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ENGEL'S THEOREM FOR A CLASS OF ALGEBRAS

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A condition of nilpotency is derived for a class of algebras which include the almost alternative algebras of A. A. Albert. This result is seen to be an extension of Engel's theorem. Some consequences are then considered.

An algebra A over a Noetherian ring R is called almost alternative if it is power associative and satisfies the following identities

I.

$$u(vw) = \alpha_1(uv)w + \alpha_2w(uv) + \alpha_3(vu)w \\ + \alpha_4w(vu) + \alpha_5(uw)v + \alpha_6v(uw) + \alpha_7(wu)v + \alpha_8v(wu)$$

and

II.

$$(vw)u = \beta_1v(wu) + \beta_2(wu)v + \beta_3v(uw) \\ + \beta_4(uw)v + \beta_5w(vu) + \beta_6(vu)w + \beta_7w(uv) + \beta_8(uv)w$$

where $u, v, w \in A$ and $\alpha_i, \beta_i \in R$. In what follows the requirement of power associativity will not be needed. Lie, alternative and (γ, δ) -algebras are contained in this class. It is the purpose of this note to give a criterion for nilpotency inspired by the Engel theorem in Lie algebras. However, it is not sufficient to assume that each multiplication is nilpotent. For let A be a 3-dimensional algebra generated by x, y and z where $xy = z$ and $zx = y$ and all other multiplications between basis elements are 0. Then A is a non-nilpotent algebra satisfying I and II with $\alpha_4 = \beta_4 = 1$ and all other $\alpha_i, \beta_i = 0$. Also each right and left multiplication by any element in A is nilpotent. Note that A is not power associative.

Let R be a Noetherian ring. All algebras and modules over R are assumed to be unital. Let A be an algebra over R satisfying I and II. For each $x \in A$, R_x and L_x will denote right and left multiplication of A by x . Let M be an A -bimodule (see [4, p. 25]) with induced representation (S, T) . Hence S and T satisfy identities derived from I and II and consequently $S_{xy}, T_{xy} \in \langle S_x, S_y, T_x, T_y \rangle$, the associative subalgebra of $\text{End}(M)$ generated by S_x, S_y, T_x, T_y . In particular, if $x \in A$ and y is in the

subalgebra of A which is generated by x , then $S_y, T_y \in \langle S_x, T_x \rangle$. This latter subalgebra will be denoted by $J(x)$. Also recall that a subset of A which is closed under multiplication is called a Lie set.

THEOREM. *Let A be an algebra over a Noetherian ring R such that A satisfies I and II. Let M be an A -bimodule such that M is a finitely generated R -module. Let C be a Lie set in A such that C generates A . Suppose that for each $x \in C$, $J(x)$ is nilpotent. Then A acts nilpotently on M .*

Proof. There exist Lie subsets G of C such that the algebra generated by G , $\langle G \rangle$, acts nilpotently on M since the algebra generated by any $x \in C$ has this property. This follows from the remarks preceding the statement of the theorem. Also $\{x \in A; Mx = xM = 0\}$ is an ideal in A . Hence assume that the representation is faithful and that A is finitely generated. Let D be a Lie subset of C such that $\langle D \rangle$ acts nilpotently on M and $\langle D \rangle$ is maximal with this property. Since we may assume that A contains $\langle D \rangle$ properly, $C \not\subseteq \langle D \rangle$. By assumption there exists n such that $M\sigma_1 \cdots \sigma_n = 0$ for all possible $\sigma_1, \dots, \sigma_n$ where $\sigma_i = S_{x_i}$ or $\sigma_i = T_{x_i}$ and each $x_i \in D$. Let $M_i = \{m \in M; m\sigma_1 \cdots \sigma_i = 0 \text{ for all possible } \sigma_1, \dots, \sigma_i\}$. Then $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$.

Now consider any element of the form $\sigma_1 \cdots \sigma_{i-1} \tau \sigma_i \cdots \sigma_{2n}$ where each σ_i is as defined above, $\tau = S_a$ or $\tau = T_a$ and $i = 1, \dots, 2n - 1$. This product must be 0. Next consider any $y \in A$ with $2n + 1$ factors of which $2n$ are from D . Both S_y and T_y are the sum of terms of the above type, hence $S_y = T_y = 0$. Hence $y = 0$. Now there exists a least positive integer m such that $C\tau_1 \cdots \tau_m \subseteq \langle D \rangle$ for all τ_1, \dots, τ_m where each $\tau_i = R_{x_i}$ or $\tau_i = L_{x_i}$ and $x_i \in D$. Then there exists $z = x\tau_1 \cdots \tau_{m-1} \notin \langle D \rangle$ and $x \in C$. Now $z \in C$ since C is a Lie set and if y is the product of z 's, no matter how associated, then $y \in C$, and using I and II, $yD, Dy \subseteq D$. If F is the union of D and all such y , then F is a Lie subset of C and $\langle F \rangle$ contains $\langle D \rangle$ properly.

It remains only to show that $\langle F \rangle$ acts nilpotently on M . Under the conditions on the representation, S_z and T_z leave each M_i invariant and $J(z)$ acts nilpotently on each factor. Furthermore $S_y, T_y \in J(z)$ where y is any product of z 's. Then the chain may be refined to one such that F , and hence $\langle F \rangle$, annihilates each factor. This contradiction establishes the result.

The following are extensions of results on Lie algebras to the present setting.

COROLLARY 1. *Let A and R be as in the theorem and let N be a nilpotent ideal of A . Let $U(N, A)$ be the associative subalgebra of $\text{End}(A)$ generated by R_x, L_x for all $x \in N$. Then $U(N, A)$ is contained in the radical of $U(A, A)$.*

Proof. Since N is a nilpotent ideal of A , $J(n)$ is nilpotent for each $n \in N$. Hence $U(N, A)$ is nilpotent by the theorem. Hence there exists a chain $0 = A_0 \subseteq \dots \subseteq A_k = A$ where $A_j = \{x \in A; Nx + xN \subseteq A_{j-1}\}$. Clearly $AA_j + A_jA \subseteq A_j$. Let $\sigma = \sigma_1 \dots \sigma_n$ where $\sigma_i = S_{x_i}$ or $\sigma_i = T_{x_i}$, $x_i \in A$, and at least one $x_i \in N$. Then T annihilates each factor in the chain. Hence the ideal of $U(A, A)$ generated by $U(N, A)$ has nilpotent length k .

Using the regular representation, the theorem becomes

COROLLARY 2. *Let A be finitely generated over the Noetherian ring R . Suppose that A satisfies I and II. Let C be a Lie set in A such that C generates A . If $J(x)$ is nilpotent for all $x \in C$, then A is nilpotent.*

The next application deals with the nilpotency of algebras which admit regular automorphisms. This is an extension of a Lie algebra result of Jacobson [3]. Note that algebras satisfying I and II remain in this class under extension of the base.

COROLLARY 3. *Let A be a finite dimensional algebra over a field. Suppose that A satisfies I and II. Let Φ be an automorphism of A such that $\Phi^p = I$ where p is a prime. Suppose that Φ has no non-zero fixed points. Then A is nilpotent.*

Proof. We may assume that the base is algebraically closed. Let $\alpha_1, \dots, \alpha_n$ be the roots of Φ , all of which are p -roots of unity other than 1, and let $A = A_{\alpha_1} \oplus \dots \oplus A_{\alpha_n}$ be the decomposition of A into characteristic subspaces. For roots α, β , $A_\alpha A_\beta \subseteq A_{\alpha\beta}$. Hence $C = UA_{\alpha_i}$ is a Lie set in A . Let $x \in A_\alpha$ and β be any p -root of unity. Then there exists i , $1 \leq i \leq p$, such that $\beta\alpha^i = 1$. If σ is the product of i terms, each of which is R_x or L_x , then $A_\beta\sigma \subseteq A_1 = 0$. Hence if σ is the product of p or more such terms, then $A_\gamma\sigma = 0$ for all roots γ . Hence $A\sigma = 0$. Now the product of p elements from $J(x)$ can be expressed as the sum of terms each with at least p factors all of the form R_x or L_x . Hence $J(x)^p = 0$. Therefore A is nilpotent.

Again let A satisfy I and II and for $x \in A$, let $E_A(x) = \{y \in A; yJ(x)^k = 0 \text{ for some } k = 1, 2, \dots\}$. Then

LEMMA. $E_A(x)$ is a subalgebra of A .

The Frattini subalgebra $\phi(A)$ of an arbitrary algebra A has been investigated by Towers [5]. In this direction we obtain a generalization of a theorem of D. W. Barnes.

COROLLARY 4. *Let A be a finite dimensional algebra over a field. Suppose that A satisfies I, II and is flexible. Let N be an ideal of A such that $N \subseteq \phi(A)$ and A/N is nilpotent. Then A is nilpotent.*

Proof. Suppose not. Then there exists $x \in A$ such that $J(x)$ is not nilpotent. For $\sigma \in J(x)$ let $A_{0\sigma}$ and $A_{1\sigma}$ be the Fitting null and one spaces of A with respect to σ . Since A is flexible, $J(x)$ is abelian and $A_{0\sigma}$ and $A_{1\sigma}$ are τ -invariant for each $\tau \in J(x)$. Since A/N is nilpotent, $A_{1\sigma} \subseteq N$ for each σ , hence $B = \bigcap A_{0\sigma}$ supplements N . Also $E_A(x) \subseteq B$. Now $J(x)$ restricted to B is a nilalgebra, hence it is nilpotent. Hence $E_A(x) = B$. Then $A = E_A(x) + N = E_A(x) + \phi(A)$, a contradiction since $E_A(x)$ is a subalgebra of A .

The final result concerns the existence of Cartan subalgebras, nilpotent self-normalizing subalgebras. Clearly these can not exist in general. However, we have the following result.

COROLLARY 5. *Let A be a finite dimensional algebra over a field. Suppose that A is solvable, flexible and satisfies I and II. Then A contains a Cartan subalgebra.*

Proof. Induct on the dimension of A . Let M be minimal ideal of A . There exists $B \subseteq A$ such that $B \supseteq M$ and B/M is a Cartan subalgebra of A/M . If $B \neq A$, then there exists a Cartan subalgebra C of B . C is nilpotent and we claim that C is self-normalizing in A . Let $D = N_A(C)$, the normalizer of C in A . Since $C + M/M$ is a Cartan subalgebra of B/M , $C + M = B$. If $x \in D$, then $x \in N_A(B) = B$. Hence $D \subseteq B$. Therefore $D = N_B(C) = C$ and C is a Cartan subalgebra of A .

Suppose $B = A$. Then A/M is nilpotent and we may assume that A is not nilpotent. By the above corollary, $M \not\subseteq \phi(A)$. Hence A contains a maximal subalgebra T which complements M in A since $T \cap M$ is an ideal

in $T + M = A$. T is nilpotent and $N_A(T) \cap M$ is an ideal in $N_A(T) + M = A$. Then either $N_A(T) \cap M = 0$, $N_A(T) = T$ and T is a Cartan subalgebra of A or $N_A(T) \cap M = M$. Then T is an ideal in A and $A = A/T \cap M$ is nilpotent, a contradiction.

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