# EXPLICIT PL SELF-KNOTTINGS AND THE STRUCTURE OF PL HOMOTOPY COMPLEX PROJECTIVE SPACES

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**We show that certain piecewise-linear homotopy complex projective spaces may be described as a union of smooth manifolds glued along their common boundaries. These boundaries are sphere bundles and the glueing homeomorphisms are piecewise-linear self-knottings on these bundles. Furthermore, we describe these self-knottings very explicitly and obtain information on the groups of concordance classes of such maps.**

A piecewise linear homotopy complex projective space $\widetilde{CP}^n$ is a compact PL manifold $M^{2n}$ homotopy equivalent to $CP^n$. In [22] Sullivan gave a complete enumeration of the set of PL isomorphism classes of these manifolds as a consequence of his Characteristic Variety theorem and his analysis of the homotopy type of $G/PL$. In [15] Madsen and Milgram have shown that these manifolds, the index 8 Milnor manifolds, and the differentiable generators of the oriented smooth bordism ring provide a complete generating set for the torsion-free part of the oriented PL bordism ring. Hence a study of the geometric structure of these exotic projective spaces $\widetilde{CP}^n$, especially with regard to their smooth singularities, may extend our understanding of the PL bordism ring. This paper begins such a study in which we obtain a geometric decomposition of $\widetilde{CP}^n$, into (for many cases) a union of smooth manifolds glued together by PL self-knottings on certain sphere bundles. We also obtain information on groups of concordance classes of PL self-knottings from these decompositions and a number of fairly explicitly constructed examples of self-knottings. I would like to thank by thesis advisor R. J. Milgram for many helpful discussions.

    I.   Sullivan's classification of PL homotopy $\widetilde{CP}^n$ proceeds as follows: Given a homotopy equivalence $h\colon \widetilde{CP}^n \to CP^n$ make $h$ transverse regular to $CP^j \subset \widetilde{CP}^n$, the standard inclusion. The restriction of $h$ to the transverse inverse image $h^{-1}(CP^j) = N^{2j} \subset \widetilde{CP}^n$ is a degree one normal map

with simply connected surgery obstruction

$$\sigma_j \in P_{2j} = \begin{cases} Z, & j \text{ even} \\ Z/2Z, & j \text{ odd} \end{cases}.$$

For $j = 2,\ldots,n-1$ these obstruction invariants yield a complete enumeration — i.e. the set of PL isomorphism classes of $\widetilde{CP}^n$ is set-isomorphic to the product $Z \times Z_2 \times Z \times \cdots \times P_{2(n-1)}$ with $n-2$ factors.

We will use the following notation to specify elements with this classification:

$$(1) \qquad\qquad \widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$$

will denote the PL homotopy $\widetilde{CP}^n$ with invariants $\sigma_j \in P_{2j}$ in Sullivan's enumeration.

We recall that a PL homeomorphism $f: M \to M$ is a "self-knotting" and $M$ is said to be "self knotted" if $f$ is homotopic but not PL isotopic to the identity. Also, PL homeomorphisms $f, g: M \to M$ are "PL concordant" (pseudo-isotopic) if we have a PL homeomorphism $F: M \times I \to M \times I$ with $F(m, 0) = (f(m), 0)$ and $F(m, 1) = (g(m), 1)$ for $m \in M$. We then define:

(2) $SK(M) = $ "the group (under composition of maps) of PL concordance classes of PL self-knottings of $M$."

Unless otherwise noted "$CP^j \subset CP^n$" means the standard embedding of $CP^j$ onto the first $(j+1)$ homogeneous coordinates of $CP^n$ or a smooth ambient isotope of this embedding. In this context we define:

(3) $\nu_N(CP^j) = $ "the smooth tubular disc bundle neighborhood of the embedding $CP^j \subset CP^N$."

Our results are as follows:

THEOREM A. *For $n \geq 4$ and $\sigma_2 \equiv 0$ (2) every $\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$ is PL homeomorphic to the identification space*

$$\left[\widehat{CP}^n - \nu_n(CP^1)\right] \cup_{\varphi_{\sigma_{n-1}}} \left[\nu_n(CP^1)\right]$$

*where $\widehat{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-2}, 0)$ and the identification is over a PL homeomorphism*

$$\varphi_{\sigma_{n-1}}: \partial\nu_n(CP^1) \to \partial\nu_n(CP^1).$$

We prove Theorem A in Part II by a careful description of Sullivan's classification and an easy $h$-cobordism argument.

An immediate consequence of Theorem A is the decomposition of $\widetilde{CP}^{n+1} \leftrightarrow (0,\ldots,0,\sigma_n)$ into

$$\widetilde{CP}^{n+1} = [CP^{n+1} - \nu(CP^1)] \cup_{\varphi_0} [\nu(CP^1)].$$

THEOREM B. *For every* $n \geq 4$ *and non-zero* $\tau \in P_{2n}$ *there is a PL self-knotting*

$$\varphi_\tau : \partial\nu_{n+1}(CP^1) \to \partial\nu_{n+1}(CP^1)$$

*which will suffice for the glueing homeomorphism in Theorem A.*

We establish this theorem by an explicit construction of $\varphi_\tau$ in Part III.

**II.** Here we prove Theorem A by beginning with a construction which shows how to obtain $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2,\ldots,\sigma_{n-1},\sigma_n)$ from $\widetilde{CP}^n \leftrightarrow (\sigma_2,\ldots,\sigma_{n-1})$ for $n \geq 4$:

Let $h: \widetilde{CP}^n \to CP^n$ be a homotopy equivalence, and let $M^{2n}$ be the compact $(n-1)$-connected Milnor or Kervaire manifold of Index $8\sigma_n$ or Kervaire-Arf invariant $\sigma_n$ as the case may be [4]. Let $r: M^{2n} \to S^{2n}$ be a degree one map. Then $h\#r: \widetilde{CP}^n\#M^{2n} \to CP^n\#S^{2n} = CP^n$ is a degree one normal map with 1-connected surgery obstruction $\sigma_n$. We define $\hat{H}$ as the $D^2$ bundle over $\widetilde{CP}^n\#M^{2n}$ induced by $h\#r$ from $H$, the disc bundle associated to the complex line bundle over $CP^n$. Let $\hat{h}: \hat{H} \to H$ be the bundle mapping. We note that the map $h\#r$ is $(n-1)$-connected with homological kernel $K_n = \pi_n(M_0^{2n})$ where $M_0^{2n} = M^{2n} - D^{2n}$. The bundle $\hat{H}$ is trivial over $M_0^{2n}$ since $M_0^{2n} = (h\#r)^{-1}(\text{point})$. In $M_0^{2n} \times D^2$ we can represent $\pi_n(M_0^{2n})$ by disjointly embedded spheres $S^n \hookrightarrow M_0^{2n} \times S^1$ with trivial normal bundles. This follows by general position and the fact that the normal bundles of the generating spheres $S^n \subset M_0^{2n}$ are the stably trivial tangent disc bundles $\tau(S^n)$. We now attach a solid handle $D^{n+1} \times D^{n+1}$ along $S^n \times D^{n+1} \subset M_0^{2n} \times S^1$ for each generator of $\pi_n(M_0^{2n})$ and extend the map $\hat{h}$ across these bundles. This is possible since the embedded spheres are in the homotopy kernel of $\hat{h}$. Call the resulting PL manifold $\tilde{H}$ and the extended map $\hat{h}: \hat{H} \to H$. In the process of extending $\hat{h}$ across the handles, we may guarantee that $\tilde{h}$ is a map of pairs $(\tilde{H}, \partial) \to (H, \partial)$. We observe, then, the:

PROPOSITION. $\tilde{h}: (\tilde{H}, \partial) \to (H, \partial)$ *is a homotopy equivalence of pairs.*

This follows directly from the construction as $\tilde{H}$ deformation retracts onto $\widetilde{CP}^n \# M^{2n} \cup \{e_\alpha^n\}$ where the $n$-cells $e_\alpha^n$ are attached so as to kill the entire homology kernel of $(h\#r)$. Hence $\tilde{h}: \tilde{H} \to H$ is a homology isomorphism, and as $\tilde{H}$ is 1-connected we have by Whitehead's theorem that it is a homotopy equivalence. The restriction of $\tilde{h}$ to the boundary is likewise a homology isomorphism as the boundaries, $D^{n+1} \times S_\alpha^n$, of the solid handles are precisely the surgeries needed to cobord $\hat{h}: \partial \hat{H} \to \partial H$ to a homotopy equivalence.

In particular as $n \geq 3$ we note that the boundary manifold, $\partial \tilde{H}$, is a PL $(2n + 1)$-sphere by the Poincaré conjecture. Thus, we attach $D^{2n+2}$ to $\tilde{H}$ as the PL cone on $\partial \tilde{H}$ and define:

$$\widetilde{CP}^{n+1} = \tilde{H} \cup c(\partial \tilde{H}) \quad \text{and} \quad h: \widetilde{CP}^{n+1} \to CP^{n+1} = H \cup c(\partial H)$$

by radial extension of $\tilde{h}$ into $c(\partial \tilde{H})$.

Observe that $h$ has 'built-in'' transverse inverse image $\widetilde{CP}^n \# M^{2n} = h^{-1}(CP^n)$ with surgery obstruction $\sigma_n$. Hence, this $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ is the space we require.

Now, given $\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1})$ let us consider a bit more closely the suspension and generalized suspension constructions described above. First, assume the homotopy equivalence

$$h: \widetilde{CP}^n \to CP^n$$

is the identity map on a disc $D^{2n} \subset \widetilde{CP}^n$. Let $\widetilde{CP}_0^n = \widetilde{CP}^n - D^{2n}$, $M_0^{2n} = M^{2n} - D^{2n}$ and observe that $\widetilde{CP}^n \# M^{2n} = \widetilde{CP}_0^n \cup_\partial M_0^{2n}$. Now, let $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0)$ be the suspension[1] of $\widetilde{CP}^n$ with homotopy equivalence

$$\tilde{h}: \widetilde{CP}^{n+1} \to CP^{n+1}$$

and $\widehat{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ be the general suspension of $\widetilde{CP}^n$ with homotopy equivalence

$$\hat{h}: \widehat{CP}^{n+1} \to CP^{n+1}.$$

Let $D^{2n} \subset CP^n$ be the image $h(D^{2n})$ and let $CP^1 = S^2 \subset CP^{n+1}$ be represented as $D_*^2 \cup c(\partial D_*^2)$ in $CP^{n+1} = H \cup c(\partial H)$ with $D_*^2$ the fiber in $H$ over the center of the disc $D^{2n}$. Then $\nu_{n+1}(CP^1) \subset CP^{n+1}$ may be represented as the set $D_*^2 \times D^{2n} \cup c(\partial H)$, a $D^{2n}$ bundle over the sphere $S^2 = D_*^2 \cup c(\partial D_*^2)$.

Now let $\tilde{V} = \tilde{h}^{-1}(\nu_{n+1}(CP^1))$ and $\hat{V} = \hat{h}^{-1}(\nu_{n+1}(CP^1))$ in $\widetilde{CP}^{n+1}$ and $\widehat{CP}^{n+1}$ respectively. We observe directly from the constructions that

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[1] We say $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, 0)$ in the "suspension" of $\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$ as it is precisely the Thom complex of the line bundle induced over $\widetilde{CP}^n$.

$\widetilde{CP}^{n+1} - \tilde{V}$ and $\widehat{CP}^{n+1} - \hat{V}$ are precisely the same spaces. To prove Theorem A we must show that $\tilde{V}$ and $\hat{V}$ are PL homeomorphic to $\nu_{n+1}(CP^1)$.

LEMMA 1. $\tilde{V} \cong \nu_{n+1}(CP^1)$ *if* $\sigma_2$ *is even.*

We observe this from PL block bundle theory as follows: by construction $\tilde{V}$ is the union of two discs $D_*^2 \times D^{2n}$ and $c(\partial \tilde{H}) = D^{2n+2}$ along $S_*^1 \times D^{2n}$. Hence $\tilde{V}$ is trivially a block bundle regular neighborhood of $CP^1 = D_*^2 \cup c(\partial D_*^2)$. Assume the obstruction $\sigma_2$ is even. Then as noted by Sullivan ([23] p. 43) the splitting obstruction of the homotopy equivalence

$$\tilde{h} \colon \widetilde{CP}^{n+1} \to CP^{n+1}$$

along $CP^1$ vanishes as it is the mod 2 reduction of $\sigma_2$. Hence, by a homotopic deformation we may conclude that the transverse inverse image of $CP^1$ by $\tilde{h}$ is $CP^1 \subset \widetilde{CP}^{n+1}$. Moreover, as any two homotopic PL embeddings of $CP^1 \subset \widetilde{CP}^{n+1}$ are ambiently PL isotopic (for $n \geq 2$ by Cor. 5.9 p. 65 [21]), we see by appeal to the uniqueness of normal block bundles (regular neighborhoods) [20] that $\tilde{V}$ is block bundle isomorphic to the bundle induced from $\nu_{n+1}(CP^1)$ by $\tilde{h}$. Conversely, the same argument on the homotopy inverse of $\tilde{h}$ implies $\nu_{n+1}(CP^1)$ is block bundle induced from $\tilde{V}$. As we are in the stable block and vector bundle range and $\pi_2 B_{\mathrm{PL}} = \pi_2 B_0 = Z_2$ we can conclude that $\tilde{C}$ and $\nu(CP^1)$ are block bundle isomorphic; hence PL homeomorphic.

LEMMA 2. $\hat{V} \simeq S^2$ (*homotopy equivalent*).

*Proof.* By construction $\hat{V} = D^2 \times M_0^{2n} \cup X \cup c(\partial H)$ where $X$ represents the solid handles we attached along $S^1 \times M_0^{2n}$ to kill the homology kernel of $\hat{h}$. The manifold $D^2 \times M_0^{2n} \cup X$ is simply-connected with simply connected boundary and the homology of a point; hence by Smale's theorem (Thm. 1.1 [22]) it is a PL disc $D^{2n+2}$. Thus, $\hat{V} = D^{2n+2} \cup_W D^{2n+2}$ where $W$ is the complement of the embedding

$$D^2 \times S^{2n-1} \subset S^{2n+1} = \partial D^{2n+2}$$

and $S^{2n-1} = \partial M_0^{2n}$. By the Mayer-Vietoris sequence we know that $W$ is a homology circle. Then, by a second application of the Mayer-Vietoris sequence to the union $D^{2n+2} \cup_W D^{2n+2}$ we see that $\hat{V}$ is a homology $S^2$. Finally, by the Van Kampen theorem $\hat{V}$ is 1-connected and we apply the Whitehead theorem for CW complexes.

LEMMA 3. $\hat{V} \cong \nu_{n+1}(CP^1)$.

*Proof.* $\partial \hat{V} = \partial [CP^{n+1} - \hat{V}] = \partial [CP^{n+1} - \tilde{V}] = \partial \tilde{V} \cong \partial \nu_{n+1}(CP^1)$ by Lemma 1. Let $S^2 \subset \hat{V}$ be a homotopy equivalence and a PL embedding via Whitney's embedding theorem. Then $S^2 \subset \hat{V} \subset \widehat{CP}^{n+1}$ is homotopic to the standard embedding $CP^1 \subset \widehat{CP}^{n+1}$, and as before, the PL block bundle neighborhoods of these two embeddings must be isomorphic. Let $\nu \subset \hat{V}$ be this block bundle. We note that

$$\partial \nu = \partial \nu_{n+1}(CP^1) \cong \partial \tilde{V} = \partial \hat{V}$$

by the previous lemmas. Hence, if

$$\hat{V} - \nu = Y$$

we have $\partial Y = \partial \hat{V} \cup. \partial \nu$, two copies of the same manifold.

We consider the Mayer-Vietoris sequence for the union $\hat{V} = Y \cup \nu$ over $\partial \nu = Y \cap \nu$:

$$\cdots \to H_1(\partial \nu) \overset{i_{1*} - i_{2*}}{\to} H_1(\nu) \oplus H_q(Y) \overset{j_{1*} - j_{2*}}{\to} H_1(\hat{V}) \to \cdots$$

where

$$i_1 : \partial \nu \hookrightarrow \nu, \quad j_1 : \nu \hookrightarrow \hat{V},$$
$$i_2 : \partial \nu \hookrightarrow Y, \quad j_2 : Y \hookrightarrow \hat{V}.$$

Since $\nu$ and $V$ are homotopy 2-spheres and $j_1$ is a homotopy equivalence, we see that for $q \neq 2$, $i_{2*} : H_q(\partial \nu) \to H_q(Y)$ must be an isomorphism. When $q = 2$ the sequence becomes:

$$Z \overset{1-i_{2*}}{\to} Z \oplus A \overset{1+j_{2*}}{\to} Z, \quad A = H_2(Y)$$

from which we obtain $i_{2*}$ are isomorphisms $Z \overset{i_{2*}}{\to} A \overset{j_{2*}}{\to} Z$. Thus, $i_2 : \partial \nu \subset Y$ is a homology isomorphism, and in fact, a homotopy equivalence since $\hat{V} = Y \cup \nu$ and $\hat{V}$, $\nu$, $\partial \nu$ are all 1-connected so that by Van Kampen's theorem $Y$ is 1-connected.

We show next that $\partial \hat{V} \overset{i}{\subset} Y$ is a homology isomorphism so that $Y$ is a $h$-cobordism from $\partial \nu$ to $\partial \hat{V}$—i.e. $Y \overset{PL}{\cong} \partial \nu \times I$ and $\hat{V} = Y \cup \nu \cong \nu \cong \tilde{V}\nu_{n+1}(CP^1)$ as required.

We know already that $\partial \hat{V} \simeq Y$ as $\partial \hat{V} \cong \partial v \simeq Y$. Moreover, $\partial v \cong \partial v_{n+1}(CP^1)$ is an $S^{2n-1}$ bundle over $S^2$. Hence, by the Serre Spectral Sequence we have

$$H_p(Y) = H_p(\partial \hat{V}) = \begin{cases} Z & \text{if } p = 0, 2, 2n-1, 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the exact sequence of the pair $(\hat{V}, \partial \hat{V})$ is:

$$0 = H_3(\hat{V}, \partial \hat{V}) \to H_2(\partial \hat{V}) \to H_2(\hat{V}) \to H_1(\hat{V}, \partial \hat{V}) = 0$$
$$\phantom{0 = H_3(\hat{V}, \partial \hat{V}) \to} \| \phantom{H_2(\partial \hat{V}) \to} \|$$
$$\phantom{0 = H_3(\hat{V}, \partial \hat{V}) \to} Z \phantom{H_2(\partial \hat{V}) \to} Z$$

where the first and last groups are 0 by Poincaré Duality. Thus, the inclusion $\partial \hat{V} \subset Y \subset \hat{V}$ is a homology isomorphism through $p = 2$.

Now, consider the composition $f \colon \partial \hat{V} \overset{i}{\hookrightarrow} Y \to \partial \hat{V}$ where the second map is a homotopy equivalence. Then $f_* \colon H_p(\partial \hat{V}) \to H_p(\partial \hat{V})$ is an isomorphism for $p \leq 2$, and by Poincaré Duality so is $f^* \colon H^1(\partial \hat{V}) \to H^1(\partial \hat{V})$ for $q = 2n-1, 2n, 2n+1$. By the Universal Coefficient Theorem $f_*$ is an isomorphism for $p = 2n-1, 2n, 2n+1$ and so for all $p$. Thus, $f$ is a homotopy equivalence, and so is $i$.

Theorem A is now an immediate consequence of the last lemma as we have:

$$\widehat{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) = [CP^{n+1} - \tilde{V}] \cup \hat{V},$$

$$\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0) = [CP^{n+1} - v_{n+1}(CP^1)] \cup_{\alpha_{\sigma_n}} v_{n+1}(CP^1)$$

where we have identified $\tilde{V}$ with $v_{n+1}(CP^1)$ by Lemma 1, and the PL homeomorphism

$$\varphi_{\sigma_n} \colon \partial[\widetilde{CP}^{n+1} - v(CP^1)] \to \partial v(CP^1)$$

comes from the restriction to the boundary of the PL homeomorphism $\hat{V} \to v_{n+1}(CP^1)$ of Lemma 3.

**III. Construction of the self-knotting $\varphi_\sigma$:** Here we construct for $n \geq 4$ a PL self-knotting

$$\varphi_\sigma \colon \partial v_{n+1}(CP^1) \to \partial v_{n+1}(CP^1)$$

with the property that it extends to a homotopy equivalence

$$\bar{\varphi}_\sigma \colon v_{n+1}(CP^1) \to v_{n+1}(CP^1)$$

which has a transverse-inverse image

$$M_0^{2n} = \overline{\varphi}_\sigma^{-1}(D^{2n})$$

on a fiber $D^{2n}$. Clearly such a $\varphi_\sigma$ will suffice for the map in Theorem A.

We begin the construction by defining

$$\Sigma_\sigma^{2n-1} \subset S^{2n+1}$$

to be the smooth Brieskorn knot represented as the link of the singularity on the hypersurface in $C^{n+1}$ defined by

$$p(Z) = \begin{cases} Z_0^{6\sigma-1} + Z_1^3 + Z_2^2 + \cdots + Z_n^2, & n \text{ even}, \\ Z_0^3 + Z_1^2 + \cdots + Z_n^2, & n \text{ odd}. \end{cases}$$

It is well-known that $S^{2n+1} - \Sigma_\sigma^{2n-1}$ is a smooth fiber bundle over the circle with fiber $M_0^{2n}$, the smooth Milnor or Kervaire manifold with surgery invariant $\sigma$.

Now, let $S^1 \subset S^{2n+1}$ be a fiber on the boundary of the smooth tubular neighborhood $D^2 \times \Sigma_\sigma^{2n-1}$ of the knot (a trivial bundle as $\pi_{2n-1}(SO(2)) = 0$ for $n > 1$). Since $n > 1$ this circle $S^1$ is smoothly un-knotted in $S^{2n+1}$ so that the complement of a small tube $S^1 \times D^{2n}$ about it is diffeomorphic to $D^2 \times S^{2n-1}$. Hence the knot $\Sigma_\sigma^{2n-1}$ lies in this complement with a trivial normal bundle and we can therefore define:

$$\beta: D^2 \times \Sigma_\sigma^{2n-1} \hookrightarrow D^2 \times S^{2n-1}$$

as this embedding. Let $W^{2n+1}$ be the complement of this smooth embedding. Then we observe:

(a) $\partial W = S^1 \times S^{2n-1} \cup S^1 \times \Sigma_\sigma^{2n-1}$.

(b) $W$ is a smooth fiber bundle over the circle $S^1$ with fiber $F^{2n} = M_0^{2n} - D^2$ and $\partial F = S^{2n-1} \cup \Sigma_\sigma^{2n-1}$.

(c) the bundle projection is trivial on $\partial W \to S^1$.

Now, using the smooth embedding $\beta$ we define a piecewise-linear embedding

$$\gamma_\sigma: D^2 \times S^{2n-1} \hookrightarrow D^2 \times S^{2n-1}$$

as the composite map

$$D^2 \times S^{2n-1} \xrightarrow{\text{id} \times \alpha_\sigma} D^2 \times \Sigma_\sigma^{2n-1} \xrightarrow{\beta} D^2 \times S^{2n-1}$$

where $\alpha_\sigma: S^{2n-1} \to \Sigma_\sigma^{2n-1}$ is a specific PL homeomorphism.

We now describe the normal bundle $\nu_{n+1}(CP^1)$ in $CP^{n+1}$ as:

$$\nu_{n+1}(CP^1) = D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1}$$

(∗) where $\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$ is a smooth bundle automorphism representing an element in $\pi_1(SO(2n)) = Z/2Z$ ($n > 1$). [We note in fact that $\gamma_{n+1}(CP^1)$ is trivial for $n$ even and non-trivial for $n$ odd as it is the Whitney sum of $n$ copies of the canonical line bundle over $CP^1 = S^2$.]

In the above description we are expressing $CP^1$ as $S^2 = D^2_- \cup D^2_+$. Using this representation we will define the self-knotting $\varphi_\sigma$ by showing that the PL embedding

$$\gamma_\sigma: D^2_+ \times S^{2n-1} \hookrightarrow D^2_+ \times S^{2n-1}$$

may be extended to a PL homeomorphism on all of $V_{n+1}(CP^1)$. We will show this using the very agreeable bundle structure on the complement $W$ of the embedding $\gamma_\sigma$.

The map

$$\varphi_\sigma: D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1} \to D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1}$$

will in fact be defined as the union of three maps —

(1) $$\gamma_\sigma: D^2_+ \times S^{2n-1} \hookrightarrow D^2_+ \times S^{2n-1},$$

(2) $$\eta: \tilde{W}^{2n+1} \to W^{2n+1},$$

(3) $$\text{id} \times \mu: D^2 \times \Sigma^{2n-1}_{-\sigma} \to D^2_- \times S^{2n-1}$$

where $\eta$ is a bundle homeomorphism of bundles over $S^1$ and $\mu: \Sigma^{2n-1}_{-\sigma} \to S^{2n-1}$ is a PL homeomorphism and

$$D^2 \times \Sigma^{2n-1}_{-\sigma} \cup \tilde{W}^{2n+1} = D^2_- \times S^{2n+1}.$$

Essentially what we are producing in this construction is a map with the symmetric property that $\varphi_\sigma$ embeds a fiber (the core of $D^2_+ \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma^{2n-1}_{-\sigma} \subset D^2_- \times S^{2n-1}$ while $\varphi_\sigma^{-1}$ embeds a fiber (the core of $D^2_- \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma^{2n-1}_\sigma \subset D^2_- \times S^{2n-1}$.

The construction will be completed by (a) defining the bundle $\tilde{W}$ and the bundle map $\eta$ in (2), (b) showing that $D^2 \times \Sigma^{2n-1}_{-\sigma} \cup \tilde{W}$ is in fact $D^2 \times S^{2n-1}$ by a PL homeomorphism which is the identity on the boundary, (c) showing that the maps (1), (2), (3) agree on boundaries after taking the defining automorphism $\rho$ into account, and finally by (d) showing that $\varphi_\sigma$ is homotopic to the identity.

We define the bundle $\tilde{W}$ over $S^1$ by defining its fiber $\tilde{F}$ and its monodromy map $\tilde{h}: \tilde{F} \to \tilde{F}$.

Recall that the $2n$-manifold $F$ (fiber of $W$) is $(n-1)$ connected and that $\partial F = S^{2n-1} \cup \Sigma_\sigma^{2n-}$ where the smooth exotic sphere is defined as $\Sigma_\sigma^{2n-1} = D_-^{2n-1} \cup_\sigma D_+^{2n+1}$ and $\sigma: S^{2n-2} \to S^{2n-2}$ is an exotic diffeomorphism.

Let $I \subset F$ be a path connecting the centers of the discs $D_+^{2n-1}$ and $D_+^{2n-1}$ of $\Sigma_\sigma^{2n-1}$ and $S^{2n-1}$. Then a tubular neighborhood of $I$ is $I \times D_+^{2n-1}$. We define $\tilde{F}$ as the smooth manifold

$$\tilde{F} = \left[ F - I \times D_+^{2n-1} \right] \cup \left[ I \times D_+^{2n-1} \right]$$

where the union is taken over the diffeomorphism

$$\mathrm{id}_I \times \sigma^{-1}: I \times S^{2n-2} \to I \times S^{2n-2}.$$

Then $\partial \tilde{F} = \Sigma_{-\sigma}^{2n-} \cup S^{2n-1}$ as a smooth manifold and we can define a PL homeomorphism

$$\hat{\eta}: \tilde{F} \to F$$

where $\hat{\eta}$ is the identity on $F - I \times D_+^{2n-1}$ and is $\mathrm{id}_I \times$ (cone extension of $\sigma$) on $I \times D_+^{2n-1}$.

Then we define the monodromy $\tilde{h}: \tilde{F} \to \tilde{F}$ as the composite map

$$\tilde{h} = \hat{\eta}^{-1} \circ h \circ \hat{\eta}$$

where $h: F \to F$ is the monodromy map defining the bundle $W$. Since $\partial W$ is a trivial bundle we know that $h$ is the identiy map on $\partial F$. Hence, $\tilde{h}$ is the identity on $\partial \tilde{F}$ and the bundle $\tilde{W}$ has the trivial boundary

$$\partial \tilde{W} = S^1 \times \Sigma_{-\sigma}^{2n-} \cup S^1 \times S^{2n-1}.$$

Since $\hat{\eta} \circ \tilde{h} = h \circ \hat{\eta}$ the PL homeomorphism $\hat{\eta}: \tilde{F} \to F$ induces a well-defined bundle homeomorphism

$$\eta: \tilde{W}^{2n+1} \to W^{2n+1}.$$

Restricted to the boundary $\eta$ is a pair of bundle maps

$$\mathrm{id}_{S^1} \times \alpha_{-\sigma}^{-1}: S^1 \times \Sigma_{-\sigma}^{2n-} \to S^1 \times S^{2n-1},$$

$$\mathrm{id}_{S^1} \times \alpha_\sigma: S^1 \times S^{2n-1} \to S^1 \times \Sigma_\sigma^{2n-1}$$

where the PL homeomorphism $\alpha_{-\sigma}$ and $\alpha_\sigma$ are the identity on $D_-^{2n-1}$ and the cone extension of $\sigma^{-1}$ and $\sigma$ respectively on $D_+^{2n-1}$.

We next embed $\tilde{W}$ in $D^2 \times S^{2n-1}$ as a knot complement which will act as an inverse to $W$:

Recall the bundle isomorphism

$(*)$ $$\rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$$

which defines $\partial \nu_{n+1}(CP^1)$. We define a PL bundle map

$$\hat{\rho}: S^1 \times \Sigma_{-\sigma}^{2n-1} \to S^1 \times \Sigma_{-\sigma}^{2n-1}$$

as the composite: $\hat{\rho} = (\mathrm{id}_{S^1} \times \alpha_{-\sigma}) \cdot \rho \cdot (\mathrm{id}_{S^1} \times \alpha_{-\sigma})^{-1}$. We consider the PL manifold

$$D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1}$$

where the union is over the appropriate component of $\partial \tilde{W}$ and show:

PROPOSITION. *The PL manifold $D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1}$ is isomorphic to $D^2 \times S^{2n-1}$ by a PL homeomorphism $\Lambda$ which restricted to the boundary $S^1 \times S^{2n-1}$ is an $S^{2n-1}$ bundle isomorphism $\lambda$.*

*Proof.* We recall from the definition of $W^{2n+1}$ that $S^1 \times D^{2n} \cup W^{2n+1}$ is the knot complement of our original Brieskorn knot and so has the homology of $S^1$. A simple exercise with the Mayer-Vietoris sequence implies then that the manifold $\tilde{W}^{2n+1} \cup S^1 \times D^{2n}$ likewise is a homology circle, and a second application of the sequence implies that the PL manifold.

$$P^{2n+1} = D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W} \cup S^1 \times D^{2n}$$

has the homology of $S^{2n+1}$. Moreover, $P^{2n+1}$ is simply connected since $\hat{W} \cup S^1 \times D^{2n}$ fibers over $S^1$ with fiber $\tilde{F}^{2n} \cup D^{2n}$ which is $(n-1)$-connected. Hence $\pi_1(\tilde{W} \cup S^1 \times D^{2n}) = Z$ and by the Van Kampen theorem on the union

$$\left[ D^2 \times \Sigma_{-\sigma}^{2n-1} \right] \cup_{S^1 \times \Sigma_{-\sigma}} \left[ \tilde{W} \cup S^1 \times D^{2n} \right]$$

we have $\pi_1(P^{2n+1}) = 0$. By the Hurewicz and Whitehead theorems any simply-connected homology sphere is a homotopy sphere, and by the generalized Poincaré conjecture $(2n + 1 \geq 9) P^{2n+1}$ is a PL sphere.

The identification $D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\tilde{W}} S^1 \times D^{2n} \cong S^{2n+1}$ provides a PL embedding $S^1 \overset{i}{\subset} S^{2n+1}$ and exhibits $i(S^1 \times D^{2n}) \subset S^{2n+1}$ as a representative for the PL normal microbundle to this embedding. We apply a

theorem due to Lashof and Rothenberg (Thm. 7.3 in [13]) to obtain a piecewise differentiable homeomorphism $g: S^{2n+1} \to S^{2n+1}$ so that $g \circ i$: $S^1 \times D^{2n} \to S^{2n+1}$ is the smooth vector bundle to the smooth embedding $g \circ i: S^1 \to S^{2n+1}$. By smoothly unknotting this circle and applying the smooth tubular neighborhood theorem we obtain a diffeomorphism $h$: $S^{2n+1} \to S^{2n+1}$ so that

$$
\begin{array}{ccc}
h \circ g \circ i: S^1 \times D^{2n} & \to & S^{2n+1} \\
\bar{\lambda} \searrow & & \uparrow j \\
& S^1 \times D^{2n} &
\end{array}
$$

commutes where $j$ is the standard embedding and $\bar{\lambda}$ is a vector bundle isomorphism. Hence, the restriction map

$$
h \circ g\,\big| : S^{2n+1} - i\big(S^1 \times D^{2n}\big) \to S^{2n+1} - j\big(S^1 \times D^{2n}\big)
$$
$$
\|
$$
$$
D^2 \times S^{2n-1}
$$

defines a piecewise differentiable homeomorphism

$$
\Lambda: \Big[ D^2 \times \Sigma^{2n-}_{-\sigma} \cup_{\hat{\rho}} \hat{W} \Big] \to D^2 \times S^{2n-1}
$$

which restricts as $\lambda = \bar{\lambda}$ on the boundary. Finally, we observe that (cf. Cor. 10.13 in [19]) we may choose a smooth triangulation of $D^2 \times S^{2n-1}$ so that $\Lambda$ is PL. Now, using the homeomorphisms $\Lambda$ and $\eta$ we define a PL homeomorphism:

$$
(1) \qquad\qquad \varphi_\sigma: \xi \to \partial \nu_{n+1}(CP^1)
$$

where $\xi$ is the $S^{2n-1}$ bundle over $CP^1 = S^2$ defined by $\lambda^{-1}$:

$$
\xi = D^2_- \times S^{2n-1} \cup_{\lambda^{-1}} D^2_+ \times S^{2n-1}
$$

$$
\overset{\Lambda^{-1} \cup \mathrm{id}}{\to} D^2 \times \Sigma^{2n-1}_{-\sigma} \cup_{\hat{\rho}} \tilde{W}^{2n+1} \cup_{\mathrm{id}} D^2_+ \times S^{2n-1}
$$

$$
\overset{(\mathrm{id} \times \alpha_{-\sigma}) \cup \eta \cup (\mathrm{id} \times \alpha_\sigma)}{\to} D^2_- \times S^{2n-1} \cup_\rho W \cup D^2 \times \Sigma^{2n-1}_{-\sigma}
$$

$$
= D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1} = \partial \nu_{n+1}(CP^1).
$$

From the next lemma to the effect that two non-isomorphic sphere bundles over $S^2$ cannot be PL homeomorphic it follows that the existence of the map $\varphi_\sigma$ itself guarantees that $\xi$ and $\partial \nu_{n+1}(CP^1)$ are the same bundle.

LEMMA. *For $m \geq 3$ the unique non-trivial orthogonal $S^m$ bundle over $S^2$, $\xi$, is not PL homeomorphic to $S^2 \times S^m$.*

*Proof.* Suppose $t: \xi \to S^2 \times S^m$ is a PL homeomorphism. Let $E$ be the non-trivial $D^{m+1}$ bundle over $S^2$ with $\partial E = \xi$ and define the PL manifold

$$M^{m+3} = E \cup_t D^3 \times S^m$$

$M$ is the union of simply connected spaces over a path connected intersection. Hence, $\pi_1(M) = \{1\}$. For $m \geq 3$ the homotopy exact sequence of the fibration $S^m \to \partial E \xrightarrow{p} S^2$ implies that $p_*: \pi_2(\partial E) \to \pi_2(S^2)$ is an isomorphism, and by the Whitehead theorem so is the inclusion $H_2(\partial E) \to H_2(E)$. Hence, in the Mayer-Vietoris sequence

$$\cdots \to H_j(S^2 \times S^m) \xrightarrow{\psi_j} H_j(E) \oplus H_j(D^3 \times S^m) \to H_j(M)$$

$$\to H_{j-1}(S^2 \times S^m) \to \cdots$$

$\psi_j$ is an isomorphism for $j \leq m + 1$. Trivially, $H_{m+2}(M) = 0$, and again we have an $(m + 2)$-connected $(m + 3)$-dimensional PL manifold which is consequently a PL sphere.

Then, $E \cup_t D^3 \times S^m \cong S^{m+3}$ defines the vector bundle $E$ as a PL normal micro-bundle to the embedding of its zero section $S^2 \hookrightarrow S^{m+3}$. By Zeeman's PL unknotting theorem and the uniqueness [7] of stable PL normal microbundles, we see that $E$ and $S^2 \times D^{m+1}$ must be micro-bundle isomorphic. Let $S^2 \xrightarrow{b} BO$ classify $E$ as a vector bundle. Then $S^2 \xrightarrow{h} BO \to BPL$ is trivial, and as by smoothing theory the fiber PL/0 is 6-connected we see that b is homotopically trivial. As $E$ was assumed non-trivial as a vector bundle the PL homeomorphism $t$ cannot exist.

Thus, we define

$$\varphi_\sigma: \partial \nu_{n+1}(CP^1) = \zeta \to \partial \nu_{n+1}(CP^1) \quad \text{from (1) as required.}$$

Next we show that the $\varphi_\sigma$ just constructed is indeed a self-knotting and that it will suffice for Theorem A.

Recalling from bundle theory that every $S^N$ bundle over $S^2$ for $N \geq 2$ has a section, we show

PROPOSITION. *Any orientation preserving PL homeomorphism* $\varphi: \nu \to \nu$, $\nu$ *an orthogonal* $S^N$ *bundle over* $S^2$, *which embeds a section* $S^2 \xrightarrow{j} \nu$ *homotopically to itself is homotopic to the identity.*

*Proof.* A tubular neighborhood of the section $j(S^2)$ is a $D^N$ bundle $U$ in the same stable bundle class as $\nu$. $\varphi(U)$ PL embeds this bundle in $\nu$ with an inherited smooth structure. By the main theorem of smoothing

theory ([8] or [13], Thm. 7.3) and the uniqueness of smoothings on $S^2$ we can piecewise differentiably isotope this embedding to a smooth embedding of $U \to \nu$. We may easily make the isotopy ambient. Next, we smoothly unknot the core sphere of $U$ and apply the smooth tubular neighborhood theorem. We have, therefore, P.D. isotoped $\varphi$ so that restricted to $U$ it is a $D^N$ bundle isomorphism. Since $\pi_2(SO(N)) = 0$ we can isotope this bundle mapping to the identity through bundle isomorphisms on $U$ all of which extend to $\nu$ as $U$ is a sub-bundle. Thus, we have isotoped $\varphi$ so that it is the identity on $U$. Now, $\nu - U \cong U$ as each fiber of $U$ is a hemisphere of a fiber in $\nu$. We isotope $\varphi \operatorname{rel}(U)$ so that it is the identity on the zero section of the bundle $\nu - U$. Finally, we homotope $\varphi$ to the identity by collapsing the fibers of $\nu - U$ to the zero-section.

We observe that the $\varphi_\sigma$ constructed above satisfies the hypothesis of this last proposition as follows: $\varphi_\sigma$ is orientation preserving by construction. Also, as the original Brieskorn knot embedded a fiber $S^{2n+1}$ homotopically to the usual embedding, we know that $\varphi_\sigma$ does also. That is $(\varphi_\sigma)_*[\partial\nu] = [\partial\nu]$ and $(\varphi_\sigma)^*(e^{2n-1}) = e^{2n-1}$, where $e^{2n-1} \in H^{2n-1}(\partial\nu)$ is the class represented by inclusion of a fiber. By Poincaré Duality, then, $(\varphi_\sigma)_*(e_2) = e_2$ for $e_2 \in H_2(\partial\nu)$ the class dual to $e^{2n-1}$. This implies by the Hurewicz Theorem that $\varphi_\sigma$ induces the identity homomorphism on $\pi_2(\partial\nu)$, which is generated by the inclusion of a section.

The map $\varphi_\sigma$ constructed in section C embeds a fiber $S^{2n-1}$ onto the image of the Brieskorn knot. Hence, in the decomposition

$$\widetilde{CP}^{n+1} = \left[ CP^{n+1} - \nu_{n+1}(CP^1) \right] \cup_{\varphi_\sigma} \left[ \nu_{n+1}(CP^1) \right]$$

the identification is in the order:

$$\varphi_\sigma \colon \partial\left[ CP^{n+1} - \nu \right] \to \partial\nu.$$

To show, therefore, that $\widetilde{CP}^{n+1} \leftrightarrow (0, \ldots, 0, \sigma)$ we must extend $\varphi_\sigma^{-1}$ to a homotopy equivalence $\overline{\varphi_\sigma}^{-1} \colon \nu \to \nu$ with transverse-inverse image of a fiber being the Milnor or Kervaire manifold $M_0^{2n}$. Note that any extension will be a homotopy equivalence as $\nu \simeq S^2$ and $\varphi_\sigma^{-1}$ induces the identity on $\pi_2(\partial\nu) = \pi_2(\nu)$.

PROPOSITION. *The PL homeomorphism* $\varphi_\sigma^{-1} \colon \partial\nu_{N+1}(CP^1) \to \partial\nu_{n+1}(CP^1)$ *constructed above extends to* $\overline{\varphi_\sigma}^{-1} \colon \nu_{n+1}(CP^1) \to \nu_{n+1}(CP^1)$ *with transverse-inverse image*

$$\left( \overline{\varphi_\sigma}^{-1} \right)^{-1}(D^{2n}) = M_0^{2n}$$

*Proof.* $(\varphi_\sigma^{-1})^{-1}(S^{2n-1}) = \varphi_\sigma(S^{2n-1}) = \Sigma_\sigma^{2n-} \subset \partial\nu$ by the construction of $\varphi_\sigma$. Moreover, the restriction $\varphi_\sigma^{-1}\mid : D^2 \times \Sigma_\sigma^{2n-1} \to D_+^2 \times S^{2n-1}$ is a product map. Now, $\Sigma_\sigma^{2n-1}$ bounds a fiber $F^{2n} \subset W^{2n+1}$ whose other boundary component is a fiber $S^{2n-1}$ of $\partial\nu$. Let $D^{2n} \subset \nu$ be the fiber whose boundary is this same sphere. Then, $F^{2n} \cup D^{2n} = M_0^{2n}$ by the definition of $F^{2n}$. By pushing $F^{2n}$ into $\nu$ along a vector field normal to $\partial\nu$ and smoothing the corner at $S^{2n-1}$ between $F^{2n}$ and $D^{2n}$ we obtain a smooth embedding $M_0^{2n} \hookrightarrow \nu$ extending

$$\partial M_0^{2n} = \Sigma_\sigma^{2n-1} \subset \partial\nu.$$

Moreover, this embedding will have trivial normal $D^2$ bundle as $H^1(M_0^{2n}, Z) = 0$. Hence, we can extend the product map

$$\varphi_\sigma^{-1}: D^2 \times \Sigma_\sigma^{2n-1} \to D_+^2 \times S^{2n-1}$$

to a bundle map $\hat\varphi_\sigma^{-1}: D^2 \times M_0^{2n} \to D_+^2 \times D^{2n}$ covering a degree one extension $M_0^{2n} \to D^{2n}$. Since $[\nu - D_+^2] \times D_-^2 \times D^{2n} = D^{2n-2}$ there are no cohomology obstructions to extending

$$\varphi_\sigma^{-1} \cup \hat\varphi_\sigma^{-1} \text{ to } \overline{\varphi_\sigma^{-1}}: \nu \to \nu$$

with the required transverse-inverse image built in.

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