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**ON THE ZETA FUNCTION FOR FUNCTION FIELDS OVER  $F_p$**

EMERY THOMAS

## ON THE ZETA FUNCTION FOR FUNCTION FIELDS OVER $F_p$

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**We consider here the zeta function for a function field defined over a finite field  $F_p$ . For each inter  $j$ ,  $\zeta(j)$  is a polynomial over  $F_p$ , as is  $\zeta'(j)$ , the "derivative" of zeta. In this note we compute the degree of these polynomials, determine when they are the constant polynomial and relate them to the polynomial gamma function.**

In a recent series of papers D. Goss has introduced the notion of a zeta function  $\zeta(j)$  for rational function fields over  $F_r$ , where  $r = p^k$ , with  $p$  a rational prime. In particular, for each positive integer  $i$ , with  $i \not\equiv 0 \pmod{r-1}$ ,  $\zeta(-i) \in F_r[t]$ . Goss also defines the "derivative" of  $\zeta$ ,  $\zeta'$ , with  $\zeta'(-i) \in F_r[t]$  if  $i \equiv 0 \pmod{r-1}$ . We combine these special values of  $\zeta$  and  $\zeta'$  into a single function  $\beta(n)$  (with  $n = -i$ ) defined by:

$$(1) \quad \beta(0) = 0, \quad \beta(1) = 1,$$

$$\beta(n) = 1 - \sum_{\substack{i=1 \\ i \equiv n(s)}}^{n-1} \binom{n}{i} t^i \beta(i), \quad n \geq 2,$$

where  $s = r - 1$ . Thus, by (3.9) and (3.10) of [2],

$$(2) \quad \beta(n) = \begin{cases} \zeta(-n), & n \not\equiv 0 \pmod{s} \\ \zeta'(-n), & n \equiv 0 \pmod{s} \end{cases}.$$

An important situation where these functions arise is in determining the class numbers of certain extension fields over  $F_r[t]$  (modeled on cyclotomic fields). If  $P$  is a prime polynomial in  $F_r[t]$ , Goss defines class numbers  $h^+(P)$  and  $h^-(P)$  associated to  $P$ , in the classical fashion, and shows that their study (à la Kummer) involves the polynomials  $\zeta(-i)$  and  $\zeta'(-i)$ . Thus it is important that we know certain facts about these functions, and hence about  $\beta(n)$ . Specifically, when is  $\beta(n) = 1$ ? What is the degree of  $\beta(n)$ ? When does  $\beta(n)$  factor? In this note we give some answers to these questions, for the case  $r = p$ .

REMARK. I am indebted to Goss for bringing this material to my attention.

**The function  $\beta(n)$ .** Let  $p$  be a rational prime, and for each integer  $n \geq 0$ , let  $\beta(n) \in F_p[t]$  be the polynomial defined above. Note that if  $0 < n \leq s (= p - 1)$ , then  $\beta(n) = 1$ . For  $n > s$  we rewrite (1) as follows: set  $k = [(n - 1)/s]$ . Then (1) becomes:

$$(3) \quad \beta(n) = 1 - \sum_{i=1}^k \binom{n}{is} t^{n-is} \beta(n - is).$$

Let  $n = \sum_i a_i p^i$  be the  $p$ -adic representation of  $n$ ; thus,  $0 \leq a_i \leq s$ , and almost all  $a_i$  are zero. Define

$$l(n) = \sum_i a_i.$$

Our first result is:

**THEOREM 1.** *Let  $n$  be a positive integer with  $l(n) \leq s$ . Then,*

$$\beta(n) = 1.$$

The proof depends upon several simple facts about binomial coefficients mod  $p$ . Recall the result of Lucas:

(4) *If  $m$  and  $n$  are given  $p$ -adically by  $m = \sum_i b_i p^i$ ,  $n = \sum_i a_i p^i$ , then*

$$\binom{n}{m} \bmod p \equiv \prod_i \binom{a_i}{b_i} \bmod p.$$

*In particular,*

$$\binom{n}{m} \not\equiv 0 \bmod p \Leftrightarrow 0 \leq b_i \leq a_i, \text{ all } i.$$

As an immediate consequence, we have:

(5) *If  $\binom{n}{m} \not\equiv 0 \bmod p$ , then  $l(n) = l(m) + l(n - m)$ . In particular, if  $1 \leq m < n$ , then  $l(n) > l(m)$ .*

Finally, note that since  $p \equiv 1 \pmod s$ , we have:

$$(6) \quad n \equiv l(n) \pmod s.$$

*Proof of Theorem 1.* Let  $j$  be any positive integer. By (6), since  $js \equiv 0 \pmod s$ ,  $l(js) \geq s$ . Thus, if  $n$  is an integer with  $js < n$  and  $\binom{n}{js} \not\equiv 0 \pmod p$ , then by (5),  $l(n) > l(js) \geq s$ . Therefore, if  $l(n) \leq s$ , then  $\binom{n}{js} \equiv 0 \pmod p$ . Thus, by (3),  $\beta(n) = 1$ , as claimed.

We suppose now that  $n$  is an integer with  $l(n) > s$ ; our goal is to calculate the degree of  $\beta(n)$  — call this simply  $D(n)$ .

Define an integer valued function  $\rho(n)$  by:

(7) *If  $l(n) \geq s$ , set  $\rho(n) = n - m$ , where  $m$  is the least positive integer such that*

$$l(m) = s \quad \text{and} \quad \binom{n}{m} \not\equiv 0 \pmod{p}.$$

Thus, if  $n$  is written  $p$ -adically in the form

(8) 
$$n = \sum_{i=0}^N p^{e_i}, \quad \text{with } e_0 \leq \dots \leq e_N,$$

and with no more than  $s$   $e_i$ 's with the same value, then

$$m = \sum_{i=0}^{s-1} p^{e_i}.$$

If  $q$  is an integer ( $\geq 0$ ) with  $l(q) < s$ , set  $\rho(q) = 0$ .

Set  $\rho^{i+1}(n) = \rho(\rho^i(n))$ , with  $\rho^0(n) = n$ . Thus, for large  $i$ ,  $\rho^i(n) = 0$ .

EXAMPLE.  $p = 5, n = 3 \cdot 1 + 4 \cdot 5 + 2 \cdot 5^3$ . Then,

$$\rho^1(n) = 3 \cdot 5 + 2 \cdot 5^3,$$

$$\rho^2(n) = 5^3,$$

$$\rho^3(n) = 0.$$

Our result is:

THEOREM 2. *Let  $n$  be an integer with  $l(n) > s$ . Then*

$$D(n) = \text{degree } \beta(n) = \sum_{i \geq 1} \rho^i(n).$$

The proof will be by induction on  $l(n)$ . Suppose first that  $l(n) = s + 1$ . If  $j$  is any positive integer with  $js < n$  and  $\binom{n}{js} \not\equiv 0 \pmod{p}$ , then by (5) and (6),  $l(n - js) = 1$ , and so by Theorem 1,  $\beta(n - js) = 1$ . Therefore, by (2),  $D(n) = n - js$ , where  $j$  is the least positive integer such that  $\binom{n}{js} \not\equiv 0 \pmod{p}$ ; i.e.,  $D(n) = \rho(n)$ , as stated in Theorem 2.

We now make the following pair of inductive hypotheses: let  $k$  be an integer  $\geq s + 1$ , and suppose that  $n$  is any integer such that

$$s + 1 \leq l(n) \leq k.$$

- (A<sub>k</sub>) For any such integer  $n$ ,  $D(n)$  is given by Theorem 2.
- (B<sub>k</sub>) Let  $n$  be any integer as above. If  $c$  is the least positive integer such that  $\binom{n}{cs} \not\equiv 0 \pmod{p}$  and  $d$  is any integer with  $cs \leq ds \leq n$  and  $\binom{n}{ds} \not\equiv 0 \pmod{p}$ ; then  $D(n - cs) \geq D(n - ds)$ .

*Claim 1. A<sub>k</sub> implies B<sub>k+1</sub>.*

*Proof.* Write  $n$  as in (8) so that  $cs = \sum_{i=0}^{s-1} p^{e_i}$ . Thus,  $n - cs = \sum_{i=0}^{N-s} p^{f_i}$ , where  $f_i = e_{i+s}$ . Similarly, write  $n - ds = \sum_{i=0}^M p^{g_i}$ , where  $M \leq N - s$ . Then, for  $i \leq M$ ,  $p^{f_i} \geq p^{g_i}$ , and so  $D(n - cs) \geq D(n - ds)$ , either by Theorem 1 or by A<sub>k</sub> and Theorem 2, since  $l(n - cs)$  and  $l(n - ds)$  are less than  $l(n)$ .

*Claim 2. A<sub>k</sub> and B<sub>k+1</sub> imply A<sub>k+1</sub>.*

*Proof.* Let  $n$  be an integer with  $l(n) = k + 1$ . Write  $n$  as in (8) and define  $cs$  as above, so that  $\rho(n) = n - cs$ . By (3) and B<sub>k+1</sub>,

$$D(n) = n - cs + D(n - cs) = \rho(n) + D(\rho(n)).$$

Since  $l(\rho(n)) < l(n) = k + 1$ , by A<sub>k</sub>

$$D(\rho(n)) = \sum_{i \geq 1} \rho^i(\rho(n)) = \sum_{i \geq 1} \rho^{i+1}(n).$$

Therefore,  $D(n) = \sum_{i \geq 1} \rho^i(n)$ , which proves A<sub>k+1</sub>.

*Proof of Theorem 2.* We showed above that A<sub>s+1</sub> holds, and so by Claims 1 and 2, A<sub>k</sub> holds for all  $k > s$ . This proves the theorem.

Note that (trivially) if  $n$  is positive, then  $\beta(n) \neq 0$ . Combining Theorems 1 and 2 we have:

**COROLLARY 1.** *If  $n$  is a positive integer, then  $\beta(n) = 1$  if, and only if,  $l(n) \leq s$ .*

For certain values of  $n$ ,  $D(n)$  can be written out explicitly.

**COROLLARY 2.** *Let  $k$  and  $m$  be positive integers, with  $m \leq s$ . Then*

$$D((m + 1)p^k - 1) = s \cdot \sum_{i=1}^{k-1} ip^i + kmp^k.$$

**Relation to the gamma function.** We are interested in comparing the function  $\beta(n)$  with the Gamma function  $\Gamma_n$  (see [1]). Combining Corollary 2 with (3.1.1) of [1], we find:

**COROLLARY 3.** *Let  $n = (m + 1)p^k - 1$ , where  $k$  and  $m$  are positive integers with  $m \leq s$ . Then,*

$$\deg \beta(n) = \deg \Gamma_n.$$

For certain values of  $n$  we have a stronger result.

**THEOREM 3.** *Suppose that  $n = (m + 1)p - 1$ , with  $1 \leq m \leq s$ . Then,*

$$\beta(n) = 1 - \Gamma_n.$$

We are especially interested in divisibility properties of  $\beta(n)$ . Thus, we have:

**COROLLARY 4.** *For  $1 \leq k \leq s/2$  and  $p$  an odd prime,*

$$\beta((2k + 1)p - 1) = (1 - \Gamma_{kp})(1 + \Gamma_{kp}).$$

*In particular,*

$$\beta(p^2 - 1) = (1 - \Gamma_{sp/2})(1 + \Gamma_{sp/2}).$$

*Proof of Theorem 3.* We will need the following (easily proved) fact:

$$\text{If } 0 \leq i \leq s, \text{ then } \binom{s}{i} \equiv (-1)^i \pmod{p}.$$

Suppose that  $n = (m + 1)p - 1$ , as above. Thus,  $n = s \cdot 1 + mp$ , and so by (3) and Theorem 1,

$$\begin{aligned} \beta(n) &= 1 - \sum_{i=0}^m \binom{n}{s-i+ip} t^{i+(m-i)p} \\ &= 1 - \sum_{i=0}^m \binom{s}{i} \binom{m}{i} t^i \cdot t^{(m-i)p} \quad \text{by (4)} \\ &= 1 - \sum_{i=0}^m (-1)^i \binom{m}{i} t^i \cdot t^{(m-i)p} \\ &= 1 - (t^p - t)^m = 1 - \Gamma_n \end{aligned}$$

by (3.1.1) of [1].

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