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**ENGEL'S THEOREM FOR A CLASS OF ALGEBRAS**

ERNEST LESTER STITZINGER

## ENGEL'S THEOREM FOR A CLASS OF ALGEBRAS

ERNEST L. STITZINGER

**A condition of nilpotency is derived for a class of algebras which include the almost alternative algebras of A. A. Albert. This result is seen to be an extension of Engel's theorem. Some consequences are then considered.**

An algebra  $A$  over a Noetherian ring  $R$  is called almost alternative if it is power associative and satisfies the following identities

I.

$$u(vw) = \alpha_1(uv)w + \alpha_2w(uv) + \alpha_3(vu)w \\ + \alpha_4w(vu) + \alpha_5(uw)v + \alpha_6v(uw) + \alpha_7(wu)v + \alpha_8v(wu)$$

and

II.

$$(vw)u = \beta_1v(wu) + \beta_2(wu)v + \beta_3v(uw) \\ + \beta_4(uw)v + \beta_5w(vu) + \beta_6(vu)w + \beta_7w(uv) + \beta_8(uv)w$$

where  $u, v, w \in A$  and  $\alpha_i, \beta_i \in R$ . In what follows the requirement of power associativity will not be needed. Lie, alternative and  $(\gamma, \delta)$ -algebras are contained in this class. It is the purpose of this note to give a criterion for nilpotency inspired by the Engel theorem in Lie algebras. However, it is not sufficient to assume that each multiplication is nilpotent. For let  $A$  be a 3-dimensional algebra generated by  $x, y$  and  $z$  where  $xy = z$  and  $zx = y$  and all other multiplications between basis elements are 0. Then  $A$  is a non-nilpotent algebra satisfying I and II with  $\alpha_4 = \beta_4 = 1$  and all other  $\alpha_i, \beta_i = 0$ . Also each right and left multiplication by any element in  $A$  is nilpotent. Note that  $A$  is not power associative.

Let  $R$  be a Noetherian ring. All algebras and modules over  $R$  are assumed to be unital. Let  $A$  be an algebra over  $R$  satisfying I and II. For each  $x \in A$ ,  $R_x$  and  $L_x$  will denote right and left multiplication of  $A$  by  $x$ . Let  $M$  be an  $A$ -bimodule (see [4, p. 25]) with induced representation  $(S, T)$ . Hence  $S$  and  $T$  satisfy identities derived from I and II and consequently  $S_{xy}, T_{xy} \in \langle S_x, S_y, T_x, T_y \rangle$ , the associative subalgebra of  $\text{End}(M)$  generated by  $S_x, S_y, T_x, T_y$ . In particular, if  $x \in A$  and  $y$  is in the

subalgebra of  $A$  which is generated by  $x$ , then  $S_y, T_y \in \langle S_x, T_x \rangle$ . This latter subalgebra will be denoted by  $J(x)$ . Also recall that a subset of  $A$  which is closed under multiplication is called a Lie set.

**THEOREM.** *Let  $A$  be an algebra over a Noetherian ring  $R$  such that  $A$  satisfies I and II. Let  $M$  be an  $A$ -bimodule such that  $M$  is a finitely generated  $R$ -module. Let  $C$  be a Lie set in  $A$  such that  $C$  generates  $A$ . Suppose that for each  $x \in C$ ,  $J(x)$  is nilpotent. Then  $A$  acts nilpotently on  $M$ .*

*Proof.* There exist Lie subsets  $G$  of  $C$  such that the algebra generated by  $G$ ,  $\langle G \rangle$ , acts nilpotently on  $M$  since the algebra generated by any  $x \in C$  has this property. This follows from the remarks preceding the statement of the theorem. Also  $\{x \in A; Mx = xM = 0\}$  is an ideal in  $A$ . Hence assume that the representation is faithful and that  $A$  is finitely generated. Let  $D$  be a Lie subset of  $C$  such that  $\langle D \rangle$  acts nilpotently on  $M$  and  $\langle D \rangle$  is maximal with this property. Since we may assume that  $A$  contains  $\langle D \rangle$  properly,  $C \not\subseteq \langle D \rangle$ . By assumption there exists  $n$  such that  $M\sigma_1 \cdots \sigma_n = 0$  for all possible  $\sigma_1, \dots, \sigma_n$  where  $\sigma_i = S_{x_i}$  or  $\sigma_i = T_{x_i}$  and each  $x_i \in D$ . Let  $M_i = \{m \in M; m\sigma_1 \cdots \sigma_i = 0 \text{ for all possible } \sigma_1, \dots, \sigma_i\}$ . Then  $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ .

Now consider any element of the form  $\sigma_1 \cdots \sigma_{i-1} \tau \sigma_i \cdots \sigma_{2n}$  where each  $\sigma_i$  is as defined above,  $\tau = S_a$  or  $\tau = T_a$  and  $i = 1, \dots, 2n - 1$ . This product must be 0. Next consider any  $y \in A$  with  $2n + 1$  factors of which  $2n$  are from  $D$ . Both  $S_y$  and  $T_y$  are the sum of terms of the above type, hence  $S_y = T_y = 0$ . Hence  $y = 0$ . Now there exists a least positive integer  $m$  such that  $C\tau_1 \cdots \tau_m \subseteq \langle D \rangle$  for all  $\tau_1, \dots, \tau_m$  where each  $\tau_i = R_{x_i}$  or  $\tau_i = L_{x_i}$  and  $x_i \in D$ . Then there exists  $z = x\tau_1 \cdots \tau_{m-1} \notin \langle D \rangle$  and  $x \in C$ . Now  $z \in C$  since  $C$  is a Lie set and if  $y$  is the product of  $z$ 's, no matter how associated, then  $y \in C$ , and using I and II,  $yD, Dy \subseteq D$ . If  $F$  is the union of  $D$  and all such  $y$ , then  $F$  is a Lie subset of  $C$  and  $\langle F \rangle$  contains  $\langle D \rangle$  properly.

It remains only to show that  $\langle F \rangle$  acts nilpotently on  $M$ . Under the conditions on the representation,  $S_z$  and  $T_z$  leave each  $M_i$  invariant and  $J(z)$  acts nilpotently on each factor. Furthermore  $S_y, T_y \in J(z)$  where  $y$  is any product of  $z$ 's. Then the chain may be refined to one such that  $F$ , and hence  $\langle F \rangle$ , annihilates each factor. This contradiction establishes the result.

The following are extensions of results on Lie algebras to the present setting.

**COROLLARY 1.** *Let  $A$  and  $R$  be as in the theorem and let  $N$  be a nilpotent ideal of  $A$ . Let  $U(N, A)$  be the associative subalgebra of  $\text{End}(A)$  generated by  $R_x, L_x$  for all  $x \in N$ . Then  $U(N, A)$  is contained in the radical of  $U(A, A)$ .*

*Proof.* Since  $N$  is a nilpotent ideal of  $A$ ,  $J(n)$  is nilpotent for each  $n \in N$ . Hence  $U(N, A)$  is nilpotent by the theorem. Hence there exists a chain  $0 = A_0 \subseteq \cdots \subseteq A_k = A$  where  $A_j = \{x \in A; Nx + xN \subseteq A_{j-1}\}$ . Clearly  $AA_j + A_jA \subseteq A_j$ . Let  $\sigma = \sigma_1 \cdots \sigma_n$  where  $\sigma_i = S_{x_i}$  or  $\sigma_i = T_{x_i}$ ,  $x_i \in A$ , and at least one  $x_i \in N$ . Then  $T$  annihilates each factor in the chain. Hence the ideal of  $U(A, A)$  generated by  $U(N, A)$  has nilpotent length  $k$ .

Using the regular representation, the theorem becomes

**COROLLARY 2.** *Let  $A$  be finitely generated over the Noetherian ring  $R$ . Suppose that  $A$  satisfies I and II. Let  $C$  be a Lie set in  $A$  such that  $C$  generates  $A$ . If  $J(x)$  is nilpotent for all  $x \in C$ , then  $A$  is nilpotent.*

The next application deals with the nilpotency of algebras which admit regular automorphisms. This is an extension of a Lie algebra result of Jacobson [3]. Note that algebras satisfying I and II remain in this class under extension of the base.

**COROLLARY 3.** *Let  $A$  be a finite dimensional algebra over a field. Suppose that  $A$  satisfies I and II. Let  $\Phi$  be an automorphism of  $A$  such that  $\Phi^p = I$  where  $p$  is a prime. Suppose that  $\Phi$  has no non-zero fixed points. Then  $A$  is nilpotent.*

*Proof.* We may assume that the base is algebraically closed. Let  $\alpha_1, \dots, \alpha_n$  be the roots of  $\Phi$ , all of which are  $p$ -roots of unity other than 1, and let  $A = A_{\alpha_1} \oplus \cdots \oplus A_{\alpha_n}$  be the decomposition of  $A$  into characteristic subspaces. For roots  $\alpha, \beta$ ,  $A_\alpha A_\beta \subseteq A_{\alpha\beta}$ . Hence  $C = UA_{\alpha_i}$  is a Lie set in  $A$ . Let  $x \in A_\alpha$  and  $\beta$  be any  $p$ -root of unity. Then there exists  $i$ ,  $1 \leq i \leq p$ , such that  $\beta\alpha^i = 1$ . If  $\sigma$  is the product of  $i$  terms, each of which is  $R_x$  or  $L_x$ , then  $A_\beta\sigma \subseteq A_1 = 0$ . Hence if  $\sigma$  is the product of  $p$  or more such terms, then  $A_\gamma\sigma = 0$  for all roots  $\gamma$ . Hence  $A\sigma = 0$ . Now the product of  $p$  elements from  $J(x)$  can be expressed as the sum of terms each with at least  $p$  factors all of the form  $R_x$  or  $L_x$ . Hence  $J(x)^p = 0$ . Therefore  $A$  is nilpotent.

Again let  $A$  satisfy I and II and for  $x \in A$ , let  $E_A(x) = \{y \in A; yJ(x)^k = 0 \text{ for some } k = 1, 2, \dots\}$ . Then

LEMMA.  $E_A(x)$  is a subalgebra of  $A$ .

The Frattini subalgebra  $\phi(A)$  of an arbitrary algebra  $A$  has been investigated by Towers [5]. In this direction we obtain a generalization of a theorem of D. W. Barnes.

COROLLARY 4. *Let  $A$  be a finite dimensional algebra over a field. Suppose that  $A$  satisfies I, II and is flexible. Let  $N$  be an ideal of  $A$  such that  $N \subseteq \phi(A)$  and  $A/N$  is nilpotent. Then  $A$  is nilpotent.*

*Proof.* Suppose not. Then there exists  $x \in A$  such that  $J(x)$  is not nilpotent. For  $\sigma \in J(x)$  let  $A_{0\sigma}$  and  $A_{1\sigma}$  be the Fitting null and one spaces of  $A$  with respect to  $\sigma$ . Since  $A$  is flexible,  $J(x)$  is abelian and  $A_{0\sigma}$  and  $A_{1\sigma}$  are  $\tau$ -invariant for each  $\tau \in J(x)$ . Since  $A/N$  is nilpotent,  $A_{1\sigma} \subseteq N$  for each  $\sigma$ , hence  $B = \bigcap A_{0\sigma}$  supplements  $N$ . Also  $E_A(x) \subseteq B$ . Now  $J(x)$  restricted to  $B$  is a nilalgebra, hence it is nilpotent. Hence  $E_A(x) = B$ . Then  $A = E_A(x) + N = E_A(x) + \phi(A)$ , a contradiction since  $E_A(x)$  is a subalgebra of  $A$ .

The final result concerns the existence of Cartan subalgebras, nilpotent self-normalizing subalgebras. Clearly these can not exist in general. However, we have the following result.

COROLLARY 5. *Let  $A$  be a finite dimensional algebra over a field. Suppose that  $A$  is solvable, flexible and satisfies I and II. Then  $A$  contains a Cartan subalgebra.*

*Proof.* Induct on the dimension of  $A$ . Let  $M$  be minimal ideal of  $A$ . There exists  $B \subseteq A$  such that  $B \supseteq M$  and  $B/M$  is a Cartan subalgebra of  $A/M$ . If  $B \neq A$ , then there exists a Cartan subalgebra  $C$  of  $B$ .  $C$  is nilpotent and we claim that  $C$  is self-normalizing in  $A$ . Let  $D = N_A(C)$ , the normalizer of  $C$  in  $A$ . Since  $C + M/M$  is a Cartan subalgebra of  $B/M$ ,  $C + M = B$ . If  $x \in D$ , then  $x \in N_A(B) = B$ . Hence  $D \subseteq B$ . Therefore  $D = N_B(C) = C$  and  $C$  is a Cartan subalgebra of  $A$ .

Suppose  $B = A$ . Then  $A/M$  is nilpotent and we may assume that  $A$  is not nilpotent. By the above corollary,  $M \not\subseteq \phi(A)$ . Hence  $A$  contains a maximal subalgebra  $T$  which complements  $M$  in  $A$  since  $T \cap M$  is an ideal

in  $T + M = A$ .  $T$  is nilpotent and  $N_A(T) \cap M$  is an ideal in  $N_A(T) + M = A$ . Then either  $N_A(T) \cap M = 0$ ,  $N_A(T) = T$  and  $T$  is a Cartan subalgebra of  $A$  or  $N_A(T) \cap M = M$ . Then  $T$  is an ideal in  $A$  and  $A = A/T \cap M$  is nilpotent, a contradiction.

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