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All left modules over a ring are flat if and only if the ring is von Neumann regular. In [7], M. Kilp showed that for a monoid S to be left absolutely flat (i.e., for all left S-sets to be flat) regularity is necessary but not sufficient. Kilp also proved [8] that every inverse union of groups is absolutely flat. In the present paper we show that in fact every inverse semigroup is absolutely flat and that the converse is not true.

1. Preliminaries. We consider a monoid to be a universal algebra $(S;\ \cdot, 1)$ of type $(2,0)$. We shall consistently denote such a monoid by S and on occasion consider it to be a semigroup via the forgetful functor. If S is a monoid S-*Ens* (respectively, *Ens*-S) will denote the class of left (right) unital S-sets. In §§1 and 2, we deal only with monoids and their associated S-sets. In §3 the considerations will be extended to arbitrary semigroups.

Let S be a monoid. For $A \in$ *Ens*-S and $B \in S$-*Ens*, let τ denote the smallest equivalence relation on $A \times B$ containing all pairs $((as, b), (a, sb))$ for $a \in A$, $b \in B$, and $s \in S$. The tensor product $A \otimes B$ (or, more precisely, $A \otimes_S B$) is defined to be the set $(A \times B)/\tau$, and possesses the customary universal mapping property with respect to balanced maps from $A \times B$ to an arbitrary set. For $a \in A$ and $b \in B$, $a \otimes b$ represents the τ-class of (a, b).

The following information will be useful in the sequel. If S is any monoid and $s, t \in S$ then $\theta(s, t)$ will denote the principal left congruence on S identifying s and t. It is easy to check that for u, v in S, $(u, v) \in \theta(s, t)$ if and only if either

$$u = v$$

or

$$\begin{gathered}\text{there exist } w_1,\dots,w_n, s_1,\dots,s_n, t_1,\dots,t_n \in S\\ \text{where } \{s_i, t_i\} = \{s, t\} \text{ for } i = 1,\dots,n, \text{ such that}\\ u = w_1 s_1,\\ w_1 t_1 = w_2 s_2,\\ \vdots\\ w_n t_n = v.\end{gathered}$$

In fact, we have

LEMMA 1.1. *Let S be a monoid, $s, t \in S$, $A \in$ Ens-S, $a, a' \in A$. Then $a \otimes \bar{1} = a' \otimes \bar{1}$ in $A \otimes_S S/\theta(s, t)$ if and only if either*

$$a = a'$$

or

there exist $a_1, \ldots, a_n \in A$, $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ where $\{s_i, t_i\} = \{s, t\}$ for $i = 1, \ldots, n$, such that

$$\begin{aligned} a &= a_1 s_1, \\ a_1 t_1 &= a_2 s_2, \\ &\vdots \\ a_n t_n &= a'. \end{aligned}$$

Proof. For $a, a' \in A$ define $a \psi a'$ if and only if $a = a'$ or a system of equalities joining a and a', such as that given in the statement of the lemma, exists. ψ is an equivalence relation on A. Define a map ϕ: $A \times S/\theta(s, t) \to A/\psi$ by $\phi(a, \bar{u}) = \overline{au}$ for $a \in A$ and $u \in S$. Check that ϕ is well-defined and balanced (i.e. $\phi(ax, \bar{u}) = \phi(a, x\bar{u})$ for each $a \in A$, $x \in S$, and $u \in S$) and that the resulting map Φ: $A \otimes_S S/\theta(s, t) \to A/\psi$ is a bijection. Thus, for $a, a' \in A$, $a \otimes \bar{1} = a' \otimes \bar{1}$ iff $\Phi(a \otimes \bar{1}) = \Phi(a' \otimes \bar{1})$ iff $a \psi a'$. □

The following lemma provides a method of determining whether two elements of a tensor product over a monoid are equal.

LEMMA 1.2. *Let S be a monoid, $A \in$ Ens-S, $a, a' \in A$, $B \in S$-Ens, and $b, b' \in B$. Then $a \otimes b = a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, \ldots, a_n \in A$, $b_2, \ldots, b_n \in B$, $s_1, \ldots, s_n \in S$ and $t_1, \ldots, t_n \in S$ such that*

$$\begin{aligned} a &= a_1 s_1, & & \\ a_1 t_1 &= a_2 s_2, & s_1 b &= t_1 b_2, \\ a_2 t_2 &= a_3 s_3, & s_2 b_2 &= t_2 b_3, \\ &\vdots & &\vdots \\ a_n t_n &= a', & s_n b_n &= t_n b'. \end{aligned}$$

Proof. Verify that the relation η on $A \times B$ defined by $(a, b)\eta(a', b')$ if and only if a system of equalities such as that appearing above exists, is, in fact, the (tensor product) relation τ presented earlier. □

We will call the above system of equalities *an* (S-) *scheme over* A *and* B *of length* n *joining* (a, b) *to* (a', b').

2. Flat S-sets over monoids.

DEFINITION 2.1. Let S be any monoid, and let B belong to S-*Ens*. Then B is called *flat* (in S-*Ens*) if and only if, for all embeddings $A \to C$ in *Ens*-S, the induced map $A \otimes B \to C \otimes B$ is an embedding. Flat right S-sets are defined analogously.

Note that flatness as defined above differs from the notion considered in [**9**] and some of the references contained therein.

LEMMA 2.2. *Let* S *be a monoid and let* B *belong to* S-*Ens. Then* B *is flat if and only if, for every right* S-*set* A, *and every* $a, a' \in A$, $b, b' \in B$ *such that there exists a scheme over* A *and* B *joining* (a, b) *to* (a', b'), *there exists a scheme* (*of possibly different length*) *over* $aS \cup a'S$ *and* B *joining* (a, b) *to* (a', b'). *A similar statement describes flat right* S-*sets.*

DEFINITION 2.3. A monoid S is called *left* (*right*) *reversible* if any two principal right (left) ideals of S intersect. (See [**1**], p. 34.) A right (left) S-set A over a monoid S is called *reversible* if any two cyclic sub-S-sets of A intersect.

LEMMA 2.4. *Let* S *be a monoid. Then the following conditions are equivalent*:

(1) *The singleton left* S-*set* $Z = \{z\}$ *is flat.*

(2) S *is left reversible.*

(3) *Every connected right* S-*set is reversible.*

(4) *Every sub*-S-*set of a connected right* S-*set is connected.*

Proof. (1) implies (2)
For any $s, t \in S$ it is clear that $s \otimes z = t \otimes z$ in $S \otimes Z$. Thus there exists a scheme over $sS \cup tS$ and Z joining (s, z) to (t, z) (by Lemmas 1.2 and 2.2). In particular there exist $s_1, \ldots, s_n$, $t_1, \ldots, t_n \in S$ and $u_1, \ldots, u_n \in sS \cup tS$ such that

$$\begin{aligned} s &= u_1 s_1, \\ u_1 t_1 &= u_2 s_2, \\ &\ \vdots \\ u_n t_n &= t. \end{aligned}$$

From this it may easily be deduced that S is left reversible.

(2) implies (3)
Suppose A is a connected right S-set and a, a' are two elements of A. Because A is connected there exist $s_1,\dots,s_n$, $t_1,\dots,t_n \in S$ and $a_1,\dots,a_n \in A$ such that

$$\begin{aligned} a &= a_1 s_1, \\ a_1 t_1 &= a_2 s_2, \\ &\vdots \\ a_n t_n &= a'. \end{aligned}$$

Because S is left reversible there exist $x_1,\dots,x_n$, $y_1,\dots,y_n \in S$ such that $s_1 x_1 = t_1 y_1$ and $(s_i y_{i-1})x_i = t_i y_i$ $(1 < i \leq n)$. From this it is easy to see that $a(x_1 \cdots x_n) = a' y_n$ and, hence, A is reversible.

(3) implies (4)
Clear, since sub-S-sets inherit reversibility.

(4) implies (1)
Suppose A is a right S-set and a, a' are two elements of A such that there exists a scheme over A and Z joining (a, z) to (a', z). This implies a and a' lie in a connected component of A of which $aS \cup a'S$ is a sub-S-set. By (4), there will exist a scheme over $aS \cup a'S$ and Z joining (a, z) to (a', z). Thus, by Lemma 2.2, Z is flat. $\square$

The following result appears in the literature.

PROPOSITION 2.5. (*Kilp* [7]) *If all cyclic left S-sets over a monoid are flat, then S is regular.*

Proof. Choose $s \in S$. $s \otimes \bar{1} = s^2 \otimes \bar{1}$ in $S \otimes_S S/\theta(s, s^2)$, hence, in $sS \otimes_S S/\theta(s, s^2)$ (by flatness of $S/\theta(s, s^2)$). By Lemma 1.1, either $s = s^2$ or there exist $u_1,\dots,u_n \in sS$, $s_1,\dots,s_n \in S$, $t_1,\dots,t_n \in S$ where $\{s_i, t_i\} = \{s, s^2\}$ for $i = 1,\dots,n$ such that

$$\begin{aligned} s &= u_1 s_1, \\ u_1 t_1 &= u_2 s_2, \\ &\vdots \\ u_n t_n &= s^2. \end{aligned}$$

In either case it is clear that $s \in sSs$. $\square$

The converse of this result is not true. In fact the band $S = \{0, e, f, 1\}$ in which $ef = e$ and $fe = f$ is both regular and left reversible but possesses a two element cyclic left S-set which is not flat.

DEFINITION 2.6. A monoid S is called *left* (*right*) *absolutely flat* if all of its left (right) S-sets are flat and *absolutely flat* if it is both left and right absolutely flat.

Clearly every left (right) absolutely flat monoid is regular and left (right) reversible.

Note. Isbell's notions of *dominion* and *absolutely closed semigroup* **[6]**, **[5]** may be formulated in terms of schemes (see **[10]** and **[4]**). Absolutely closed monoids need not be absolutely flat: they may not even be regular. The authors thank the referee for demonstrating, however, that every left (or right) absolutely flat monoid is absolutely closed.

PROPOSITION 2.7. *Homomorphic images of* (*left*, *right*) *absolutely flat monoids are* (*left*, *right*) *absolutely flat*.

Proof. If f: $S \to T$ is a monoid homomorphism onto T then any $A \in Ens\text{-}T$ ($B \in T\text{-}Ens$) may be considered a right (left) S-set via the action $(a, s) \to af(s)$ $((s, b) \to f(s)b)$ and furthermore, $A \otimes_S B = A \otimes_T B$. The result follows easily. □

DEFINITION 2.8. A submonoid F of a monoid S is called a *filter* of S if for all $x, y \in S$, $xy \in F$ implies $x, y \in F$.

PROPOSITION 2.9. *A filter F of a* (*left*, *right*) *absolutely flat monoid S is* (*left*, *right*) *absolutely flat*.

Proof. We will assume S is a left absolutely flat monoid, and that $F \neq S$. Then, by Proposition 2.7 the Rees factor semigroup S/P, where $P = S \setminus F$, is left absolutely flat. Now $S/P \cong F \mathbin{\mathring{\cup}} \{0\} = F^0$. If X is a left F-set, X^* will denote the left F^0-set obtained by adjoining a new element $*$ to X and extending the action by defining $OX^* = \{*\}$ and $F* = \{*\}$. Y^* will denote the right F^0-set obtained by performing a similar construction on $Y \in Ens\text{-}F$.

Suppose $B \in F\text{-}Ens$, $A \in Ens\text{-}F$, $a, a' \in A$, $b, b' \in B$ and there exists an F-scheme over A and B joining (a, b) to (a', b'). This may be interpreted as an F^0-scheme over A^* and B^* joining (a, b) to (a', b').

Since F^0 is left absolutely flat, there exists an F^0-scheme over $aF^0 \cup a'F^0$ and B^* joining (a, b) to (a', b'). It is easily checked that this last scheme is actually an F-scheme over $aF \cup a'F$ and B joining (a, b) to (a', b'). Thus, B is flat and F is left absolutely flat. □

3. Flat S-sets over semigroups. If S is any semigroup, let S^1 denote the monoid $S \mathring{\cup} \{1\}$, obtained by adjoining a new identity element to S, even in the case in which S is already a monoid.

DEFINITION 3.1. A semigroup S is called (*left*, *right*) *absolutely flat* if S^1 is a (left, right) absolutely flat monoid.

The concept of absolute flatness of a semigroup S may also be developed in terms of the tensor product of S-sets in a manner similar to that which has been outlined for monoids. This approach is consistent with the definition above. Note that if S is a semigroup which possesses an identity element, then S is absolutely flat as a semigroup if and only if S is absolutely flat as a monoid. Finally, the semigroup analogues of Lemma 2.4 and Propositions 2.5, 2.7, and 2.9 are clearly valid.

4. Inverse semigroups. In this section we prove that every inverse semigroup is absolutely flat. Without loss of generality, assume S is an inverse monoid and B belongs to S-*Ens*. We shall use Lemma 2.2 to show B is flat. Let A belong to *Ens*-S, a, $a' \in A$, b, $b' \in B$ and suppose that the following scheme over A and B joins (a, b) to (a', b'):

$$\begin{aligned} a &= a_1 s_1, & & \\ a_1 t_1 &= a_2 s_2, & s_1 b &= t_1 b_2, \\ a_2 t_2 &= a_3 s_3, & s_2 b_2 &= t_2 b_3, \\ &\vdots & &\vdots \\ a_n t_n &= a', & s_n b_n &= t_n b'. \end{aligned}$$

It will be convenient and will impose no added restriction to assume n is even throughout this section. With reference to the above scheme, let

$$x_0 = 1, \qquad x_i = s_1^{-1} t_1 s_2^{-1} t_2 \cdots s_i^{-1} t_i \quad (1 \le i \le n)$$

and let

$$y_0 = 1, \qquad y_i = t_n^{-1} s_n t_{n-1}^{-1} s_{n-1} \cdots t_{n-i+1}^{-1} s_{n-i+1} \quad (1 \le i \le n).$$

LEMMA 4.1.

$$x_{n-i}y_i^{-1} = x_n \qquad (0 \le i \le n), \tag{1}$$
$$y_i x_{n-i}^{-1} = y_n \qquad (0 \le i \le n), \tag{2}$$
$$ax_i = a_i t_i x_i^{-1} x_i \qquad (1 \le i \le n), \tag{3}$$
$$a'y_i = a_{n-i+1} s_{n-i+1} y_i^{-1} y_i \qquad (1 \le i \le n). \tag{4}$$

Proof. (1) and (2) follow immediately from the definition of the x's and y's. We employ induction on i to establish (3). If $i = 1$, then, since the idempotents of S commute,

$$a_1 t_1 x_1^{-1} x_1 = a_1 t_1 t_1^{-1} s_1 s_1^{-1} t_1 = a_1 s_1 s_1^{-1} t_1 = ax_1$$

as required. Assuming (3) holds for some k, $1 \le k < n$,

$$\begin{aligned} a_{k+1} t_{k+1} x_{k+1}^{-1} x_{k+1} &= a_{k+1} t_{k+1} t_{k+1}^{-1} s_{k+1} x_k^{-1} x_k s_{k+1}^{-1} t_{k+1} \\ &= a_{k+1} s_{k+1} x_k^{-1} x_k s_{k+1}^{-1} t_{k+1} \qquad \text{(idempotents commute)} \\ &= a_k t_k x_k^{-1} x_k s_{k+1}^{-1} t_{k+1} \\ &= ax_k s_{k+1}^{-1} t_{k+1} \qquad \text{(inductive hypotheses)} \\ &= ax_{k+1}, \quad \text{which is the desired result.} \end{aligned}$$

The proof of (4) is similar to that of (3). □

It will now be convenient to use the notation $s_i^{-1} s_i = e_i$ and $t_{n-i+1}^{-1} t_{n-i+1} = f_i$ for $i = 1, 2, \dots, n$. We shall verify that the following is a scheme (of length $3n$) over $aS \cup a'S$ and B joining (a, b) to (a', b'):

$$\begin{array}{ll} ax_0 = ax_0 e_1, & \\ ax_1 = ax_1 e_2, & s_1 b = t_1 b_2, \\ ax_2 = ax_2 e_3, & s_2 b_2 = t_2 b_3, \\ \vdots & \vdots \\ ax_{n-1} = ax_{n-1} e_n, & s_{n-1} b_{n-1} = t_{n-1} b_n, \\ ax_n y_0 = ax_n y_0 f_1, & s_n b_n = t_n b', \\ ax_n y_1 = ax_n y_1 f_2, & t_n b' = s_n b_n, \end{array}$$

$$
\begin{array}{rlrl}
ax_n y_2 &= ax_n y_2 f_3, & t_{n-1}b_n &= s_{n-1}b_{n-1}, \\
&\vdots & &\vdots \\
ax_n y_{n/2-1} &= ax_n y_{n/2-1} f_{n/2}, & t_{n/2+2}b_{n/2+3} &= s_{n/2+2}b_{n/2+2} \\
ax_n y_{n/2} &= a' y_n x_{n/2}, & t_{n/2+1}b_{n/2+2} &= s_{n/2+1}b_{n/2+1}, \\
a' y_n x_{n/2-1} e_{n/2} &= a' y_n x_{n/2-1}, & t_{n/2}b_{n/2+1} &= s_{n/2}b_{n/2}, \\
&\vdots & &\vdots \\
a' y_n x_2 e_3 &= a' y_n x_2, & t_3 b_4 &= s_3 b_3, \\
a' y_n x_1 e_2 &= a' y_n x_1, & t_2 b_3 &= s_2 b_2, \\
a' y_n x_0 e_1 &= a' y_n x_0, & t_1 b_2 &= s_1 b, \\
a' y_{n-1} f_n &= a' y_{n-1}, & s_1 b &= t_1 b_2, \\
&\vdots & &\vdots \\
a' y_2 f_3 &= a' y_2, & s_{n-2}b_{n-2} &= t_{n-2}b_{n-1}, \\
a' y_1 f_2 &= a' y_1, & s_{n-1}b_{n-1} &= t_{n-1}b_n, \\
a' y_0 f_1 &= a' y_0, & s_n b_n &= t_n b'.
\end{array}
$$

We begin by checking that these equalities hold. Because the equalities on the right appear in the original scheme we need only consider those on the left.

(1) $\quad ax_i = ax_i e_{i+1} \qquad (0 \leq i \leq n-1).$

For $i = 0$, $\quad ax_0 e_1 = ae_1 = a_1 s_1 s_1^{-1} s_1 = a_1 s_1 = a = ax_0.$

For $0 < i \leq n-1$,
$$
\begin{aligned}
ax_i e_{i+1} &= a_i t_i x_i^{-1} x_i e_{i+1} && \text{(Lemma 4.1(3))} \\
&= a_{i+1} s_{i+1} x_i^{-1} x_i e_{i+1} \\
&= a_{i+1} s_{i+1} x_i^{-1} x_i && \text{(idempotents commute)} \\
&= a_i t_i x_i^{-1} x_i \\
&= ax_i && \text{(Lemma 4.1(3))}.
\end{aligned}
$$

(2) $\quad ax_n y_i = ax_n y_i f_{i+1} \qquad (0 \leq i \leq n/2 - 1).$

$$
\begin{aligned}
ax_n y_i &= ax_{n-i} y_i^{-1} y_i && \text{(Lemma 4.1(1))} \\
&= ax_{n-i} f_{i+1} y_i^{-1} y_i \\
&= ax_{n-i} y_i^{-1} y_i f_{i+1} && \text{(idempotents commute)} \\
&= ax_n y_i f_{i+1} && \text{(Lemma 4.1(1))}.
\end{aligned}
$$

(3) $ax_n y_{n/2} = a' y_n x_{n/2}$

$$
\begin{aligned}
ax_n y_{n/2} &= ax_{n/2} y_{n/2}^{-1} y_{n/2} && \text{(Lemma 4.1(1))}\\
&= a_{n/2} t_{n/2} x_{n/2}^{-1} x_{n/2} y_{n/2}^{-1} y_{n/2} && \text{(Lemma 4.1(3))}\\
&= a_{n/2+1} s_{n/2+1} y_{n/2}^{-1} y_{n/2} x_{n/2}^{-1} x_{n/2} && \text{(idempotents commute)}\\
&= a' y_{n/2} x_{n/2}^{-1} x_{n/2} && \text{(Lemma 4.1(4))}\\
&= a' y_n x_{n/2} && \text{(Lemma 4.1(2))}.
\end{aligned}
$$

The remaining two groups of equalities in the left hand column are similar to the second and first groups respectively and the proofs that the equalities hold are analogous to the proofs given in (2) and (1) above.

Finally, it is necessary to verify that successive equalities "match up" properly. For example, the first n equalities on the left and $n-1$ equalities on the right may be written as follows:

$$
\begin{array}{rcl}
a = \left(as_1^{-1}\right)s_1 & & \\
\left(as_1^{-1}\right)t_1 = \left(ax_1 s_2^{-1}\right)s_2, & & s_1 b = t_1 b_2,\\
\left(ax_1 s_2^{-1}\right)t_2 = \left(ax_2 s_3^{-1}\right)s_3, & & s_2 b_2 = t_2 b_3,\\
\vdots & & \vdots\\
\left(ax_{n-2} s_{n-1}^{-1}\right)t_{n-1} = \left(ax_{n-1} s_n^{-1}\right)s_n, & & s_{n-1} b_{n-1} = t_{n-1} b_n.
\end{array}
$$

By continuing in this way it is easy to see that the equalities are correctly connected and, therefore, constitute a proper scheme.

We have proven in the above discussion that every inverse semigroup is left absolutely flat and may now state the main theorem.

THEOREM 4.2. *Inverse semigroups are* (*left*, *right*) *absolutely flat*.

Completely injective semigroups (monoids all of whose left and right S-sets are injective) are inverse (see **[3]**) and hence, by the theorem above, absolutely flat.

The referee has pointed out that the proof of Theorem 4.2 in fact establishes the following stronger result: if T is a submonoid of an inverse monoid S, then any embedding of right (left) S-sets is preserved on forming tensor products over T with any left (right) T-set.

Among unions of groups the absolutely flat semigroups are exactly those which are inverse.

THEOREM 4.3. *Let S be a union of groups. Then S is absolutely flat iff S is a semilattice of groups.*

Proof. Without loss of generality, assume S is a monoid. If S is a semilattice of groups, S is inverse and, hence, absolutely flat. Suppose S is an absolutely flat union of groups. Then S is a semilattice of completely simple semigroups ([**1**], p. 126), i.e. $S = \bigcup\{S_\gamma \mid \gamma \in \Gamma\}$ where Γ is a semilattice and S_γ is completely simple for each $\gamma \in \Gamma$. Choose any $\delta \in \Gamma$. Then $S_{[\delta)} = \bigcup\{S_\gamma \mid \gamma \in \Gamma,\ \gamma \geq \delta\}$ is also absolutely flat because it is a filter in S (Proposition 2.9). Hence, $S_{[\delta)}$ and, therefore, S_δ is left and right reversible. However, because S_δ is completely simple it can be left and right reversible only if it is a group. Thus, each S_γ is a group and S is, therefore, a semilattice of groups. □

Note. For an alternative proof that every semilattice of groups is absolutely flat, see Kilp [**8**].

COROLLARY 4.4. *A band is absolutely flat iff it is a semilattice.*

COROLLARY 4.5. *A completely simple semigroup is absolutely flat iff it is a group.*

In the next section we demonstrate that absolutely flat semigroups need not be inverse.

5. Primitive regular semigroups. In this section, we will characterize a class of semigroups with 0 which are absolutely flat because, very roughly speaking, every scheme behaves like a scheme over a group or reduces to a trivial scheme involving 0.

Recall that a regular semigroup with 0 is called *primitive* if each of its non-zero idempotents is primitive (i.e., minimal non-zero with respect to the usual partial order on idempotents: $e \leq f$ iff $e = ef = fe$).

DEFINITION 5.1. Suppose S is a semigroup with 0, and $x \in S$. Then
$\mathrm{ann}_l(x) = \{s \in S \mid sx = 0\}$ is the *left annihilator of* x and
$\mathrm{stab}_l(x) = \{s \in S \mid sx = x\}$ is the *left stabilizer of* x.
$\mathrm{ann}_r(x)$ and $\mathrm{stab}_r(x)$ are defined similarly.

THEOREM 5.2. *Let S be a primitive regular semigroup. Then S is left* (*resp. right*) *absolutely flat iff S satisfies the condition*
(Ann_l): $(\forall x, y \in S)$ $(\mathrm{ann}_l(x) = \mathrm{ann}_l(y)$ *implies* $xS = yS)$
(resp. (Ann_r)).

Proof. Suppose that S satisfies (Ann_l). We shall prove that S^1 is a left absolutely flat monoid, and hence S is a left absolutely flat semigroup. To this end, note first that S^1 satisfies the condition

$$(*):\qquad (\forall\, x, y \in S^1)(x = 1 \text{ or } x \in yS \text{ or } (\mathrm{ann}_l(x) \cap \mathrm{stab}_l(y)) \cup (\mathrm{ann}_l(y) \cap \mathrm{stab}_l(x)) \neq \varnothing).$$

Indeed, for x, $y \in S^1$, if $x \neq 1$ and $x \notin yS$ (so $y \neq 1$), then x and y are elements of S for which $xS \neq yS$. By (Ann_l) there exists an element $w \in (\mathrm{ann}_l(x)\backslash\mathrm{ann}_l(y)) \cup (\mathrm{ann}_l(y)\backslash\mathrm{ann}_l(x))$. Since S is left 0-stratified (see for example [2], pp. 23 ff.), either $y \in Swy$ (if $w \in \mathrm{ann}_l(x)\backslash\mathrm{ann}_l(y)$) or $x \in Swx$ (if $w \in \mathrm{ann}_l(y)\backslash\mathrm{ann}_l(x)$). In the first case, if $y = uwy$ for $u \in S$, then $uw \in \mathrm{ann}_l(x) \cap \mathrm{stab}_l(y)$; in the second case, if $x = vwx$ for $v \in S$, then $vw \in \mathrm{ann}_l(y) \cap \mathrm{stab}_l(x)$.

Suppose now $A \in$ *Ens*-S^1 and $B \in S^1$-*Ens*. We shall prove by induction on n that, for a, $a' \in A$ and b, $b' \in B$, the existence of a scheme of length n over A and B joining (a, b) to (a', b') implies the existence of a scheme over $aS^1 \cup a'S^1$ and B joining these two pairs. Then, by Lemma 2.2, B will be flat and hence the proof of the sufficiency of (Ann_l) will be complete. Throughout this proof, if $s \in S$ then s' will denote any inverse of s in S.

If $n = 1$ we must consider schemes of the form

$$\begin{aligned} a &= a_1 s_1, & & \\ a_1 t_1 &= a', & s_1 b &= t_1 b' \end{aligned}$$

where s_1, $t_1 \in S^1$ and $a_1 \in A$. If $t_1 = 1$ then $a_1 = a' \in aS^1 \cup a'S^1$ and so the original scheme itself is of the required type. If $t_1 \in s_1 S$ then $t_1 = s_1 u$ for some $u \in S$, so we may calculate $as_1't_1 = a_1 s_1 s_1' s_1 u = a_1 s_1 u = a_1 t_1 = a'$. Hence, in this case, the scheme

$$\begin{aligned} a &= (as_1')s_1, & & \\ (as_1')t_1 &= a', & s_1 b &= t_1 b' \end{aligned}$$

establishes the result. Finally, if $zt_1 = t_1$ and $zs_1 = 0$, *or* $zt_1 = 0$ and $zs_1 = s_1$ for some $z \in S$ (by $(*)$), we have $zs_1 b = zt_1 b'$, from which it follows (in either case) that $0b = t_1 b' = s_1 b = 0b'$. The scheme

$$\begin{aligned} a &= (as_1')s_1, & & \\ (as_1')0 &= (a't_1')0, & s_1 b &= 0b', \\ (a't_1')t_1 &= a', & 0b' &= t_1 b' \end{aligned}$$

furnishes the desired conclusion.

Assume now that appropriate new schemes may be found for all schemes of length k for $1 \leq k < n$. Consider any scheme

$$\begin{array}{rlrl} a &= a_1 s_1, & & \\ a_1 t_1 &= a_2 s_2, & s_1 b &= t_1 b_2, \\ a_2 t_2 &= a_3 s_3, & s_2 b_2 &= t_2 b_3, \\ & \vdots & & \vdots \\ a_{n-1} t_{n-1} &= a_n s_n, & s_{n-1} b_{n-1} &= t_{n-1} b_n, \\ a_n t_n &= a', & s_n b_n &= t_n b' \end{array}$$

of length n over A and B joining (a, b) to (a', b').

If $t_1 = 1$

$$\begin{array}{rlrl} a &= a_2(s_2 s_1), & & \\ a_2 t_2 &= a_3 s_3, & (s_2 s_1) b &= t_2 b_3, \\ & \vdots & & \vdots \\ a_n t_n &= a', & s_n b_n &= t_n b' \end{array}$$

is a scheme of length $n - 1$ over A and B joining (a, b) to (a', b'), and the inductive hypothesis gives the result. By symmetry, the case in which $s_n = 1$ is handled similarly. If $t_1 \in s_1 S$ then $t_1 = s_1 u$ for some $u \in S$. It follows that $a_1 t_1 = a_1(s_1 u) = au$ and so, since the pairs $(a_1 t_1, b_2)$ and (a', b') are joined over $a_1 t_1 S^1 \cup a' S^1$ and B by some scheme (using the inductive hypothesis again), they are a fortiori joined by a scheme over $aS^1 \cup a'S^1$ and B. Moreover, the pairs (a, b) and $(a_1 t_1, b_2)$ (by the $n = 1$ case) are also joined by such a scheme. The latter two schemes may be spliced together to join (a, b) and (a', b') over $aS^1 \cup a'S^1$ and B as required. By symmetry, the case in which $s_n \in t_n S$ is handled similarly. Finally, using $(*)$ if $z \in S$ exists for which $zt_1 = t_1$ and $zs_1 = 0$ *or* $zt_1 = 0$ and $zs_1 = s_1$, and $w \in S$ exists for which $ws_n = s_n$ and $wt_n = 0$ *or* $ws_n = 0$ and $wt_n = t_n$, then we have $zs_1 b = zt_1 b_2$ (implying $0b = t_1 b_2 = s_1 b = 0b_2 = \cdots = 0b_n = 0b'$) and $ws_n b_n = wt_n b'$ (implying $0b' = s_n b_n = t_n b' = 0b_n = \cdots = 0b_2 = 0b$). In this case, the scheme

$$\begin{array}{rlrl} a &= (as_1')s_1, & & \\ (as_1')0 &= (a't_n')0, & s_1 b &= 0b', \\ (a't_n')t_n &= a', & 0b' &= t_n b' \end{array}$$

joins (a, b) and (a', b') over $aS^1 \cup a'S^1$ and B. Thus, B is flat and S is left absolutely flat.

Conversely, assume that S does not satisfy the condition (Ann_l), and so there exist $x, y \in S$ such that $\mathrm{ann}_l(x) = \mathrm{ann}_l(y)$ but $xS \neq yS$ (and, hence, $xS \cap yS = \{0\}$). Without loss of generality, we assume $x \neq 0$. We prove $S^1/\theta(x, y)$ (see §1) is not flat in S^1-*Ens* by showing that the induced map $(xS \cup yS) \otimes_{S^1} S^1/\theta(x, y) \to S^1 \otimes_{S^1} S^1/\theta(x, y)$ is not an embedding.

Clearly $x \otimes \bar{1} = y \otimes \bar{1}$ in $S^1 \otimes_{S^1} S^1/\theta(x, y)$. Assume for the moment that $x \otimes \bar{1} = y \otimes \bar{1}$ in $(xS \cup yS) \otimes_{S^1} S^1/\theta(x, y)$. Then by Lemma 1.1 there exist $a_1, \dots, a_n \in xS \cup yS$, $s_1, \dots, s_n \in S^1$, $t_1, \dots, t_n \in S^1$ where $\{s_i, t_i\} = \{x, y\}$ for $i = 1, \dots, n$ such that

$$\begin{aligned} x &= a_1 s_1, \\ a_1 t_1 &= a_2 s_2, \\ a_2 t_2 &= a_3 s_3, \\ &\vdots \\ a_n t_n &= y. \end{aligned}$$

Now $a_1 \notin yS$ for otherwise $x \in yS$. Hence, $a_1 t_1 \in xS$. $a_1 t_1 \neq 0$ because otherwise $a_1 \in \mathrm{ann}_l(t_1) = \mathrm{ann}_l(s_1)$ implying $a_1 s_1 = x = 0$ which is a contradiction. Thus $0 \neq a_1 t_1 \in xS$. By induction it may be established that $0 \neq a_i t_i \in xS$ for $i = 1, \dots, n$. In particular $0 \neq a_n t_n = y \in xS$ which is impossible since $xS \cap yS = \{0\}$. This contradiction concludes the proof that $S^1/\theta(x, y)$ is not flat in S^1-*Ens*, and, therefore, S is not left absolutely flat. □

Primitive regular semigroups may be characterized as those semigroups with 0 which are 0-direct unions of completely 0-simple semigroups ([**2**], p. 28). These latter semigroups are Rees matrix semigroups $\mathfrak{M}^0[G; I, \Lambda; P]$ where P is a regular $\Lambda \times I$ sandwich matrix with entries in G^0. We will denote by $s(P)$ (the support of P) the $\Lambda \times I$ matrix obtained by replacing all of the non-zero entries of P by the symbol 1.

COROLLARY 5.3. *A Rees matrix semigroup* $S = \mathfrak{M}^0[G; I, \Lambda; P]$ *is left* (*right*) *absolutely flat iff no two columns* (*rows*) *of* $s(P)$ *are identical.*

Proof. S has condition (Ann_l) ((Ann_r)) of Theorem 5.2 iff $s(P)$ does not possess two identical columns (rows). □

The example following Proposition 2.5 is isomorphic to S^1, where $S = \mathfrak{M}^0[\{1\}; \{1,2\}, \{1\}; [1\ 1]]$, and thus is a right absolutely flat monoid which is not (as also noted earlier) left absolutely flat.

Any finite congruence-free semigroup with 0 is absolutely flat (see [**4**], p. 84). Furthermore, if $S = \mathring{\bigcup} \{S_\gamma \mid \gamma \in \Gamma\}$ is a 0-direct union decomposition of a primitive regular semigroup S into completely 0-simple semigroups S_γ ($\gamma \in \Gamma$), S is absolutely flat iff S_γ is absolutely flat for each $\gamma \in \Gamma$.

Any regular semigroup S for which $|S| \leq 4$ is completely regular i.e. a union of groups. Therefore, using Theorem 4.3, it is easy to see that if $|S| \leq 4$, S is absolutely flat iff S is inverse. It follows that a non-inverse absolutely flat semigroup must have at least 5 elements. Consider the semigroup $S = \{0, e, f, g, s\}$ with the following multiplication table.

	0	e	f	g	s
0	0	0	0	0	0
e	0	e	f	e	f
f	0	e	f	0	0
g	0	g	s	g	s
s	0	g	s	0	0

This semigroup is isomorphic to $\mathfrak{M}^0[G; I, \Lambda; P]$ where $G = \{1\}$ is the one element group, $I = \Lambda = \{1, 2\}$ and $P = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. S is a 5-element non-inverse congruence-free semigroup with 0 which, by Corollary 5.3, is absolutely flat. In fact for any natural number $n \geq 5$, there exists a non-inverse, absolutely flat semigroup with cardinality n. (One could, for example, adjoin successive new identity elements to the semigroup provided above.)

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