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EXPLICIT PL SELF-KNOTTINGS AND THE STRUCTURE OF PL HOMOTOPY COMPLEX PROJECTIVE SPACES

DOUGLAS SHELBY MEADOWS

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EXPLICIT PL SELF-KNOTTINGS AND THE STRUCTURE OF PL HOMOTOPY COMPLEX PROJECTIVE SPACES

DOUGLAS MEADOWS

We show that certain piecewise-linear homotopy complex projective spaces may be described as a union of smooth manifolds glued along their common boundaries. These boundaries are sphere bundles and the glueing homeomorphisms are piecewise-linear self-knottings on these bundles. Furthermore, we describe these self-knottings very explicitly and obtain information on the groups of concordance classes of such maps.

A piecewise linear homotopy complex projective space \widetilde{CP}^n is a compact PL manifold M^{2n} homotopy equivalent to CP^n . In [22] Sullivan gave a complete enumeration of the set of PL isomorphism classes of these manifolds as a consequence of his Characteristic Variety theorem and his analysis of the homotopy type of G/PL. In [15] Madsen and Milgram have shown that these manifolds, the index 8 Milnor manifolds, and the differentiable generators of the oriented smooth bordism ring provide a complete generating set for the torsion-free part of the oriented PL bordism ring. Hence a study of the geometric structure of these exotic projective spaces \widetilde{CP}^n , especially with regard to their smooth singularities, may extend our understanding of the PL bordism ring. This paper begins such a study in which we obtain a geometric decomposition of CP^n , into (for many cases) a union of smooth manifolds glued together by PL self-knottings on certain sphere bundles. We also obtain information on groups of concordance classes of PL self-knottings from these decompositions and a number of fairly explicitly constructed examples of self-knottings. I would like to thank by thesis advisor R. J. Milgram for many helpful discussions.

I. Sullivan's classification of PL homotopy \widetilde{CP}^n proceeds as follows: Given a homotopy equivalence $h: \widetilde{CP}^n \to CP^n$ make h transverse regular to $CP^j \subset \widetilde{CP}^n$, the standard inclusion. The restriction of h to the transverse inverse image $h^{-1}(CP^j) = N^{2j} \subset \widetilde{CP}^n$ is a degree one normal map with simply connected surgery obstruction

$$\sigma_j \in P_{2j} = \begin{cases} Z, & j \text{ even} \\ Z/2Z, & j \text{ odd} \end{cases}.$$

For j = 2, ..., n - 1 these obstruction invariants yield a complete enumeration—i.e. the set of PL isomorphism classes of \widehat{CP}^n is set-isomorphic to the product $Z \times Z_2 \times Z \times \cdots \times P_{2(n-1)}$ with n - 2 factors.

We will use the following notation to specify elements with this classification:

(1)
$$\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \dots, \sigma_{n-1})$$

will denote the PL homotopy \widetilde{CP}^n with invariants $\sigma_j \in P_{2j}$ in Sullivan's enumeration.

We recall that a PL homeomorphism $f: M \to M$ is a "self-knotting" and M is said to be "self knotted" if f is homotopic but not PL isotopic to the identity. Also, PL homeomorphisms $f, g: M \to M$ are "PL concordant" (pseudo-isotopic) if we have a PL homeomorphism $F: M \times I \to M \times I$ with F(m, 0) = (f(m), 0) and F(m, 1) = (g(m), 1) for $m \in M$. We then define:

(2) SK(M) = "the group (under composition of maps) of PL concordance classes of PL self-knottings of M."

Unless otherwise noted " $CP^{j} \subset CP^{n}$ " means the standard embedding of CP^{j} onto the first (j + 1) homogeneous coordinates of CP^{n} or a smooth ambient isotope of this embedding. In this context we define:

(3) $\nu_N(CP^j) =$ "the smooth tubular disc bundle neighborhood of the embedding $CP^j \subset CP^N$."

Our results are as follows:

THEOREM A. For $n \ge 4$ and $\sigma_2 \equiv 0$ (2) every $\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \dots, \sigma_{n-1})$ is PL homeomorphic to the identification space

$$\left[\widehat{CP}^n-\nu_n(CP^1)\right]\cup_{\varphi_{\sigma_{n-1}}}\left[\nu_n(CP^1)\right]$$

where $\widehat{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \dots, \sigma_{n-2}, 0)$ and the identification is over a PL homeomorphism

$$\varphi_{\sigma_{n-1}}: \partial \nu_n(CP^1) \to \partial \nu_n(CP^1).$$

We prove Theorem A in Part II by a careful description of Sullivan's classification and an easy *h*-cobordism argument.

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An immediate consequence of Theorem A is the decomposition of $\widetilde{CP}^{n+1} \leftrightarrow (0, \dots, 0, \sigma_n)$ into

$$\widetilde{CP}^{n+1} = \left[CP^{n+1} - \nu(CP^1)\right] \cup_{\varphi_0} \left[\nu(CP^1)\right].$$

THEOREM B. For every $n \ge 4$ and non-zero $\tau \in P_{2n}$ there is a PL self-knotting

$$\varphi_{\tau} \colon \partial \nu_{n+1}(CP^1) \to \partial \nu_{n+1}(CP^1)$$

which will suffice for the glueing homeomorphism in Theorem A.

We establish this theorem by an explicit construction of φ_{τ} in Part III.

II. Here we prove Theorem A by beginning with a construction which shows how to obtain $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ from $\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1})$ for $n \ge 4$:

Let $h: \widetilde{CP}^n \to CP^n$ be a homotopy equivalence, and let M^{2n} be the compact (n-1)-connected Milnor or Kervaire manifold of Index $8\sigma_n$ or Kervaire-Arf invariant σ_n as the case may be [4]. Let $r: M^{2n} \to S^{2n}$ be a degree one map. Then $h \# r: \widetilde{CP}^n \# M^{2n} \to CP^n \# S^{2n} = CP^n$ is a degree one normal map with 1-connected surgery obstruction σ_n . We define \hat{H} as the D^2 bundle over $\widetilde{CP}^n \# M^{2n}$ induced by h # r from H, the disc bundle associated to the complex line bundle over CP^n . Let $\hat{h}: \hat{H} \to H$ be the bundle mapping. We note that the map h # r is (n-1)-connected with homological kernel $K_n = \pi_n(M_0^{2n})$ where $M_0^{2n} = M^{2n} - D^{2n}$. The bundle \hat{H} is trivial over M_0^{2n} since $M_0^{2n} = (h \# r)^{-1}$ (point). In $M_0^{2n} \times D^2$ we can represent $\pi_n(M_0^{2n})$ by disjointly embedded spheres $S^n \hookrightarrow M_0^{2n} \times S^1$ with trivial normal bundles. This follows by general position and the fact that the normal bundles of the generating spheres $S^n \subset M_0^{2n}$ are the stably trivial tangent disc bundles $\tau(S^n)$. We now attach a solid handle $D^{n+1} \times D^{n+1}$ along $S^n \times D^{n+1} \subset M_0^{2n} \times S^1$ for each generator of $\pi_n(M_0^{2n})$ and extend the map \hat{h} across these bundles. This is possible since the embedded spheres are in the homotopy kernel of \hat{h} . Call the resulting PL manifold \tilde{H} and the extended map $\hat{h}: \hat{H} \to H$. In the process of extending \hat{h} across the handles, we may guarantee that \tilde{h} is a map of pairs $(\tilde{H}, \partial) \to (H, \partial)$. We observe, then, the:

PROPOSITION. \tilde{h} : $(\tilde{H}, \partial) \rightarrow (H, \partial)$ is a homotopy equivalence of pairs.

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This follows directly from the construction as \tilde{H} deformation retracts onto $\widetilde{CP}^n \# M^{2n} \cup \{e_{\alpha}^n\}$ where the *n*-cells e_{α}^n are attached so as to kill the entire homology kernel of (h # r). Hence $\tilde{h}: \tilde{H} \to H$ is a homology isomorphism, and as \tilde{H} is 1-connected we have by Whitehead's theorem that it is a homotopy equivalence. The restriction of \tilde{h} to the boundary is likewise a homology isomorphism as the boundaries, $D^{n+1} \times S_{\alpha}^n$, of the solid handles are precisely the surgeries needed to cobord $\hat{h}: \partial \hat{H} \to \partial H$ to a homotopy equivalence.

In particular as $n \ge 3$ we note that the boundary manifold, $\partial \hat{H}$, is a PL (2n + 1)-sphere by the Poincaré conjecture. Thus, we attach D^{2n+2} to \hat{H} as the PL cone on $\partial \hat{H}$ and define:

$$\widetilde{CP}^{n+1} = \widetilde{H} \cup c(\partial \widetilde{H}) \text{ and } h: \widetilde{CP}^{n+1} \to CP^{n+1} = H \cup c(\partial H)$$

by radial extension of \tilde{h} into $c(\partial \tilde{H})$.

Observe that *h* has 'built-in' transverse inverse image $\widetilde{CP}^n \# M^{2n} = h^{-1}(CP^n)$ with surgery obstruction σ_n . Hence, this $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ is the space we require.

Now, given $\widetilde{CP}^n \leftrightarrow (\sigma_2, \dots, \sigma_{n-1})$ let us consider a bit more closely the suspension and generalized suspension constructions described above. First, assume the homotopy equivalence

$$h: \widetilde{CP}^n \to CP^n$$

is the identity map on a disc $D^{2n} \subset \widetilde{CP}^n$. Let $\widetilde{CP}_0^n = \widetilde{CP}^n - D^{2n}$, $M_0^{2n} = M^{2n} - D^{2n}$ and observe that $\widetilde{CP}^n \# M^{2n} = \widetilde{CP}_0^n \cup_{\partial} M_0^{2n}$. Now, let $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0)$ be the suspension¹ of \widetilde{CP}^n with homotopy equivalence

$$\tilde{h} \colon \widetilde{CP}^{n+1} \to CP^{n+1}$$

and $\widehat{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ be the general suspension of \widetilde{CP}^n with homotopy equivalence

$$\hat{h}:\widehat{CP}^{n+1}\to CP^{n+1}.$$

Let $D^{2n} \subset CP^n$ be the image $h(D^{2n})$ and let $CP^1 = S^2 \subset CP^{n+1}$ be represented as $D^2_* \cup c(\partial D^2_*)$ in $CP^{n+1} = H \cup c(\partial H)$ with D^2_* the fiber in H over the center of the disc D^{2n} . Then $\nu_{n+1}(CP^1) \subset CP^{n+1}$ may be represented as the set $D^2_* \times D^{2n} \cup c(\partial H)$, a D^{2n} bundle over the sphere $S^2 = D^2_* \cup c(\partial D^2_*)$.

Now let $\tilde{V} = \tilde{h}^{-1}(\nu_{n+1}(CP^1))$ and $\hat{V} = \hat{h}^{-1}(\nu_{n+1}(CP^1))$ in \widetilde{CP}^{n+1} and \widehat{CP}^{n+1} respectively. We observe directly from the constructions that

¹We say $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \sigma_3, \dots, \sigma_{n-1}, 0)$ in the "suspension" of $\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \dots, \sigma_{n-1})$ as it is precisely the Thom complex of the line bundle induced over \widetilde{CP}^n .

 $\widetilde{CP}^{n+1} - \widetilde{V}$ and $\widehat{CP}^{n+1} - \widehat{V}$ are precisely the same spaces. To prove Theorem A we must show that \widetilde{V} and \widehat{V} are PL homeomorphic to $\nu_{n+1}(CP^1)$.

LEMMA 1. $\tilde{V} \cong v_{n+1}(CP^1)$ if σ_2 is even.

We observe this from PL block bundle theory as follows: by construction \tilde{V} is the union of two discs $D_*^2 \times D^{2n}$ and $c(\partial \tilde{H}) = D^{2n+2}$ along $S_*^1 \times D^{2n}$. Hence \tilde{V} is trivially a block bundle regular neighborhood of $CP^1 = D_*^2 \cup c(\partial D_*^2)$. Assume the obstruction σ_2 is even. Then as noted by Sullivan ([23] p. 43) the splitting obstruction of the homotopy equivalence

$$\tilde{h}: \widetilde{CP}^{n+1} \to CP^{n+1}$$

along CP^1 vanishes as it is the mod 2 reduction of σ_2 . Hence, by a homotopic deformation we may conclude that the transverse inverse image of CP^1 by \tilde{h} is $CP^1 \subset \widetilde{CP}^{n+1}$. Moreover, as any two homotopic PL embeddings of $CP^1 \subset \widetilde{CP}^{n+1}$ are ambiently PL isotopic (for $n \ge 2$ by Cor. 5.9 p. 65 [21]), we see by appeal to the uniqueness of normal block bundles (regular neighborhoods) [20] that \tilde{V} is block bundle isomorphic to the bundle induced from $\nu_{n+1}(CP^1)$ by \tilde{h} . Conversely, the same argument on the homotopy inverse of \tilde{h} implies $\nu_{n+1}(CP^1)$ is block bundle induced from \tilde{V} . As we are in the stable block and vector bundle range and $\pi_2 B_{PL} = \pi_2 B_0 = Z_2$ we can conclude that \tilde{C} and $\nu(CP^1)$ are block bundle isomorphic; hence PL homeomorphic.

LEMMA 2. $\hat{V} \simeq S^2$ (homotopy equivalent).

Proof. By construction $\hat{V} = D^2 \times M_0^{2n} \cup X \cup c(\partial H)$ where X represents the solid handles we attached along $S^1 \times M_0^{2n}$ to kill the homology kernel of \hat{h} . The manifold $D^2 \times M_0^{2n} \cup X$ is simply-connected with simply connected boundary and the homology of a point; hence by Smale's theorem (Thm. 1.1 [22]) it is a PL disc D^{2n+2} . Thus, $\hat{V} = D^{2n+2} \cup_W D^{2n+2}$ where W is the complement of the embedding

$$D^2 \times S^{2n-1} \subset S^{2n+1} = \partial D^{2n+2}$$

and $S^{2n-1} = \partial M_0^{2n}$. By the Mayer-Vietoris sequence we know that W is a homology circle. Then, by a second application of the Mayer-Vietoris sequence to the union $D^{2n+2} \cup_W D^{2n+2}$ we see that \hat{V} is a homology S^2 . Finally, by the Van Kampen theorem \hat{V} is 1-connected and we apply the Whitehead theorem for CW complexes.

Lemma 3. $\hat{V} \cong \nu_{n+1}(CP^1)$.

Proof. $\partial \hat{V} = \partial [CP^{n+1} - \hat{V}] = \partial [CP^{n+1} - \tilde{V}] = \partial \tilde{V} \cong \partial \nu_{n+1}(CP^1)$ by Lemma 1. Let $S^2 \subset \hat{V}$ be a homotopy equivalence and a PL embedding via Whitney's embedding theorem. Then $S^2 \subset \hat{V} \subset \widehat{CP}^{n+1}$ is homotopic to the standard embedding $CP^1 \subset \widehat{CP}^{n+1}$, and as before, the PL block bundle neighborhoods of these two embeddings must be isomorphic. Let $\nu \subset \hat{V}$ be this block bundle. We note that

$$\partial \nu = \partial \nu_{n+1}(CP^1) \cong \partial \tilde{V} = \partial \tilde{V}$$

by the previous lemmas. Hence, if

$$\hat{V} - \nu = Y$$

we have $\partial Y = \partial \hat{V} \cup \partial \nu$, two copies of the same manifold.

We consider the Mayer-Vietoris sequence for the union $\hat{V} = Y \cup \nu$ over $\partial \nu = Y \cap \nu$:

$$\cdots \to H_1(\partial \nu) \stackrel{i_1,-i_2}{\to} H_1(\nu) \oplus H_q(Y) \stackrel{j_1,-j_2}{\to} H_1(\hat{V}) \to \cdots$$

where

$$\begin{split} i_1 \colon \partial \nu &\hookrightarrow \nu, \quad j_1 \colon \nu \hookrightarrow \hat{V}, \\ i_2 \colon \partial \nu \hookrightarrow Y, \quad j_2 \colon Y \hookrightarrow \hat{V}. \end{split}$$

Since ν and V are homotopy 2-spheres and j_1 is a homotopy equivalence, we see that for $q \neq 2$, i_{2_*} : $H_q(\partial \nu) \rightarrow H_q(Y)$ must be an isomorphism. When q = 2 the sequence becomes:

$$Z \xrightarrow{1-i_{2*}} Z \oplus A \xrightarrow{1+j_{2*}} Z, \qquad A = H_2(Y)$$

from which we obtain i_{2_*} are isomorphisms $Z \xrightarrow{i_{2_*}} A \xrightarrow{j_{2_*}} Z$. Thus, $i_2: \partial \nu \subset Y$ is a homology isomorphism, and in fact, a homotopy equivalence since $\hat{V} = Y \cup \nu$ and \hat{V} , ν , $\partial \nu$ are all 1-connected so that by Van Kampen's theorem Y is 1-connected.

We show next that $\partial \hat{V} \subset Y$ is a homology isomorphism so that Y is a *h*-cobordism from $\partial \nu$ to $\partial \hat{V}$ —i.e. $Y \cong \partial \nu \times I$ and $\hat{V} = Y \cup \nu \cong \nu \cong \tilde{V}\nu_{n+1}(CP^1)$ as required.

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We know already that $\partial \hat{V} \simeq Y$ as $\partial \hat{V} \simeq \partial \nu \simeq Y$. Moreover, $\partial \nu \simeq \partial \nu_{n+1}(CP^1)$ is an S^{2n-1} bundle over S^2 . Hence, by the Serre Spectral Sequence we have

$$H_p(Y) = H_p(\partial \hat{V}) = \begin{cases} Z & \text{if } p = 0, 2, 2n - 1, 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the exact sequence of the pair $(\hat{V}, \partial \hat{V})$ is:

$$0 = H_3(\hat{V}, \partial \hat{V}) \to H_2(\partial \hat{V}) \to H_2(\hat{V}) \to H_1(\hat{V}, \partial \hat{V}) = 0$$

$$\| \qquad \|$$

$$Z \qquad Z$$

where the first and last groups are 0 by Poincaré Duality. Thus, the inclusion $\partial \hat{V} \subset Y \subset \hat{V}$ is a homology isomorphism through p = 2.

Now, consider the composition $f: \partial \hat{V} \stackrel{i}{\hookrightarrow} Y \rightarrow \partial \hat{V}$ where the second map is a homotopy equivalence. Then $f_*: H_p(\partial \hat{V}) \rightarrow H_p(\partial \hat{V})$ is an isomorphism for $p \leq 2$, and by Poincaré Duality so is $f^*: H^1(\partial \hat{V}) \rightarrow H^1(\partial \hat{V})$ for q = 2n - 1, 2n, 2n + 1. By the Universal Coefficient Theorem f_* is an isomorphism for p = 2n - 1, 2n, 2n + 1 and so for all p. Thus, f is a homotopy equivalence, and so is i.

Theorem A is now an immediate consequence of the last lemma as we have:

$$\widehat{CP}^{n+1} \leftrightarrow (\sigma_2, \dots, \sigma_{n-1}, \sigma_n) = [CP^{n+1} - \tilde{V}] \cup \hat{V},$$

$$\widetilde{CP}^n \leftrightarrow (\sigma_2, \dots, \sigma_{n-1}, 0) = [CP^{n+1} - \nu_{n+1}(CP^1)] \cup_{\alpha_{\sigma_n}} \nu_{n+1}(CP^1)$$

where we have identified \tilde{V} with $\nu_{n+1}(CP^1)$ by Lemma 1, and the PL homeomorphism

$$\varphi_{\sigma_n}: \partial \big[\widetilde{CP}^{n+1} - \nu(CP^1)\big] \to \partial \nu(CP^1)$$

comes from the restriction to the boundary of the PL homeomorphism $\hat{V} \rightarrow \nu_{n+1}(CP^1)$ of Lemma 3.

III. Construction of the self-knotting φ_{σ} : Here we construct for $n \ge 4$ a PL self-knotting

$$\varphi_{\sigma}: \partial \nu_{n+1}(CP^1) \to \partial \nu_{n+1}(CP^1)$$

with the property that it extends to a homotopy equivalence

$$\overline{\varphi}_{\sigma} \colon \nu_{n+1}(CP^1) \to \nu_{n+1}(CP^1)$$

which has a transverse-inverse image

$$M_0^{2n} = \overline{\varphi}_{\sigma}^{-1}(D^{2n})$$

on a fiber D^{2n} . Clearly such a φ_{σ} will suffice for the map in Theorem A.

We begin the construction by defining

$$\Sigma_{\sigma}^{2n-1} \subset S^{2n+1}$$

to be the smooth Brieskorn knot represented as the link of the singularity on the hypersurface in C^{n+1} defined by

$$p(Z) = \begin{cases} Z_0^{6\sigma-1} + Z_1^3 + Z_2^2 + \dots + Z_n^2, & n \text{ even,} \\ Z_0^3 + Z_1^2 + \dots + Z_n^2, & n \text{ odd.} \end{cases}$$

It is well-known that $S^{2n+1} - \Sigma_{\sigma}^{2n-1}$ is a smooth fiber bundle over the circle with fiber M_0^{2n} , the smooth Milnor or Kervaire manifold with surgery invariant σ .

Now, let $S^1 \subset S^{2n+1}$ be a fiber on the boundary of the smooth tubular neighborhood $D^2 \times \Sigma_{\sigma}^{2n-1}$ of the knot (a trivial bundle as $\pi_{2n-1}(SO(2)) = 0$ for n > 1). Since n > 1 this circle S^1 is smoothly unknotted in S^{2n+1} so that the complement of a small tube $S^1 \times D^{2n}$ about it is diffeomorphic to $D^2 \times S^{2n-1}$. Hence the knot Σ_{σ}^{2n-1} lies in this complement with a trivial normal bundle and we can therefore define:

$$\beta: D^2 \times \Sigma^{2n-1}_{\sigma} \hookrightarrow D^2 \times S^{2n-1}$$

as this embedding. Let W^{2n+1} be the complement of this smooth embedding. Then we observe:

(a) $\partial W = S^1 \times S^{2n-1} \cup S^1 \times \Sigma^{2n-1}_{\sigma}$.

(b) W is a smooth fiber bundle over the circle S^1 with fiber $F^{2n} = M_0^{2n} - D^2$ and $\partial F = S^{2n-1} \cup \Sigma_{\sigma}^{2n-1}$.

(c) the bundle projection is trivial on $\partial W \to S^1$.

Now, using the smooth embedding β we define a piecewise-linear embedding

$$\gamma_{\sigma}: D^2 \times S^{2n-1} \hookrightarrow D^2 \times S^{2n-1}$$

as the composite map

$$D^2 \times S^{2n-1} \xrightarrow{\mathrm{id} \times \alpha_{\sigma}} D^2 \times \Sigma^{2n-1}_{\sigma} \xrightarrow{\beta} D^2 \times S^{2n-1}$$

where $\alpha_{\sigma}: S^{2n-1} \to \Sigma_{\sigma}^{2n-1}$ is a specific PL homeomorphism.

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We now describe the normal bundle $v_{n+1}(CP^1)$ in CP^{n+1} as:

$$v_{n+1}(CP^1) = D^2_- \times S^{2n-1} \cup_{\rho} D^2_+ \times S^{2n-1}$$

(*) where $\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$ is a smooth bundle automorphism representing an element in $\pi_1(SO(2n)) = Z/2Z$ (n > 1). [We note in fact that $\gamma_{n+1}(CP^1)$ is trivial for *n* even and non-trivial for *n* odd as it is the Whitney sum of *n* copies of the canonical line bundle over $CP^1 = S^2$.]

In the above description we are expressing CP^1 as $S^2 = D_-^2 \cup D_+^2$. Using this representation we will define the self-knotting φ_{σ} by showing that the PL embedding

$$\gamma_{n}: D^{2}_{+} \times S^{2n-1} \hookrightarrow D^{2}_{+} \times S^{2n-1}$$

may be extended to a PL homeomorphism on all of $V_{n+1}(CP^1)$. We will show this using the very agreeable bundle structure on the complement W of the embedding γ_{σ} .

The map

$$\varphi_{\sigma} \colon D^2_- \times S^{2n-1} \cup_{\rho} D^2_+ \times S^{2n-1} \to D^2_- \times S^{2n-1} \cup_{\rho} D^2_+ \times S^{2n-1}$$

will in fact be defined as the union of three maps --

(1)
$$\gamma_{\sigma}: D^2_+ \times S^{2n-1} \hookrightarrow D^2_+ \times S^{2n-1}$$

(2) $\eta \colon \tilde{W}^{2n+1} \to W^{2n+1},$

(3)
$$\operatorname{id} \times \mu \colon D^2 \times \Sigma^{2n-1}_{-\sigma} \to D^2_- \times S^{2n-1}$$

where η is a bundle homeomorphism of bundles over S^1 and $\mu: \sum_{-\sigma}^{2n-1} \to S^{2n-1}$ is a PL homeomorphism and

$$D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W}^{2n+1} = D_{-}^2 \times S^{2n+1}$$

Essentially what we are producing in this construction is a map with the symmetric property that φ_{σ} embeds a fiber (the core of $D^2_+ \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\sum_{-\sigma}^{2n-1} \subset D^2_- \times S^{2n-1}$ while φ_{σ}^{-1} embeds a fiber (the core of $D^2_- \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\sum_{\sigma}^{2n-1} \subset D^2_- \times S^{2n-1}$.

The construction will be completed by (a) defining the bundle \tilde{W} and the bundle map η in (2), (b) showing that $D^2 \times \sum_{-\sigma}^{2n-1} \cup \tilde{W}$ is in fact $D^2 \times S^{2n-1}$ by a PL homeomorphism which is the identity on the boundary, (c) showing that the maps (1), (2), (3) agree on boundaries after taking the defining automorphism ρ into account, and finally by (d) showing that φ_{σ} is homotopic to the identity. We define the bundle \tilde{W} over S^1 by defining its fiber \tilde{F} and its monodromy map $\tilde{h}: \tilde{F} \to \tilde{F}$.

Recall that the 2*n*-manifold F (fiber of W) is (n-1) connected and that $\partial F = S^{2n-1} \cup \sum_{\sigma}^{2n-}$ where the smooth exotic sphere is defined as $\sum_{\sigma}^{2n-1} = D_{-}^{2n-1} \cup_{\sigma} D_{+}^{2n+1}$ and $\sigma: S^{2n-2} \to S^{2n-2}$ is an exotic diffeomorphism.

Let $I \subset F$ be a path connecting the centers of the discs D_+^{2n-1} and D_+^{2n-1} of Σ_{σ}^{2n-1} and S^{2n-1} . Then a tubular neighborhood of I is $I \times D_+^{2n-1}$. We define \tilde{F} as the smooth manifold

$$\tilde{F} = \left[F - I \times D_+^{2n-1}\right] \cup \left[I \times D_+^{2n-1}\right]$$

where the union is taken over the diffeomorphism

$$\operatorname{id}_{I} \times \sigma^{-1} \colon I \times S^{2n-2} \to I \times S^{2n-2}.$$

Then $\partial \tilde{F} = \sum_{-\sigma}^{2n-} \bigcup_{\sigma} S^{2n-1}$ as a smooth manifold and we can define a PL homeomorphism

 $\hat{\eta} \colon \tilde{F} \to F$

where $\hat{\eta}$ is the identity on $F - I \times D_+^{2n-1}$ and is $\mathrm{id}_I \times (\mathrm{cone} \ \mathrm{extension} \ \mathrm{of} \ \sigma)$ on $I \times D_+^{2n-1}$.

Then we define the monodromy $\tilde{h}: \tilde{F} \to \tilde{F}$ as the composite map

$$ilde{h} = \hat{\eta}^{-1} \circ h \circ \hat{\eta}$$

where $h: F \to F$ is the monodromy map defining the bundle W. Since ∂W is a trivial bundle we know that h is the identity map on ∂F . Hence, \tilde{h} is the identity on $\partial \tilde{F}$ and the bundle \tilde{W} has the trivial boundary

$$\partial \tilde{W} = S^1 \times \Sigma^{2n-}_{-\sigma} \cup S^1 \times S^{2n-1}.$$

Since $\hat{\eta} \circ \tilde{h} = h \circ \hat{\eta}$ the PL homeomorphism $\hat{\eta}: \tilde{F} \to F$ induces a well-defined bundle homeomorphism

$$\eta\colon \tilde{W}^{2n+1}\to W^{2n+1}.$$

Restricted to the boundary η is a pair of bundle maps

$$\begin{split} & \mathrm{id}_{S^{1}} \times \alpha_{-\sigma}^{-1} \colon S^{1} \times \Sigma_{-\sigma}^{2n-} \to S^{1} \times S^{2n-1}, \\ & \mathrm{id}_{S^{1}} \times \alpha_{\sigma} \colon S^{1} \times S^{2n-1} \to S^{1} \times \Sigma_{\sigma}^{2n-1} \end{split}$$

where the PL homeomorphism $\alpha_{-\sigma}$ and α_{σ} are the identity on D_{-}^{2n-1} and the cone extension of σ^{-1} and σ respectively on D_{+}^{2n-1} .

We next embed \tilde{W} in $D^2 \times S^{2n-1}$ as a knot complement which will act as an inverse to W:

Recall the bundle isomorphism

(*)
$$\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$$

which defines $\partial v_{n+1}(CP^1)$. We define a PL bundle map

$$\hat{\rho}: S^1 \times \Sigma^{2n-1}_{-\sigma} \to S^1 \times \Sigma^{2n-1}_{-\sigma}$$

as the composite: $\hat{\rho} = (\mathrm{id}_{S^1} \times \alpha_{-\sigma}) \cdot \rho \cdot (\mathrm{id}_{S^1} \times \alpha_{-\sigma})^{-1}$. We consider the PL manifold

$$D^2 imes \Sigma^{2n-1}_{-\sigma} \cup_{\hat{o}} ilde{W}^{2n+1}$$

where the union is over the appropriate component of $\partial \tilde{W}$ and show:

PROPOSITION. The PL manifold $D^2 \times \sum_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1}$ is isomorphic to $D^2 \times S^{2n-1}$ by a PL homeomorphism Λ which restricted to the boundary $S^1 \times S^{2n-1}$ is an S^{2n-1} bundle isomorphism λ .

Proof. We recall from the definition of W^{2n+1} that $S^1 \times D^{2n} \cup W^{2n+1}$ is the knot complement of our original Brieskorn knot and so has the homology of S^1 . A simple exercise with the Mayer-Vietoris sequence implies then that the manifold $\tilde{W}^{2n+1} \cup S^1 \times D^{2n}$ likewise is a homology circle, and a second application of the sequence implies that the PL manifold.

$$P^{2n+1} = D^2 \times \sum_{-\sigma}^{2n-1} \cup_{\hat{\sigma}} \tilde{W} \cup S^1 \times D^{2n}$$

has the homology of S^{2n+1} . Moreover, P^{2n+1} is simply connected since $\hat{W} \cup S^1 \times D^{2n}$ fibers over S^1 with fiber $\tilde{F}^{2n} \cup D^{2n}$ which is (n-1)-connected. Hence $\pi_1(\tilde{W} \cup S^1 \times D^{2n}) = Z$ and by the Van Kampen theorem on the union

$$\left[D^2 \times \Sigma^{2n-1}_{-\sigma}\right] \cup_{S^1 \times \Sigma_{-\sigma}} \left[\tilde{W} \cup S^1 \times D^{2n}\right]$$

we have $\pi_1(P^{2n+1}) = 0$. By the Hurewicz and Whitehead theorems any simply-connected homology sphere is a homotopy sphere, and by the generalized Poincaré conjecture $(2n + 1 \ge 9)P^{2n+1}$ is a PL sphere.

The identification $D^2 \times \sum_{-\sigma}^{2n-1} \cup \tilde{W}S^1 \times D^{2n} \cong S^{2n+1}$ provides a PL embedding $S^1 \subset S^{2n+1}$ and exhibits $i(S^1 \times D^{2n}) \subset S^{2n+1}$ as a representative for the PL normal microbundle to this embedding. We apply a

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theorem due to Lashof and Rothenberg (Thm. 7.3 in [13]) to obtain a piecewise differentiable homeomorphism $g: S^{2n+1} \to S^{2n+1}$ so that $g \circ i$: $S^1 \times D^{2n} \to S^{2n+1}$ is the smooth vector bundle to the smooth embedding $g \circ i$: $S^1 \to S^{2n+1}$. By smoothly unknotting this circle and applying the smooth tubular neighborhood theorem we obtain a diffeomorphism h: $S^{2n+1} \to S^{2n+1}$ so that

$$\begin{array}{ccc} h \circ g \circ i \colon S^1 \times D^{2n} & \to & S^{2n+1} \\ & \bar{\lambda} \searrow & \uparrow j \\ & & S^1 \times D^{2n} \end{array}$$

commutes where j is the standard embedding and $\overline{\lambda}$ is a vector bundle isomorphism. Hence, the restriction map

$$h \circ g \mid : S^{2n+1} - i(S^1 \times D^{2n}) \to S^{2n+1} - j(S^1 \times D^{2n})$$
$$\parallel$$
$$D^2 \times S^{2n-1}$$

defines a piecewise differentiable homeomorphism

$$\Lambda: \left[D^2 \times \Sigma^{2n-}_{-\sigma} \cup_{\hat{\rho}} \hat{W} \right] \to D^2 \times S^{2n-1}$$

which restricts as $\lambda = \overline{\lambda}$ on the boundary. Finally, we observe that (cf. Cor. 10.13 in [19]) we may choose a smooth triangulation of $D^2 \times S^{2n-1}$ so that Λ is PL. Now, using the homeomorphisms Λ and η we define a PL homeomorphism:

(1)
$$\varphi_{\sigma}: \xi \to \partial \nu_{n+1}(CP^1)$$

where ξ is the S^{2n-1} bundle over $CP^1 = S^2$ defined by λ^{-1} :

$$\begin{split} \xi &= D_{-}^{2} \times S^{2n-1} \cup_{\lambda^{-1}} D_{+}^{2} \times S^{2n-1} \\ \xrightarrow{\Lambda^{-1} \cup \mathrm{id}} D^{2} \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1} \cup_{\mathrm{id}} D_{+}^{2} \times S^{2n-1} \\ \xrightarrow{(\mathrm{id} \times \alpha_{-\sigma}) \cup \eta \cup (\mathrm{id} \times \alpha_{\sigma})} D_{-}^{2} \times S^{2n-1} \cup_{\rho} W \cup D^{2} \times \Sigma_{-\sigma}^{2n-1} \\ &= D_{-}^{2} \times S^{2n-1} \cup_{\rho} D_{+}^{2} \times S^{2n-1} = \partial \nu_{n+1} (CP^{1}). \end{split}$$

From the next lemma to the effect that two non-isomorphic sphere bundles over S^2 cannot be PL homeomorphic it follows that the existence of the map φ_{σ} itself guarantees that ξ and $\partial \nu_{n+1}(CP^1)$ are the same bundle.

LEMMA. For $m \ge 3$ the unique non-trivial orthogonal S^m bundle over S^2 , ξ , is not PL homeomorphic to $S^2 \times S^m$.

Proof. Suppose $t: \xi \to S^2 \times S^m$ is a PL homeomorphism. Let E be the non-trivial D^{m+1} bundle over S^2 with $\partial E = \xi$ and define the PL manifold

$$M^{m+3} = E \cup, D^3 \times S^m$$

M is the union of simply connected spaces over a path connected intersection. Hence, $\pi_1(M) = \{1\}$. For $m \ge 3$ the homotopy exact sequence of the fibration $S^m \to \partial E \xrightarrow{p} S^2$ implies that $p_*: \pi_2(\partial E) \to \pi_2(S^2)$ is an isomorphism, and by the Whitehead theorem so is the inclusion $H_2(\partial E) \to H_2(E)$. Hence, in the Mayer-Vietoris sequence

$$\cdots \to H_j(S^2 \times S^m) \xrightarrow{\Psi_j} H_j(E) \oplus H_j(D^3 \times S^m) \to H_j(M)$$
$$\to H_{j-1}(S^2 \times S^m) \to \cdots$$

 ψ_j is an isomorphism for $j \le m + 1$. Trivally, $H_{m+2}(M) = 0$, and again we have an (m + 2)-connected (m + 3)-dimensional PL manifold which is consequently a PL sphere.

Then, $E \cup_t D^3 \times S^m \cong S^{m+3}$ defines the vector bundle *E* as a PL normal micro-bundle to the embedding of its zero section $S^2 \hookrightarrow S^{m+3}$. By Zeeman's PL unknotting theorem and the uniqueness [7] of stable PL normal microbundles, we see that *E* and $S^2 \times D^{m+1}$ must be micro-bundle isomorphic. Let $S^2 \xrightarrow{b} BO$ classify *E* as a vector bundle. Then $S^2 \xrightarrow{c} BO \xrightarrow{h} BO$ see that *b* is homotopically trivial. As *E* was assumed non-trivial as a vector bundle the PL homeomorphism *t* cannot exist.

Thus, we define

$$\varphi_{\sigma}: \partial \nu_{n+1}(CP^1) = \zeta \to \partial \nu_{n+1}(CP^1)$$
 from (1) as required.

Next we show that the φ_{σ} just constructed is indeed a self-knotting and that it will suffice for Theorem A.

Recalling from bundle theory that every S^N bundle over S^2 for $N \ge 2$ has a section, we show

PROPOSITION. Any orientation preserving PL homeomorphism $\varphi: \nu \to \nu$, ν an orthogonal S^N bundle over S^2 , which embeds a section $S^2 \stackrel{j}{\to} \nu$ homotopically to itself is homotopic to the identity.

Proof. A tubular neighborhood of the section $j(S^2)$ is a D^N bundle U in the same stable bundle class as ν . $\varphi(U)$ PL embeds this bundle in ν with an inherited smooth structure. By the main theorem of smoothing

theory ([8] or [13], Thm. 7.3) and the uniqueness of smoothings on S^2 we can piecewise differentially isotope this embedding to a smooth embedding of $U \rightarrow \nu$. We may easily make the isotopy ambient. Next, we smoothly unknot the core sphere of U and apply the smooth tubular neighborhood theorem. We have, therefore, P.D. isotoped φ so that restricted to U it is a D^N bundle isomorphism. Since $\pi_2(SO(N)) = 0$ we can isotope this bundle mapping to the identity through bundle isomorphisms on U all of which extend to ν as U is a sub-bundle. Thus, we have isotoped φ so that it is the identity on U. Now, $\nu - U \cong U$ as each fiber of U is a hemisphere of a fiber in ν . We isotope φ rel(U) so that it is the identity on the zero section of the bundle $\nu - U$. Finally, we homotope φ to the identity by collapsing the fibers of $\nu - U$ to the zero-section.

We observe that the φ_{σ} constructed above satisfies the hypothesis of this last proposition as follows: φ_{σ} is orientation preserving by construction. Also, as the original Brieskorn knot embedded a fiber S^{2n+1} homotopically to the usual embedding, we know that φ_{σ} does also. That is $(\varphi_{\sigma})_*[\partial \nu] = [\partial \nu]$ and $(\varphi_{\sigma})^*(e^{2n-1}) = e^{2n-1}$, where $e^{2n-1} \in H^{2n-1}(\partial \nu)$ is the class represented by inclusion of a fiber. By Poincaré Duality, then, $(\varphi_{\sigma})_*(e_2) = e_2$ for $e_2 \in H_2(\partial \nu)$ the class dual to e^{2n-1} . This implies by the Hurewicz Theorem that φ_{σ} induces the identity homomorphism on $\pi_2(\partial \nu)$, which is generated by the inclusion of a section.

The map φ_{σ} constructed in section C embeds a fiber S^{2n-1} onto the image of the Brieskorn knot. Hence, in the decomposition

$$\widetilde{CP}^{n+1} = \left[CP^{n+1} - \nu_{n+1}(CP^1)\right] \cup_{\varphi_{\sigma}} \left[\nu_{n+1}(CP^1)\right]$$

the identification is in the order:

$$\varphi_{\sigma}: \partial \big[CP^{n+1} - \nu \big] \to \partial \nu.$$

To show, therefore, that $\widetilde{CP}^{n+1} \leftrightarrow (0, \dots, 0, \sigma)$ we must extend φ_{σ}^{-1} to a homotopy equivalence $\overline{\varphi_{\sigma}^{-1}}$: $\nu \to \nu$ with transverse-inverse image of a fiber being the Milnor or Kervaire manifold M_0^{2n} . Note that any extension will be a homotopy equivalence as $\nu \simeq S^2$ and φ_{σ}^{-1} induces the identity on $\pi_2(\partial \nu) = \pi_2(\nu)$.

PROPOSITION. The PL homeomorphism φ_{σ}^{-1} : $\partial \nu_{N+1}(CP^1) \rightarrow \partial \nu_{n+1}(CP^1)$ constructed above extends to $\overline{\varphi}_{\sigma}^{-1}$: $\nu_{n+1}(CP^1) \rightarrow \nu_{n+1}(CP^1)$ with transverse-inverse image

$$\left(\overline{\varphi}_{\sigma}^{-1}\right)^{-1}(D^{2n})=M_0^{2n}$$

Proof. $(\varphi_{\sigma}^{-1})^{-1}(S^{2n-1}) = \varphi_{\sigma}(S^{2n-1}) = \sum_{\sigma}^{2n-1} \subset \partial \nu$ by the construction of φ_{σ} . Moreover, the restriction $\varphi_{\sigma}^{-1} | : D^2 \times \sum_{\sigma}^{2n-1} \to D_+^2 \times S^{2n-1}$ is a product map. Now, \sum_{σ}^{2n-1} bounds a fiber $F^{2n} \subset W^{2n+1}$ whose other boundary component is a fiber S^{2n-1} of $\partial \nu$. Let $D^{2n} \subset \nu$ be the fiber whose boundary is this same sphere. Then, $F^{2n} \cup D^{2n} = M_0^{2n}$ by the definition of F^{2n} . By pushing F^{2n} into ν along a vector field normal to $\partial \nu$ and smoothing the corner at S^{2n-1} between F^{2n} and D^{2n} we obtain a smooth embedding $M_0^{2n} \Leftrightarrow \nu$ extending

$$\partial M_0^{2n} = \Sigma_{\sigma}^{2n-1} \subset \partial \nu.$$

Moreover, this embedding will have trivial normal D^2 bundle as $H^1(M_0^{2n}, Z) = 0$. Hence, we can extend the product map

$$\varphi_{\sigma}^{-1}: D^2 \times \Sigma_{\sigma}^{2n-1} \to D^2_+ \times S^{2n-1}$$

to a bundle map $\hat{\varphi}_{\sigma}^{-1}$: $D^2 \times M_0^{2n} \to D_+^2 \times D^{2n}$ covering a degree one extension $M_0^{2n} \to D^{2n}$. Since $[\nu - D_+^2] \times D_-^2 \times D^{2n} = D^{2n-2}$ there are no cohomology obstructions to extending

$$\varphi_{\sigma}^{-1} \cup \hat{\varphi}_{\sigma}^{-1}$$
 to $\overline{\varphi_{\sigma}^{-1}}$: $\nu \to \nu$

with the required transverse-inverse image built in.

References

- [1] Y. S. Akbulut, Algebraic Equations for a Class of P. L. Manifolds, Ph.D. Thesis, Berkeley, 1974.
- Y. S. Akbulut, and H. C. King, *Real algebraic variety structures on PL manifolds*, Bull. Amer. Math. Soc., 83 (1977), 281–282.
- [3] W. Browder, *Manifolds with* $\pi_1 = Z$, Bull. Amer. Math. Soc., **72** (1966), 238–244.
- [4] _____, Surgery on Simply-Connected Manifolds, Springer-Verlag, New York, 1972.
- [5] G. Brumfiel, I. Madsen, and R. J. Milgram, *PL characteristic classes and cobordism*, Annals of Math., **97** (1973), 82–159.
- [6] A. Haefliger, Plongemonts differentiables de varietés dans varietés, Comm. Math. Helv., 36 (1961), 47–82.
- [7] A. Haefliger and C. T. C. Wall, *Piecewise linear bundles in the stable range*, Topology, 4 (1965), 209-214.
- [8] M. W. Hirsch and B. Mazur, Smoothings of Piecewise Linear Manifolds, Princeton, 1974.
- [9] F. Hirzebruch, Singularities and Exotic Spheres, Seminar Bourbaki No. 314, 1966.
- [10] M. Kato, A concordance classification of PL homeomorphisms of $S^p \times S^q$, Topology, **8** (1969), 371–383.
- [11] M. Kervaire and J. Milnor, Groups of homotopy spheres I, Annals of Math., 77 (1963), 504-537.
- [12] S. Lang, Introduction to Differentiable Manifolds, Interscience, New York, 1962.
- [13] R. Lashof and M. Rothenberg, *Microbundles and Smoothing*, Topology, **3** (1965), 357–388.

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- [14] I. Madsen and R. J. Milgram, The oriented topological and PL cobordism rings, Bull. Amer. Math. Soc., 80 (1974), 855–860.
- [15] _____, The Classifying Spaces for Surgery and Cobordism of Manifolds, Princeton University Press, Princton New Jersey, 1979.
- [16] D. Meadows, Some groups of PL self-knotting, Houston J. Math., (to appear).
- [17] J. Milnor, Singular Points of Complex Hypersurfaces, Princeton University Press, Princeton, New Jersey, 1968.
- [18] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, New Jersey, 1974.
- [19] J. Munkres, Elementary Differential Topology, Princeton, 1966.
- [20] C. P. Rourke and B. J. Sanderson, Block bundles I, II, and III, Ann. of Math., 87 (1968), 1–28, 256–278, 431–483.
- [21] _____, Introduction to Piecewise-Linear Topology, Springer-Verlag, New York, 1972.
- [22] S. Smale, *Differentiable and combinatorial structures on manifolds*, Ann. of Math., 74 (1961), 498–502.
- [23] D. Sullivan, *Triangulating and smoothing homotopy equivalences*, mimeographed notes, Princeton University.
- [24] C. T. C. Wall, Surgery on Compact Manifolds, Academic Press, New York, 1970.
- [25] G. W. Whitehead, Homotopy Theory, M.I.T. Press, Cambridge, Massachusetts, 1966.
- [26] R. E. Williamson Jr., Cobordism of combinatorial manifolds, Ann. of Math., 83 (1966), 1–133.
- [27] H. Winkelnkemper, Manifolds as open books, Bull. Amer. Math. Soc., 79 (1973), 45–51.
- [28] E.-C. Zeeman, Seminar on combinatorial topology, I.H.E.S., 1963.

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