point; hence by Smale’s theorem (Thm. 1.1 [22]) it is a PL disc \( D^{2n+2} \). Thus, \( \hat{V} = D^{2n+2} \cup W D^{2n+2} \) where \( W \) is the complement of the embedding

\[
D^2 \times S^{2n-1} \subset S^{2n+1} = \partial D^{2n+2}
\]

and \( S^{2n-1} = \partial M_0^{2n} \). By the Mayer-Vietoris sequence we know that \( W \) is a homology circle. Then, by a second application of the Mayer-Vietoris sequence to the union \( D^{2n+2} \cup W D^{2n+2} \) we see that \( \hat{V} \) is a homology \( S^2 \). Finally, by the Van Kampen theorem \( \hat{V} \) is 1-connected and we apply the Whitehead theorem for CW complexes.
LEMMA 3. $\hat{V} \cong v_{n+1}(CP^1)$.

Proof. $\partial \hat{V} = \partial [CP^{n+1} - \hat{V}] = \partial \hat{V} \cong \partial v_{n+1}(CP^1)$ by Lemma 1. Let $S^2 \subset \hat{V}$ be a homotopy equivalence and a PL embedding via Whitney's embedding theorem. Then $S^2 \subset \hat{V} \subset \hat{CP}^{n+1}$ is homotopic to the standard embedding $CP^1 \subset \hat{CP}^{n+1}$, and as before, the PL block bundle neighborhoods of these two embeddings must be isomorphic. Let $\nu \subset \hat{V}$ be this block bundle. We note that

$$\partial \nu = \partial v_{n+1}(CP^1) \cong \partial \hat{V} = \partial \hat{V}$$

by the previous lemmas. Hence, if

$$\hat{V} - \nu = Y$$

we have $\partial Y = \partial \hat{V} \cup \partial \nu$, two copies of the same manifold.

We consider the Mayer-Vietoris sequence for the union $\hat{V} = Y \cup \nu$ over $\partial \nu = Y \cap \nu$:

$$\cdots \rightarrow H_1(\partial \nu) \xrightarrow{i_1 - i_2} H_1(\nu) \oplus H_q(Y) \xrightarrow{j_1 - j_2} H_1(\hat{V}) \rightarrow \cdots$$

where

$$i_1 : \partial \nu \Rightarrow \nu, \quad j_1 : \nu \Rightarrow \hat{V},$$

$$i_2 : \partial \nu \Rightarrow Y, \quad j_2 : Y \Rightarrow \hat{V}.$$ 

Since $\nu$ and $V$ are homotopy 2-spheres and $j_1$ is a homotopy equivalence, we see that for $q \neq 2$, $i_{2*} : H_q(\partial \nu) \rightarrow H_q(Y)$ must be an isomorphism. When $q = 2$ the sequence becomes:

$$Z \xrightarrow{1 - i_{2*}} Z \oplus A \xrightarrow{i_{2*} + j_{2*}} Z, \quad A = H_2(Y)$$

from which we obtain $i_{2*}$ are isomorphisms $Z \xrightarrow{i_{2*}} A \xrightarrow{j_{2*}} Z$. Thus, $i_2 : \partial \nu \subset Y$ is a homology isomorphism, and in fact, a homotopy equivalence since $\hat{V} = Y \cup \nu$ and $\hat{V}$, $\nu$, $\partial \nu$ are all 1-connected so that by Van Kampen's theorem $Y$ is 1-connected.

We show next that $\partial \hat{V} \subset Y$ is a homology isomorphism so that $Y$ is a $h$-cobordism from $\partial \nu$ to $\partial \hat{V}$—i.e. $Y \cong \nu \times I$ and $\hat{V} = Y \cup \nu \cong \nu \cong \hat{V} v_{n+1}(CP^1)$ as required.
We know already that $\partial \hat{V} \approx Y$ as $\partial \hat{V} \approx \partial v \approx Y$. Moreover, $\partial v \approx \partial v_{n+1}(CP^1)$ is an $S^{2n-1}$ bundle over $S^2$. Hence, by the Serre Spectral Sequence we have

$$H_p(Y) = H_p(\partial \hat{V}) = \begin{cases} Z & \text{if } p = 0, 2, 2n - 1, 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the exact sequence of the pair $(\hat{V}, \partial \hat{V})$ is:

$$0 = H_3(\hat{V}, \partial \hat{V}) \to H_2(\partial \hat{V}) \to H_2(\hat{V}) \to H_1(\hat{V}, \partial \hat{V}) = 0$$

where the first and last groups are 0 by Poincaré Duality. Thus, the inclusion $\partial \hat{V} \subset Y \subset \hat{V}$ is a homology isomorphism through $p = 2$.

Now, consider the composition $f: \partial \hat{V} \to Y \to \partial \hat{V}$ where the second map is a homotopy equivalence. Then $f_*: H_p(\partial \hat{V}) \to H_p(\partial \hat{V})$ is an isomorphism for $p \leq 2$, and by Poincaré Duality so is $f^*: H^q(\partial \hat{V}) \to H^q(\partial \hat{V})$ for $q = 2n - 1, 2n, 2n + 1$. By the Universal Coefficient Theorem $f_*$ is an isomorphism for $p = 2n - 1, 2n, 2n + 1$ and so for all $p$. Thus, $f$ is a homotopy equivalence, and so is $i$.

Theorem A is now an immediate consequence of the last lemma as we have:

$$\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) = [CP^{n+1} - \hat{V}] \cup \hat{V},$$

$$\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0) = [CP^{n+1} - v_{n+1}(CP^1)] \cup_{a_{\hat{V}}} v_{n+1}(CP^1)$$

where we have identified $\hat{V}$ with $v_{n+1}(CP^1)$ by Lemma 1, and the PL homeomorphism

$$\varphi_{\hat{V}}: \partial[\widetilde{CP}^{n+1} - v(CP^1)] \to \partial v(CP^1)$$

comes from the restriction to the boundary of the PL homeomorphism $\hat{V} \to v_{n+1}(CP^1)$ of Lemma 3.

III. Construction of the self-knotting $\varphi_{\hat{V}}$: Here we construct for $n \geq 4$ a PL self-knotting

$$\varphi_{\hat{V}}: \partial v_{n+1}(CP^1) \to \partial v_{n+1}(CP^1)$$

with the property that it extends to a homotopy equivalence

$$\varphi_{\hat{V}}: v_{n+1}(CP^1) \to v_{n+1}(CP^1)$$
which has a transverse-inverse image

\[ M_0^{2n} = \varphi_{\sigma}^{-1}(D^{2n}) \]

on a fiber \( D^{2n} \). Clearly such a \( \varphi_{\sigma} \) will suffice for the map in Theorem A.

We begin the construction by defining

\[ \Sigma_{\sigma}^{2n-1} \subset S^{2n+1} \]

to be the smooth Brieskorn knot represented as the link of the singularity on the hypersurface in \( C^{n+1} \) defined by

\[ p(Z) = \begin{cases} Z_0^{n-1} + Z_1^3 + \cdots + Z_n^2, & \text{even,} \\ Z_0^3 + Z_1^2 + \cdots + Z_n^2, & \text{odd.} \end{cases} \]

It is well-known that \( S^{2n+1} - \Sigma_{\sigma}^{2n-1} \) is a smooth fiber bundle over the circle with fiber \( M_0^{2n} \), the smooth Milnor or Kervaire manifold with surgery invariant \( \sigma \).

Now, let \( S^1 \subset S^{2n+1} \) be a fiber on the boundary of the smooth tubular neighborhood \( D^2 \times \Sigma_{\sigma}^{2n-1} \) of the knot (a trivial bundle as \( \pi_{2n-1}(SO(2)) = 0 \) for \( n > 1 \)). Since \( n > 1 \) this circle \( S^1 \) is smoothly unknotted in \( S^{2n+1} \) so that the complement of a small tube \( S^1 \times D^{2n} \) about it is diffeomorphic to \( D^2 \times S^{2n-1} \). Hence the knot \( \Sigma_{\sigma}^{2n-1} \) lies in this complement with a trivial normal bundle and we can therefore define:

\[ \beta: D^2 \times \Sigma_{\sigma}^{2n-1} \hookrightarrow D^2 \times S^{2n-1} \]

as this embedding. Let \( W^{2n+1} \) be the complement of this smooth embedding. Then we observe:

(a) \( \partial W = S^1 \times S^{2n-1} \cup S^1 \times \Sigma_{\sigma}^{2n-1} \).

(b) \( W \) is a smooth fiber bundle over the circle \( S^1 \) with fiber \( F^{2n} = M_0^{2n} - D^2 \) and \( \partial F = S^{2n-1} \cup \Sigma_{\sigma}^{2n-1} \).

(c) The bundle projection is trivial on \( \partial W \to S^1 \).

Now, using the smooth embedding \( \beta \) we define a piecewise-linear embedding

\[ \gamma_{\sigma}: D^2 \times S^{2n-1} \hookrightarrow D^2 \times S^{2n-1} \]

as the composite map

\[ D^2 \times S^{2n-1} \xrightarrow{\text{id} \times \alpha_{\sigma}} D^2 \times \Sigma_{\sigma}^{2n-1} \xrightarrow{\beta} D^2 \times S^{2n-1} \]

where \( \alpha_{\sigma}: S^{2n-1} \to \Sigma_{\sigma}^{2n-1} \) is a specific PL homeomorphism.
We now describe the normal bundle $v_{n+1}(CP^1)$ in $CP^{n+1}$ as:

$$v_{n+1}(CP^1) = D_+^2 \times S^{2n-1} \cup _{\rho} D_-^2 \times S^{2n-1}$$

(*) where $\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$ is a smooth bundle automorphism representing an element in $\pi_1(SO(2n)) = Z/2Z$ ($n > 1$). [We note in fact that $\gamma_{n+1}(CP^1)$ is trivial for $n$ even and non-trivial for $n$ odd as it is the Whitney sum of $n$ copies of the canonical line bundle over $CP^1 = S^2$.]

In the above description we are expressing $CP^1$ as $S^2 \cup D_+^2$. Using this representation we will define the self-knotting $\varphi_{\sigma}$ by showing that the PL embedding

$$\gamma_{\sigma}: D_+^2 \times S^{2n-1} \to D_+^2 \times S^{2n-1}$$

may be extended to a PL homeomorphism on all of $V_{n+1}(CP^1)$. We will show this using the very agreeable bundle structure on the complement $W$ of the embedding $\gamma_{\sigma}$.

The map

$$\varphi_{\sigma}: D_+^2 \times S^{2n-1} \cup _{\rho} D_-^2 \times S^{2n-1} \to D_+^2 \times S^{2n-1} \cup _{\rho} D_-^2 \times S^{2n-1}$$

will in fact be defined as the union of three maps—

1. $\gamma_{\sigma}: D_+^2 \times S^{2n-1} \to D_+^2 \times S^{2n-1}$,
2. $\eta: \tilde{W}^{2n+1} \to W^{2n+1}$,
3. $\text{id} \times \mu: D^2 \times \Sigma_{-\sigma}^{2n-1} \to D_+^2 \times S^{2n-1}$

where $\eta$ is a bundle homeomorphism of bundles over $S^1$ and $\mu: \Sigma_{-\sigma}^{2n-1} \to S^{2n-1}$ is a PL homeomorphism and

$$D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W}^{2n+1} = D_+^2 \times S^{2n+1}.$$  

Essentially what we are producing in this construction is a map with the symmetric property that $\varphi_{\sigma}$ embeds a fiber (the core of $D_+^2 \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma_{-\sigma}^{2n-1} \subset D_+^2 \times S^{2n-1}$ while $\varphi_{\sigma}^{-1}$ embeds a fiber (the core of $D_-^2 \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma_{\sigma}^{2n-1} \subset D_-^2 \times S^{2n-1}$.

The construction will be completed by (a) defining the bundle $\tilde{W}$ and the bundle map $\eta$ in (2), (b) showing that $D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W}$ is in fact $D_+ \times S^{2n-1}$ by a PL homeomorphism which is the identity on the boundary, (c) showing that the maps (1), (2), (3) agree on boundaries after taking the defining automorphism $\rho$ into account, and finally by (d) showing that $\varphi_{\sigma}$ is homotopic to the identity.
We define the bundle $\tilde{W}$ over $S^1$ by defining its fiber $\tilde{F}$ and its monodromy map $\tilde{h}: \tilde{F} \to \tilde{F}$.

Recall that the $2n$-manifold $F$ (fiber of $W$) is $(n-1)$ connected and that $\partial F = S^{2n-1}_+ \cup \Sigma^{2n-1}_-$ where the smooth exotic sphere is defined as $\Sigma^{2n-1}_o = D^{2n-1}_+ \cup_o D^{2n+1}_+$ and $\sigma: S^{2n-2} \to S^{2n-2}$ is an exotic diffeomorphism.

Let $I \subset F$ be a path connecting the centers of the discs $D^{2n-1}_+$ and $D^{2n-1}_+$ of $\Sigma^{2n-1}_o$ and $S^{2n-1}$. Then a tubular neighborhood of $I$ is $I \times D^{2n-1}_+$. We define $\tilde{F}$ as the smooth manifold

$$\tilde{F} = \left[ F - I \times D^{2n-1}_+ \right] \cup \left[ I \times D^{2n-1}_- \right]$$

where the union is taken over the diffeomorphism $\text{id}_I \times \sigma^{-1}: I \times S^{2n-2} \to I \times S^{2n-2}$. Then $\partial \tilde{F} = \Sigma^{2n-1}_o \cup S^{2n-1}$ as a smooth manifold and we can define a PL homeomorphism

$$\hat{\eta}: \tilde{F} \to F$$

where $\hat{\eta}$ is the identity on $F - I \times D^{2n-1}_+$ and is $\text{id}_I \times (\text{cone extension of } \sigma)$ on $I \times D^{2n-1}_+$.

Then we define the monodromy $\tilde{h}: \tilde{F} \to \tilde{F}$ as the composite map

$$\tilde{h} = \hat{\eta}^{-1} \circ \eta \circ \hat{\eta}$$

where $\eta: F \to F$ is the monodromy map defining the bundle $W$. Since $\partial W$ is a trivial bundle we know that $\eta$ is the identity map on $\partial F$. Hence, $\tilde{h}$ is the identity on $\partial \tilde{F}$ and the bundle $\tilde{W}$ has the trivial boundary

$$\partial \tilde{W} = S^1 \times \Sigma^{2n-1}_o \cup S^1 \times S^{2n-1}.$$ 

Since $\hat{\eta} \circ \tilde{h} = \eta \circ \hat{\eta}$ the PL homeomorphism $\hat{\eta}: \tilde{F} \to F$ induces a well-defined bundle homeomorphism

$$\eta: \tilde{W}^{2n+1} \to W^{2n+1}.$$ 

Restricted to the boundary $\eta$ is a pair of bundle maps

$$\text{id}_{S^1} \times \alpha^{-1}_o: S^1 \times \Sigma^{2n-1}_o \to S^1 \times S^{2n-1},$$

$$\text{id}_{S^1} \times \alpha_o: S^1 \times S^{2n-1} \to S^1 \times \Sigma^{2n-1}_o$$

where the PL homeomorphism $\alpha^{-1}_o$ and $\alpha_o$ are the identity on $D^{2n-1}_-$ and the cone extension of $\sigma^{-1}$ and $\sigma$ respectively on $D^{2n-1}_+$. 
We next embed $\tilde{W}$ in $D^2 \times S^{2n-1}$ as a knot complement which will act as an inverse to $W$:

Recall the bundle isomorphism

$\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$

which defines $\partial_{n+1}(CP^1)$. We define a PL bundle map

$\hat{\rho}: S^1 \times \Sigma_{-\sigma} \to S^1 \times \Sigma_{-\sigma}$

as the composite: $\hat{\rho} = (\text{id}_{S^1} \times \alpha_{-\sigma}) \cdot \rho \cdot (\text{id}_{S^1} \times \alpha_{-\sigma})^{-1}$. We consider the PL manifold

$D^2 \times \Sigma_{-\sigma} \cup \tilde{W}^{2n+1}$

where the union is over the appropriate component of $\partial \tilde{W}$ and show:

**Proposition.** The PL manifold $D^2 \times \Sigma_{-\sigma} \cup \tilde{W}^{2n+1}$ is isomorphic to $D^2 \times S^{2n-1}$ by a PL homeomorphism $\Delta$ which restricted to the boundary $S^1 \times S^{2n-1}$ is an $S^{2n-1}$ bundle isomorphism $\lambda$.

**Proof.** We recall from the definition of $W^{2n+1}$ that $S^1 \times D^{2n} \cup W^{2n+1}$ is the knot complement of our original Brieskorn knot and so has the homology of $S^1$. A simple exercise with the Mayer-Vietoris sequence implies then that the manifold $\tilde{W}^{2n+1} \cup S^1 \times D^{2n}$ likewise is a homology circle, and a second application of the sequence implies that the PL manifold.

$P^{2n+1} = D^2 \times \Sigma_{-\sigma} \cup \tilde{W} \cup S^1 \times D^{2n}$

has the homology of $S^{2n+1}$. Moreover, $P^{2n+1}$ is simply connected since $\tilde{W} \cup S^1 \times D^{2n}$ fibers over $S^1$ with fiber $\tilde{F}^{2n} \cup D^{2n}$ which is $(n-1)$-connected. Hence $\pi_1(\tilde{W} \cup S^1 \times D^{2n}) = Z$ and by the Van Kampen theorem on the union

$[D^2 \times \Sigma_{-\sigma}] \cup S^1 \times \Sigma_{-\sigma} [\tilde{W} \cup S^1 \times D^{2n}]$

we have $\pi_1(P^{2n+1}) = 0$. By the Hurewicz and Whitehead theorems any simply-connected homology sphere is a homotopy sphere, and by the generalized Poincaré conjecture $(2n + 1 \geq 9)P^{2n+1}$ is a PL sphere.

The identification $D^2 \times \Sigma_{-\sigma} \cup i \tilde{W} S^1 \times D^{2n} \cong S^{2n+1}$ provides a PL embedding $S^1 \subset S^{2n+1}$ and exhibits $i(S^1 \times D^{2n}) \subset S^{2n+1}$ as a representative for the PL normal microbundle to this embedding. We apply a
theorem due to Lashof and Rothenberg (Thm. 7.3 in [13]) to obtain a piecewise differentiable homeomorphism $g: S^{2n+1} \to S^{2n+1}$ so that $g \circ i: S^1 \times D^{2n} \to S^{2n+1}$ is the smooth vector bundle to the smooth embedding $g \circ i: S^1 \to S^{2n+1}$. By smoothly unknotting this circle and applying the smooth tubular neighborhood theorem we obtain a diffeomorphism $h: S^{2n+1} \to S^{2n+1}$ so that

\[
\begin{array}{c}
h \circ g \circ i: S^1 \times D^{2n} \\
\downarrow \lambda \downarrow j \\
S^1 \times D^{2n}
\end{array}
\]

commutes where $j$ is the standard embedding and $\lambda$ is a vector bundle isomorphism. Hence, the restriction map

\[
h \circ g \mid : S^{2n+1} - i(S^1 \times D^{2n}) \to S^{2n+1} - j(S^1 \times D^{2n})
\]

defines a piecewise differentiable homeomorphism

\[
\Lambda: \left[ D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \hat{W} \right] \to D^2 \times S^{2n-1}
\]

which restricts as $\lambda = \lambda$ on the boundary. Finally, we observe that (cf. Cor. 10.13 in [19]) we may choose a smooth triangulation of $D^2 \times S^{2n-1}$ so that $\Lambda$ is PL. Now, using the homeomorphisms $\Lambda$ and $\eta$ we define a PL homeomorphism:

(1) $\varphi_\sigma: \xi \to \partial r_{n+1}(CP^1)$

where $\xi$ is the $S^{2n-1}$ bundle over $CP^1 = S^2$ defined by $\lambda^{-1}$:

\[
\xi = D^2_\sigma \times S^{2n-1} \cup \lambda^{-1} \cup D^2_+ \times S^{2n-1}
\]

\[
\Lambda^{-1} \cup \text{id} \rightarrow D^2_\sigma \times S^{2n-1} \cup \hat{W} \cup \text{id} D^2_+ \times S^{2n-1}
\]

\[
(\text{id} \times \alpha_{-\sigma}) \cup \eta \cup (\text{id} \times \alpha_{+}) \rightarrow D^2_\sigma \times S^{2n-1} \cup \hat{W} \cup D^2_+ \times S^{2n-1}
\]

\[
= D^2_\sigma \times S^{2n-1} \cup \hat{W} \cup D^2_+ \times S^{2n-1} = \partial r_{n+1}(CP^1).
\]

From the next lemma to the effect that two non-isomorphic sphere bundles over $S^2$ cannot be PL homeomorphic it follows that the existence of the map $\varphi_\sigma$ itself guarantees that $\xi$ and $\partial r_{n+1}(CP^1)$ are the same bundle.

**Lemma.** For $m \geq 3$ the unique non-trivial orthogonal $S^m$ bundle over $S^2$, $\xi$, is not PL homeomorphic to $S^2 \times S^m$. 

Proof. Suppose $t: \xi \to S^2 \times S^m$ is a PL homeomorphism. Let $E$ be the non-trivial $D^{m+1}$ bundle over $S^2$ with $\partial E = \xi$ and define the PL manifold $M^{m+3} = E \cup_i D^3 \times S^m$.

$M$ is the union of simply connected spaces over a path connected intersection. Hence, $\pi_1(M) = \{1\}$. For $m \geq 3$ the homotopy exact sequence of the fibration $S^m \to \partial E \to S^2$ implies that $p_*: \pi_2(\partial E) \to \pi_2(S^2)$ is an isomorphism, and by the Whitehead theorem so is the inclusion $H_2(\partial E) \to H_2(E)$. Hence, in the Mayer-Vietoris sequence

$$\cdots \to H_j(S^2 \times S^m) \to H_j(E) \oplus H_j(D^3 \times S^m) \to H_j(M)$$

$$\to H_{j-1}(S^2 \times S^m) \to \cdots$$

$\psi_j$ is an isomorphism for $j \leq m + 1$. Trivially, $H_{m+2}(M) = 0$, and again we have an $(m+2)$-connected $(m+3)$-dimensional PL manifold which is consequently a PL sphere.

Then, $E \cup_i D^3 \times S^m \cong S^{m+3}$ defines the vector bundle $E$ as a PL normal micro-bundle to the embedding of its zero section $S^2 \leadsto S^{m+3}$. By Zeeman's PL unknotting theorem and the uniqueness [7] of stable PL normal microbundles, we see that $E$ and $S^2 \times D^{m+1}$ must be micro-bundle isomorphic. Let $S^2 \xrightarrow{b} BO$ classify $E$ as a vector bundle. Then $S^2 \xrightarrow{h} BO \to BPL$ is trivial, and as by smoothing theory the fiber $PL/0$ is 6-connected we see that $b$ is homotopically trivial. As $E$ was assumed non-trivial as a vector bundle the PL homeomorphism $t$ cannot exist.

Thus, we define

$$\varphi_\sigma: \partial v_{n+1}(CP^1) = \xi \to \partial v_{n+1}(CP^1) \quad \text{from (1) as required.}$$

Next we show that the $\varphi_\sigma$ just constructed is indeed a self-knotting and that it will suffice for Theorem A.

Recalling from bundle theory that every $S^N$ bundle over $S^2$ for $N \geq 2$ has a section, we show

**Proposition.** Any orientation preserving PL homeomorphism $\varphi: v \to v$, $v$ an orthogonal $S^N$ bundle over $S^2$, which embeds a section $S^2 \xrightarrow{f} v$ homotopically to itself is homotopic to the identity.

**Proof.** A tubular neighborhood of the section $j(S^2)$ is a $D^N$ bundle $U$ in the same stable bundle class as $v$. $\varphi(U)$ PL embeds this bundle in $v$ with an inherited smooth structure. By the main theorem of smoothing
theory ([8] or [13], Thm. 7.3) and the uniqueness of smoothings on $S^2$ we can piecewise differentially isotope this embedding to a smooth embedding of $U \to v$. We may easily make the isotopy ambient. Next, we smoothly unknot the core sphere of $U$ and apply the smooth tubular neighborhood theorem. We have, therefore, P.D. isotoped $\varphi$ so that restricted to $U$ it is a $D^N$ bundle isomorphism. Since $\pi_2(SO(N)) = 0$ we can isotope this bundle mapping to the identity through bundle isomorphisms on $U$ all of which extend to $v$ as $U$ is a sub-bundle. Thus, we have isotoped $\varphi$ so that it is the identity on $U$. Now, $v - U \cong U$ as each fiber of $U$ is a hemisphere of a fiber in $v$. We isotope $\varphi_{rel(U)}$ so that it is the identity on the zero section of the bundle $v - U$. Finally, we homotope $\varphi$ to the identity by collapsing the fibers of $v - U$ to the zero-section.

We observe that the $\varphi$ constructed above satisfies the hypothesis of this last proposition as follows: $\varphi$ is orientation preserving by construction. Also, as the original Brieskorn knot embedded a fiber $S^{2n+1}$ homotopically to the usual embedding, we know that $\varphi$ does also. That is $(\varphi_{\ast})_\ast [\partial v] = [\partial v]$ and $(\varphi_{\ast})_{\ast}(e^{2n-1}) = e^{2n-1}$, where $e^{2n-1} \in H^{2n-1}(\partial v)$ is the class represented by inclusion of a fiber. By Poincaré Duality, then, $(\varphi_{\ast})_\ast(e) = e_2$ for $e_2 \in H_2(\partial v)$ the class dual to $e^{2n-1}$. This implies by the Hurewicz Theorem that $\varphi$ induces the identity homomorphism on $\pi_2(\partial v)$, which is generated by the inclusion of a section.

The map $\varphi$ constructed in section C embeds a fiber $S^{2n-1}$ onto the image of the Brieskorn knot. Hence, in the decomposition

$$\widetilde{CP}^{n+1} = [CP^{n+1} - v_{n+1}(CP^1)] \cup \varphi_{\ast}[v_{n+1}(CP^1)]$$

the identification is in the order:

$$\varphi_{\ast}: \partial[CP^{n+1} - v] \to \partial v.$$ 

To show, therefore, that $\widetilde{CP}^{n+1} \leftrightarrow (0, \ldots, 0, \sigma)$ we must extend $\varphi_{\ast}^{-1}$ to a homotopy equivalence $\varphi_{\ast}^{-1}: v \to v$ with transverse-inverse image of a fiber being the Milnor or Kervaire manifold $M_{0}^{2n}$. Note that any extension will be a homotopy equivalence as $v \cong S^2$ and $\varphi_{\ast}^{-1}$ induces the identity on $\pi_2(\partial v) = \pi_2(v)$.

**Proposition.** The PL homeomorphism $\varphi_{\ast}^{-1}: \partial v_{N+1}(CP^1) \to \partial v_{n+1}(CP^1)$ constructed above extends to $\varphi_{\ast}^{-1}: v_{n+1}(CP^1) \to v_{n+1}(CP^1)$ with transverse-inverse image

$$(\varphi_{\ast}^{-1})_{\ast}(D^{2n}) = M_{0}^{2n}$$
Proof. $(\varphi^{-1}_o)^{-1}(S^{2n-1}) = \varphi_o(S^{2n-1}) = \Sigma_o^{2n-1} \subset \partial v$ by the construction of $\varphi_o$. Moreover, the restriction $\varphi^{-1}_o | : D^2 \times \Sigma^{2n-1}_o \to D^2_+ \times S^{2n-1}$ is a product map. Now, $\Sigma^{2n-1}_o$ bounds a fiber $F^{2n}_o \subset W^{2n+1}$ whose other boundary component is a fiber $S^{2n-1}$ of $\partial v$. Let $D^{2n} \subset v$ be the fiber whose boundary is this same sphere. Then, $F^{2n}_o \cup D^{2n} = M^{2n}_o$ by the definition of $F^{2n}_o$. By pushing $F^{2n}_o$ into $v$ along a vector field normal to $\partial v$ and smoothing the corner at $S^{2n-1}$ between $F^{2n}$ and $D^{2n}$ we obtain a smooth embedding $M^{2n}_o \to v$ extending

$$\partial M^{2n}_o = \Sigma^{2n-1}_o \subset \partial v.$$ 

Moreover, this embedding will have trivial normal $D^2$ bundle as $H^1(M^{2n}_o, \mathbb{Z}) = 0$. Hence, we can extend the product map

$$\varphi^{-1}_o | : D^2 \times \Sigma^{2n-1}_o \to D^2_+ \times S^{2n-1}$$

to a bundle map $\hat{\varphi}^{-1}_o | : D^2 \times M^{2n}_o \to D^2_+ \times D^{2n}$ covering a degree one extension $M^{2n}_o \to D^{2n}$. Since $[v - D^2_+] \times D^-_+ \times D^{2n} = D^{2n-2}$ there are no cohomology obstructions to extending

$$\varphi^{-1}_o \cup \hat{\varphi}^{-1}_o \text{ to } \varphi^{-1}_o | : v \to v$$

with the required transverse-inverse image built in.

**References**


Received March 24, 1980.

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