PRODUCTS OF POSITIVE REFLECTIONS IN REAL ORTHOGONAL GROUPS

Dragomir Z. Djokovic
PRODUCTS OF POSITIVE REFLECTIONS
IN REAL ORTHOGONAL GROUPS

Dragomir Ž. Djoković

Let $O(f)$ be the orthogonal group of a symmetric bilinear form $f$ defined on a finite-dimensional real vector space $V$. If $f$ is indefinite then $O(f)$ has two conjugacy classes of reflections, one of which consists of so called positive reflections. We denote by $G^+$ the subgroup of $O(f)$ generated by all positive reflections. In this paper we describe this subgroup and solve the length problem in $G^+$ with respect to the distinguished set of generators. When $f$ is non-degenerate this problem was solved by J. Malzan. Our proof (in the case of arbitrary $f$) is shorter and completely different from his proof.

**Introduction.** Let $O(f)$ be the orthogonal group of a symmetric bilinear form $f$ defined on a finite-dimensional real vector space $V$. If $f$ is indefinite then $O(f)$ has two conjugacy classes of reflections, one of which consists of so called positive reflections. We denote by $G^+$ the subgroup of $O(f)$ generated by all positive reflections. In this paper we solve the length problem in $G^+$ with respect to the distinguished set of generators. When $f$ is non-degenerate this problem was solved by J. Malzan. Our proof (in the case of arbitrary $f$) is shorter and completely different from his proof.

A non-isotropic vector $a$ determines a unique orthogonal reflection $R_a$ and we say that $R_a$ is positive if $f(a, a) > 0$. The weak orthogonal group $O^*(f)$ consists of all isometries which fix every vector in Rad $V$. To avoid trivial and known cases let us assume that $f$ is indefinite, i.e., that $f(x, x)$ takes both positive and negative values. Then $O^*(f) \supset G^+ \supset O^*_1(f)$ where $O^*_1(f)$ denotes the identity component of $O^*(f)$. Moreover $O^*(f)/O^*_1(f) \cong Z_2 \times Z_2$ and $G^+/O^*_1(f) \cong Z_2$.

Our main theorem (Theorem 2) gives explicit formulas for the length of any $u \in G^+$ with respect to the generating set consisting of all positive reflections. When $f$ is nondegenerate this result is due to J. Malzan [5]. The proof is based on some earlier results of M. Götzky [3] on $O^*(f)$. One should point out that Götzky considers also weak unitary groups and his underlying field $F$ is arbitrary (char $F \neq 2$ in the case of $O^*(f)$).

The main idea of the proof is to take a shortest representation of $u \in G^+$ as a product of reflections and then try to convert all reflections
into positive ones. This method is effective in the generic case; the exceptional cases are treated separately.

1. Weak orthogonal groups in general. Let $V$ be a finite-dimensional vector space over a field $F$, char $F \neq 2$, and let $f$ be a symmetric bilinear form on $V$. An automorphism $u$ of $V$ is called an isometry if $f(u(x), u(y)) = f(x, y)$ for all $x, y \in V$. The group of all isometries will be denoted by $O(f)$ and we refer to it as the orthogonal group of the form $f$. (Note that we allow $f$ to be degenerate.)

The weak orthogonal group $O^*(f)$ is the subgroup of $O(f)$ consisting of all isometries which fix every vector in the radical $\text{Rad} V = \{ x \in V : f(x, y) = 0, \forall y \in V \}$.

For $u \in O(f)$ we define its fixed space $\text{Fix } u$ and its residual space $\text{Res } u$ by

$$\text{Fix } u = \ker (u - 1), \quad \text{Res } u = \text{im} (u - 1).$$

We also define the residue $r(u)$ and the radical residue $r_0(u)$ of $u$ to be

$$r(u) = \dim \text{Res } u, \quad r_0(u) = \dim (\text{Res } u \cap \text{Rad } V).$$

If $a$ is a non-isotropic vector, i.e., $f(a, a) \neq 0$, then the transformation $R_a : V \to V$ defined by

$$R_a(x) = x - 2f(a, x)f(a, a)^{-1}a$$

belongs to $O^*(f)$ and is called a reflection. We have

$$\text{Fix } R_a = \langle a \rangle^\perp, \quad \text{Res } R_a = \langle a \rangle$$

and $R_a(a) = -a$. (For any subspace $W$ of $V$ we denote by $W^\perp$ the orthogonal complement of $W$ with respect to the form $f$.)

We shall now state some results of M. Götzky [3] concerning the group $O^*(f)$. (In his paper he also treats the weak unitary groups but we shall not need those results.) For further results and generalizations we refer the reader to a paper of E. Ellers [2].

Every $u \in O^*(f)$ can be expressed as a product of reflections

$$u = R_{a_1}R_{a_2} \cdots R_{a_m}. \quad (1)$$

Since $\det R_a = -1$ for every reflection $R_a$, it follows that $\det u = \pm 1$ for all $u \in O^*(f)$. Moreover the subgroup

$$SO^*(f) = \{ u \in O^*(f) : \det u = 1 \}$$

has index 2 in $O^*(f)$. 

For \( u \in O^*(f) \) we shall denote by \( l(u) \) the length of \( u \) with respect to
the generating set consisting of all reflections. Thus \( l(u) \) is the smallest
integer \( m(\geq 0) \) for which a factorization (1) exists.

**Theorem 1.** (M. Götzky) For \( u \in O^*(f) \) we have \( l(u) = r(u) + r_0(u) \)
except when \((\text{Fix } u)^\perp \) is totally isotropic and \( u \neq 1 \). In the exceptional case
we have \( l(u) = r(u) + r_0(u) + 2 \).

When \( f \) is non-degenerate, i.e., \( \text{Rad } V = 0 \); this theorem is due to P.
Scherk [6].

2. **Real case and the statement of the main result.** From now on we
shall assume that \( F \) is the real field \( R \). A vector \( x \) is called *positive* (resp. *negative*)
if \( f(x, x) > 0 \) (resp. \( f(x, x) < 0 \)). We shall denote by \( n \) the
dimension of \( V \) and by \( (p, q, s) \) the signature of \( f \). This means that every
orthogonal basis of \( V \) consists of \( p \) positive vectors, \( q \) negative vectors, and
\( s \) isotropic vectors.

A reflection \( R_a \) is *positive* (resp. *negative*) if \( a \) is positive (resp. negative).
It follows from Witt's theorem that all positive (resp. negative)
reflections are conjugate in \( O^*(f) \). We shall denote by \( G^+ \) (resp. \( G^- \)) the
subgroup of \( O^*(f) \) generated by all positive (resp. negative) reflections. If
\( p = 0 \), i.e., \( f \) is negative semidefinite then there are no positive reflections
and we have \( G^+ = \{1\} \) and \( G^- = O^*(f) \). If \( q = 0 \) then \( G^+ = O^*(f) \) and
\( G^- = \{1\} \).

In view of these remarks and Theorem 1 we *shall assume throughout
that \( f \) is indefinite, i.e., \( p \geq 1 \) and \( q \geq 1 \). Clearly \( O(f) \) and \( O^*(f) \) are real
algebraic groups and so Lie groups. Let \( O_1^*(f) \) be the identity component
of \( O^*(f) \) viewed as a Lie group.

Let \( V = V_1 \oplus \text{Rad } V \) and let \( f_1 \) be the restriction of \( f \) to \( V_1 \times V_1 \).
Clearly \( f_1 \) is a non-degenerate symmetric bilinear form on \( V_1 \) of signature
\((p, q, 0)\). Then the elements \( u \) of \( O(f) \) are represented by matrices

\[
    u = \begin{pmatrix} u_1 & 0 \\ v & u_0 \end{pmatrix}
\]

where \( u_1 \in O(f_1) \), \( u_0 \) is an automorphism of \( \text{Rad } V \) and \( v : V_1 \to \text{Rad } V \) is
an arbitrary linear map. We have \( u \in O^*(f) \) if and only if \( u_0 = 1 \).

**Lemma 1.** \( O^*(f)/O_1^*(f) = Z_2 \times Z_2 \).
Proof. If $s = 0$ this is well known, see e.g. [4, Lemma 2.4(b), p. 451]. In general the assertion follows from this special case and the above matrix description of elements of $O^*(f)$.

**Corollary.** $G^+ \cdot O^*_1(f)/O^*_1(f)$ and $G^- \cdot O^*_1(f)/O^*_1(f)$ are cyclic groups of order two. The three subgroups $G^+ O^*_1(f)$, $G^- O^*_1(f)$, and $SO^*(f)$ are distinct.

Proof. Since all positive (resp. negative) reflections are conjugate in $O^*(f)$, they lie in a single connected component of $O^*(f)$. This implies the first assertion. We have $G^+ O^*_1(f) \neq G^- O^*_1(f)$ because $O^*(f)$ is generated by reflections. These two groups are different from $SO^*(f)$ because $\det R = -1$ for each reflection $R$.

For $u \in G^+$ we shall denote by $l^+(u)$ the length of $u$ with respect to the generating set consisting of all positive reflections. We can now state our main result.

**Theorem 2.** We have $G^+ \supset O^*_1(f)$. For $u \in G^+$ we have $l^+(u) = r(u) + r_0(u)$ except in the following cases:

(i) The subspace $(\text{Fix } u)^\perp$ is negative semidefinite and $u \neq 1$,

(ii) $u^2 = 1$ and $u(x) = -x$ for some negative vector $x$.

In the exceptional cases we have $l^+(u) = r(u) + r_0(u) + 2$.

When $f$ is non-degenerate this theorem is due to J. Malzan [5]. Our proof below even in the more general case is simpler and more elementary than his. For instance we do not need the detailed knowledge of the conjugacy classes of $O(f)$, which is heavily used in [5] in the case when $f$ is non-degenerate.

3. **Proofs.** We shall assume that the reader is familiar with Götzky's paper [3] and we shall use some of his technical lemmas in addition to Theorem 1. The main tool in our proof is the following technical lemma.

**Lemma 2.** Let $a$, $b$, $c$ be linearly independent vectors with a positive and $b$ and $c$ negative. If the sequence $a$, $b$, $c$ is not orthogonal then the isometry $u = R_a R_b R_c$ can be written as a product of three positive reflections.

Proof. Without any loss of generality we may assume that $f(a, a) = 1$ and $f(b, b) = f(c, c) = -1$. Set $f(a, b) = \alpha$, $f(a, c) = \beta$, and $f(b, c) = \gamma$. By hypothesis at least one of $\alpha$, $\beta$, $\gamma$ is non-zero. Since $R_b R_c = R_d R_b$, we have $R_a R_b R_c = R_d R_b$. Therefore, $u = R_a R_b R_c$ can be written as a product of three positive reflections.
where $d = R_b(c)$, we may assume that in fact $\beta$ or $\gamma$ is non-zero. Then for $e = (\eta - \alpha \xi)a + \xi b$ we have
\[
f(e, e) = (\eta - \alpha \xi)^2 - \xi^2 + 2\alpha \xi(\eta - \alpha \xi) = \eta^2 - (1 + \alpha^2)\xi^2,
\]
and
\[
\Delta = \begin{vmatrix} f(c, c) & f(c, e) \\ f(e, c) & f(e, e) \end{vmatrix} = (1 + \alpha^2)\xi^2 - \eta^2 - (\beta \eta + (\gamma - \alpha \beta)\xi)^2.
\]
Since $\beta$ or $\gamma$ is not zero, we can choose $\xi$ and $\eta$ so that $f(e, e) = -1$ and $\Delta < 0$. By Dreispiegelungssatz [1, Proposition 6.1] the product $R = R_aR_bR_e$ is a reflection. Since $b$ and $e$ are negative vectors, we have $R_bR_e \in O^+_1(f)$ and so $R$ must be a positive reflection by Lemma 1, Cor. We have $u = RR_eR_c$ where $R_e$ and $R_c$ are negative reflections. Since $\Delta < 0$ the space $W = \langle c, e \rangle$ is a hyperbolic plane. We claim that $R_eR_c$ is a product of two positive reflections. To prove this it suffices to consider the restrictions of $R_e$ and $R_c$ to $W$. Then in $W$ the operators $-R_e$ and $-R_c$ are positive reflections whose product is $R_eR_c$. This completes the proof.

Proof of Theorem 2. Let $u \in G^+ \cdot O^+_1(f)$.

Case 1. $u$ is not exceptional, i.e., neither (i) nor (ii) holds.

Clearly $l^+(u) \geq l(u)$ and by Theorem 1, $l(u) = r(u) + r_0(u)$. Write $m = l(u)$ and let (1) be a factorization of $u$ into a product of $m$ reflections containing a maximal number, say $k$, of positive reflections. We have to prove that $k = m$.

This is clear if $m = 0$, i.e., $u = 1$. Otherwise we prove first that $k \geq 1$. Since (i) does not hold there exists a positive vector $a \in (\text{Fix } u \perp$. It follows from [3, Hilfssatz 2.1, p. 385] that for $v = R_a u$ we have $r(v) = r(u)$ and $r_0(v) = r_0(u) - 1$. By Theorem 1 $l(v) = m - 1$ and since $u = R_a v$ we have $k \geq 1$. We may assume that the vectors $a_i$ are positive for $1 \leq i \leq k$ and negative for $k < i \leq m$.

Now assume that $k < m$. By Lemma 1, Cor. $m - k$ must be even, and so $k \leq m - 2$. Assume that for every pair of indices $(i, j)$ such that $1 \leq i < j \leq m$ and $j > k$ we have $a_i \perp a_j$. Since (ii) does not hold there must exist a pair of indices $(i, j)$ such that $1 \leq i < j \leq k$ and $f(a_i, a_j) \neq 0$. Without any loss of generality we may assume that $f(a_{k-1}, a_k) \neq 0$. Let
Let $b \in \langle a_k, a_{k+1} \rangle$ be a positive vector such that $b \notin \langle a_k \rangle$. By Dreispiegelungssatz the product $R_b R_{a_k} R_{a_{k+1}}$ is a reflection, say $R_c$, and by Lemma 1, Cor. it is a negative reflection. Thus we can replace in (1) the product $R_{a_k} R_{a_{k+1}}$ by $R_b R_c$. Note that $f(a_{k-1}, c) \neq 0$. This shows that we may assume that there exists a pair of indices $(i, j)$ such that $1 \leq i < j \leq m$, $j > k$ and $f(a_i, a_j) \neq 0$. Without any loss of generality we may in fact assume that the sequence $a_k, a_{k+1}, a_{k+2}$ is not orthogonal. By Lemma 2 the product $R_{a_k} R_{a_{k+1}} R_{a_{k+2}}$ can be replaced by a product of three positive reflections. This contradicts the maximality of $k$.

Hence we have shown that $k = m$, and in particular $u \in G^+$. 

Case 2. (i) or (ii) holds. Let $m = r(u) + r_0(u)$. We prove first that $l^+(u) \geq m + 2$. This is clear if $l(u) = m + 2$. Otherwise we have $l(u) = m$ and since $\det u = (-1)^m$, it suffices to show that $u$ cannot be written as a product of $m$ positive reflections. Assume that it can and let (1) be such a factorization.

We claim that $a_k \in (\text{Fix } u)^\perp$ for all $k$. It suffices to prove this for $k = 1$. Thus let us assume that $a_1 \notin (\text{Fix } u)^\perp$. Then by [3, Proposition 2.1.3] for $v = R_{a_1} u$ we have $\text{Res } v = \text{Res } u \oplus \langle a_1 \rangle$, and consequently $r(v) = r(u) + 1$ and $r_0(v) = r_0(u)$. It follows that

$$l(v) = r(v) + r_0(v) = r(u) + r_0(u) + 1 = m + 1.$$ 

This is a contradiction since $v$ is a product of $m - 1$ reflections. Hence our claim is proved.

If (i) holds then since $a_k \in (\text{Fix } u)^\perp$ for all $k$, we conclude that all reflections in (1) are negative, contrary to our hypothesis. Thus if (i) holds then $l^+(u) \geq m + 2$.

Now assume that (ii) holds. Since $u^2 = 1$ we have $V = \text{Fix } u \oplus \text{Res } u$ and $\text{Fix } u \perp \text{Res } u$. Since $\text{Rad } V \subseteq \text{Fix } u$, it follows that $\text{Res } u$ is non-degenerate, $r_0(u) = 0$, and so $m = r(u)$. From (1) it follows that $\text{Res } u \subseteq \langle a_1, \ldots, a_m \rangle$, see e.g. [2, §3]. Since $r(u) = m$, we conclude that $a_1, \ldots, a_m$ is a basis of $\text{Res } u$.

We claim that this basis is orthogonal. It suffices to show that $a_i \perp a_i$ for $2 \leq i \leq m$. Let $b$ be a non-zero vector in $\text{Res } u$ such that $b \perp a_i$ for $2 \leq i \leq m$. Since $u$ is $-1$ on $\text{Res } u$, we have $u(b) = -b$. On the other hand it follows from (1) that $u(b) = R_{a_1}(b)$. Hence we have $R_{a_1}(b) = -b$ and so $a_1 \in \langle b \rangle$. This proves our claim.

Since the basis $a_1, \ldots, a_m$ of $\text{Res } u$ is orthogonal and each of these vectors is positive, we conclude that $\text{Res } u$ is a positive definite subspace.
This contradicts (ii). Hence also in the case (ii) we must have \( l^+(u) \geq m + 2 \).

It remains to show that \( l^+(u) \leq m + 2 \), i.e., that \( u \) can be written as a product of \( m + 2 \) positive reflections.

Assume first that (i) holds. Since the positive vectors form an open set in \( V \), we can choose a positive vector \( a \) such that \( a \not\in \text{Fix} \, u \). Since (i) holds we have also \( a \not\in (\text{Fix} \, u)^{\perp} \). Therefore \( \text{Fix} \, u \) is not invariant under \( R_a \). Hence we can choose \( x \in \text{Fix} \, u \) such that \( R_a(x) \not\in \text{Fix} \, u \). Let \( v = R_a(u) \) and note that

\[
v^2(x) = R_a u R_a(x) \neq R_a R_a(x) = x,
\]

and so \( v^2 \neq 1 \). By [3, Proposition 2.1.3] we have \( \text{Res} \, v = \text{Res} \, u \oplus \langle a \rangle \), and so \( r(v) = r(u) + 1 \) and \( r_0(v) = r_0(u) \). Thus \( v \) is non-exceptional and by the result of Case 1 we have

\[
l^+(v) = l(v) = r(v) + r_0(v) = m + 1.
\]

Since \( u = R_a v, u \) is a product of \( m + 2 \) positive reflections.

Now assume that (ii) holds. Choose an orthogonal basis \( a_1, \ldots, a_m \) of \( \text{Res} \, u \) such that \( a_1, \ldots, a_k \) are positive and \( a_{k+1}, \ldots, a_m \) are negative vectors. It follows from (ii) that \( k < m \). Let

\[
v = R_{a_1} \cdots R_{a_k} u.
\]

This \( v \) satisfies (i) and we have \( l(v) = m - k \). Hence \( l^+(v) = m - k + 2 \) by the result just proved above, and so \( l^+(u) \leq m + 2 \).

This completes the proof of Theorem 2.

**Remark.** It is easy to modify Theorem 2 so that it applies to the case when \( V \) is infinite-dimensional. Clearly if \( u \in G^+ \) then \( r(u) < \infty \). The length formulas of Theorem 2 remain valid.

**References**


Received September 10, 1981.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA N2L 3G1
PACIFIC JOURNAL OF MATHEMATICS
EDITORS
DONALD BABBITT (Managing Editor) J. DUGUNDJI
University of California
Los Angeles, CA 90024
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113
HUGO ROSSI
University of Utah
Salt Lake City, UT 84112
R. FINN and H. SAMELSON
Stanford University
Stanford, CA 94305
C. C. MOORE and ARTHUR OGUS
University of California
Berkeley, CA 94720

ASSOCIATE EDITORS
R. ARENS E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA
(1906–1982)

SUPPORTING INSTITUTIONS
UNIVERSITY OF ARIZONA UNIVERSITY OF OREGON
UNIVERSITY OF BRITISH COLUMBIA UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA UNIVERSITY OF HAWAII
MONTANA STATE UNIVERSITY UNIVERSITY OF TOKYO
UNIVERSITY OF NEVADA, RENO UNIVERSITY OF UTAH
NEW MEXICO STATE UNIVERSITY WASHINGTON STATE UNIVERSITY
OREGON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $132.00 a year (6 Vol., 12 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Copyright © 1983 by Pacific Journal of Mathematics
Driss Abouabdillah, Topologies de corps $A$ linéaires .......................... 257
Patrick Robert Ahern, On the behavior near a torus of functions holomorphic in the ball ......................................................... 267
Donald Werner Anderson, There are no phantom cohomology operations in $K$-theory ................................................................. 279
Peter Bloomfield, Nicolas P. Jewell and Eric Hayashi, Characterizations of completely nondeterministic stochastic processes ......................... 307
Sydney Dennis Bulman-Fleming and K. McDowell, Absolutely flat semigroups ................................................................. 319
C. Debiève, On a Radon-Nikodým problem for vector-valued measures .... 335
Dragomir Z. Djokovic, Products of positive reflections in real orthogonal groups ................................................................. 341
Thomas Farmer, The dual of the nilradical of the parabolic subgroups of symplectic groups ................................................................. 349
Gary R. Greenfield, Uniform distribution in subgroups of the Brauer group of an algebraic number field ................................................ 369
Paul Daniel Hill, When Tor($A$, $B$) is a direct sum of cyclic groups .......... 383
Hiroshi Maehara, Regular embeddings of a graph .................................. 393
Nikolaos S. Papageorgiou, Nonsmooth analysis on partially ordered vector spaces. I. Convex case .................................................. 403
Louis Jackson Ratliff, Jr., Powers of ideals in locally unmixed Noetherian rings ................................................................. 459
F. Dennis Sentilles and Robert Francis Wheeler, Pettis integration via the Stonian transform ................................................................. 473