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For an arbitrary parabolic subgroup P of the real or complex symplectic group, let N be the nilradical. Using Kirillov theory, a subset of the dual of N is found, whose complement has Plancherel measure zero. It is shown how these representations extend by combining with the oscillator representation of a lower rank symplectic group. A result is obtained concerning the commuting algebra of the restrictions to P of the principal series representation of the symplectic group induced from a unitary character of the opposite parabolic.

Introduction. In [6] and [2] there are irreducibility theorems for principal series representations of symplectic groups induced from unitary characters of certain maximal parabolic subgroups. Such a representation can be realized to act in the L^2-space of a nilpotent subgroup, the nilradical of the opposite parabolic. The irreducibility results are obtained in two stages. In the first stage the representation T is restricted to the opposite parabolic and the commuting algebra of the restriction is computed using nilpotent harmonic analysis. Since these parabolics are maximal subgroups, the full symplectic group is generated by the opposite parabolic together with a single element, say p. The commuting algebra of T is, therefore, the subalgebra of the commuting algebra of the restriction consisting of operators that commute with $T(p)$. The second stage is the difficult determination of which operators these are. For arbitrary parabolic subgroups of the symplectic groups, even for arbitrary maximal ones, it appears that the second stage of this program is not feasible and the irreducibility theorems must come from more powerful methods in semisimple representation theory. However, the first stage can be carried out in full generality, and it is of interest for the way in which the oscillator representation occurs and because of the computations involved in the nilpotent harmonic analysis. This is the topic of this paper.

To be more specific, let P be a parabolic subgroup of the symplectic group $\mathrm{Sp}(n, F)$, where $F = \mathbf{R}$ or $\mathbf{C}$. It is known that the principal series representation T of $\mathrm{Sp}(n, F)$ induced up from a unitary character of P can be realized to act in $L^2(N)$, where N is a nilpotent subgroup of $\mathrm{Sp}(n, F)$ and $N \cap P$ is trivial. Let M be the normalizer of N in P, then the semidirect product NM is a parabolic subgroup conjugate to P (NM is

the opposite parabolic) and N is the nilradical of NM. These groups are further described in §1. In §2, the theory of Kirillov is used to compute, except for a set of measure zero, the dual object of the nilradical of an arbitrary parabolic subgroup of $\mathrm{Sp}(n, F)$. Denote by $\mathfrak{N}$ the Lie algebra of the nilradical N and let $\mathfrak{N}^*$ be the real vector space dual of $\mathfrak{N}$. Sufficiently many coadjoint orbits in $\mathfrak{N}^*$ are found so that the complement of their union has measure zero in $\mathfrak{N}^*$. The irreducible representations corresponding to these orbits are computed and they form a subset of the dual object of N whose complement must have Plancherel measure zero.

Let $F = \mathbf{C}$, to simplify the remainder of this introduction, and let λ be an element of the dual of N, acting in a space $L^2(V)$. Define M_λ to be the subgroup of M consisting of all $m \in M$ such that the representations $z \to \lambda(m^{-1}zm)$ and λ are unitarily equivalent. Depending on the choice of P, we will see that M is isomorphic to a direct product of the form

$$\mathrm{Sp}(n_0, \mathbf{C}) \times \mathrm{Gl}(n_1, \mathbf{C}) \times \cdots \times \mathrm{Gl}(n_r, \mathbf{C}).$$

where $n = n_0 + n_1 + \cdots + n_r$, n_0 is nonnegative and $n_1, \ldots, n_r$ are positive. It turns out that M_λ is isomorphic to

$$\mathrm{Sp}(n_0, \mathbf{C}) \times O(n_1, \mathbf{C}) \times \cdots \times O(n_r, \mathbf{C}).$$

In §3, we extend λ to a unitary representation of NM_λ acting in the same Hilbert space $L^2(V)$. For this we need a representation D_λ of M_λ acting in $L^2(V)$ and satisfying

$$\lambda(m^{-1}zm) = D_\lambda(m)^{-1}\lambda(z)D_\lambda(m) \quad (z \in N, m \in M_\lambda).$$

The operators $D_\lambda(m)$ are given in terms of the oscillator representation of $\mathrm{Sp}(n_0, \mathbf{C})$ and a certain natural representation of $O(n_1, \mathbf{C}) \times \cdots \times O(n_r, \mathbf{C})$.

Finally, in §4, we indicate that there is a *-isomorphism between the commuting algebras of $T|_{NM}$ and D_λ. Throughout the paper we give the analogous results for the real field, which are more complicated to state.

1. Definitions. In this section, we define some of the groups and introduce some of the notation that will be used throughout the paper.

Let F be the real field or the complex field, $\mathbf{R}$ or C. We choose to define the symplectic groups over F as is done in [5] in order to have the most appealing treatment of the parabolic subgroups. Because of this choice, a notation for "secondary transpose" becomes very useful. For any positive integer p, let

$$J_p = \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot} & \\ & \cdot^{\cdot} & & \\ 1 & & & \end{bmatrix}$$

be the $p \times p$ matrix with ones along the secondary diagonal and zeros elsewhere. The subscript can be omitted when the context determines the dimensions of J_p. If x is a $p \times q$ matrix and x^t denotes the usual transpose of x, let the secondary transpose x^s be defined by

$$x^s = J_q X^t J_p.$$

It is easily verified that $(x^s)^t = (x^t)^s$ and, in the case of invertible matrices, $(x^s)^{-1} = (x^{-1})^s$. Thus, we can write x^{st-1} for $((x^s)^t)^{-1}$ without confusion. Also, $(xy)^s = y^s x^s$ whenever x and y are compatible for multiplication.

Fix a positive integer n and define the symplectic group

$$\mathrm{Sp}(n, F) = \left\{ g \in F^{2n \times 2n} : g \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} g^t = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} \right\}. \tag{1.1}$$

Another way of expressing (1.1) is the following: denote the elements of $F^{1 \times 2n}$ by $[xy]$ with $x, y \in F^{1 \times n}$. Then $\mathrm{Sp}(n, F)$ is the subgroup of $\mathrm{Gl}(2n, F)$ consisting of all linear transformations of $F^{1 \times 2n}$ which leave invariant the bilinear form

$$B([x_1 y_1], [x_2 y_2]) = y_1 x_2^s - x_1 y_2^s. \tag{1.2}$$

The parabolic subgroups of $\mathrm{Sp}(n, F)$, up to conjugacy, can be worked out from (1.2) as in [**8**] (§8). Let

$$\{n_1, n_2, \ldots, n_r, 2n_0, n_r, n_{r-1}, \ldots, n_1\}$$

be any sequence of integers with the indicated symmetry and satisfying the conditions that n_i is positive for $i = 1, \ldots, r$, n_0 is nonnegative, and $n = n_0 + n_1 + \cdots + n_r$. Consider the blocking scheme for $2n \times 2n$ matrices in which the diagonal blocks have dimensions

$$n_1 \times n_1, n_2 \times n_2, \ldots, n_r \times n_r, 2n_0 \times 2n_0, n_r \times n_r, \ldots, n_1 \times n_1,$$

respectively, from upper left to lower right. Corresponding to this blocking scheme we have

$$P = \{ g \in \mathrm{Sp}(n, F) : g \text{ is upper block-triangular} \}. \tag{1.3}$$

That is, $g \in P$ if and only if $g \in \mathrm{Sp}(n, F)$ and, with respect to the blocking scheme, the blocks in g below the block-diagonal are zero. As defined by (1.3), P is a parabolic subgroup of $\mathrm{Sp}(n, F)$. Moreover, each conjugacy class of parabolics in $\mathrm{Sp}(n, F)$ contains exactly one member of this form.

Also associated with the above blocking scheme are the subgroups of $\mathrm{Sp}(n, F)$ called N and M, which we now describe. First, denote by W the subgroup of $\mathrm{Gl}(n - n_0, F)$ consisting of all matrices that are lower

block-triangular with identity blocks along the block-diagonal; here the blocking scheme for $(n - n_0) \times (n - n_0)$ matrices has diagonal blocks of dimensions

$$n_r \times n_r, \ldots, n_2 \times n_2, n_1 \times n_1,$$

respectively, from upper left to lower right. Define N to be the subgroup consisting of all lower block-triangular elements of $\mathrm{Sp}(n, F)$ with identity blocks along the diagonal. Using (1.1), a matrix computation shows that

$$(1.4)\quad N = \left\{ \begin{bmatrix} w^{s-1} & 0 & 0 & 0 \\ -x^s w^{s-1} & I & 0 & 0 \\ y^s w^{s-1} & 0 & I & 0 \\ t & y & x & w \end{bmatrix} : w \in W;\ x, y \in F^{(n-n_0)\times n_0};\ wt^s - tw^s = yx^s - xy^s \right\}.$$

We will abbreviate the notation for elements of N by using (t, y, x, w). The multiplication law for N can then be written as

$$(1.5)\quad (t_1, y_1, x_1, w_1)(t_2, y_2, x_2, w_2) = \left((t_1 - y_1 x_2^s + x_1 y_2^s + w_1 t_2 w_2^s) w_2^{s-1}, y_1 + w_1 y_2, x_1 + w_1 x_2, w_1 w_2\right).$$

Let M be the block-diagonal subgroup of $\mathrm{Sp}(n, F)$. If we denote by $\mathrm{Gl}(n_r, F) \times \cdots \times \mathrm{Gl}(n_1, F)$ the subgroup of $\mathrm{Gl}(n - n_0, F)$ consisting of block-diagonal matrices

$$a = \begin{bmatrix} a_r & & \\ & \ddots & \\ & & a_1 \end{bmatrix},$$

with $a_i \in \mathrm{Gl}(n_i, F)$ for $i = 1, \ldots, r$, then

$$(1.6)\quad M = \left\{ \begin{bmatrix} a^{s-1} & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & a \end{bmatrix} : s \in \mathrm{Sp}(n_0, F),\ a \in \mathrm{Gl}(n_r, F) \times \cdots \times \mathrm{Gl}(n_1, F) \right\}.$$

In (1.4), (1.6), and throughout the paper, we use the convention that any matrix with n_0 as a dimension is suppressed if $n_0 = 0$.

It is easy to verify that NM is a semidirect product with M normalizing N and that NM is conjugate to P in $\mathrm{Sp}(n, F)$. Let χ be a unitary character of P. The unitary representation of $\mathrm{Sp}(n, F)$ induced by χ is a member of the principal series associated with P. The formula for this representation T realized to act in $L^2(N)$ can be found in [5] (§33.1). We shall only present, here, the formula for the restriction to NM of the principal series representations of $\mathrm{Sp}(n, F)$. For $f \in L^2(N)$, $z \in N$, and $m \in M$,

$$T(zm)f(\zeta) = \delta(m)^{-1/2}\chi(m)f(m^{-1}\zeta zm), \tag{1.7}$$

where $\delta(m) = d(m\zeta m^{-1})/d\zeta$ and $d\zeta$ is Haar measure on N.

2. The dual of N. Let the nilpotent group N have the form (1.4). In particular, this means we are fixing a blocking scheme for $2n \times 2n$ matrices, in which the diagonal blocks have dimensions

$$n_1 \times n_1, \ldots, n_r \times n_r, 2n_0 \times 2n_0, n_r \times n_r, \ldots, n_1 \times n_1,$$

respectively, from upper left to lower right. This induces blocking schemes for the various submatrices of $z \in N$. Notice that if $(t, y, x, w) \in N$ then the blocking scheme induced on t has blocks along the secondary diagonal of dimensions $n_1 \times n_1, \ldots, n_r \times n_r$ from lower left to upper right.

In this section we use the method of Kirillov to obtain the dual object of N (up to a set of Plancherel measure zero). For this we need the Lie algebra $\mathfrak{N}$, its real vector space dual $\mathfrak{N}^*$, and the coadjoint action of N on $\mathfrak{N}^*$. The dual of N is taken to be a set of irreducible unitary representations of N which correspond to the orbits of maximal dimension in $\mathfrak{N}^*$ under the coadjoint action of N.

As a linear Lie algebra, $\mathfrak{N}$ is given by

$$\mathfrak{N} = \left\{ \begin{bmatrix} -w_1^s & 0 & 0 & 0 \\ -x_1^s & 0 & 0 & 0 \\ y_1^s & 0 & 0 & 0 \\ t_1 & y_1 & x_1 & w_1 \end{bmatrix} : w_1 \in \mathfrak{W}; \right. \tag{2.1}$$

$$\left. x_1, y_1 \in F^{(n-n_0)\times n_0};\ t_1 = t_1^s \right\},$$

where $\mathfrak{W}$, the Lie algebra of W, is the set of $(n - n_0) \times (n - n_0)$ lower block-triangular matrices with zero blocks along the diagonal. The best way to express $\mathfrak{N}^*$ seems to be as a set of equivalence classes in $F^{2n\times 2n}$.

For $X, Y \in F^{2n\times 2n}$, define

$$\langle X \mid Y \rangle = \operatorname{Re} \operatorname{tr} XY^t$$

(the real part of the trace of XY^t). Let two elements, X' and Y', of $F^{2n\times 2n}$ be equivalent if $\langle X \mid X' \rangle = \langle X \mid Y' \rangle$ for all $X \in \mathfrak{N}$. Denote the equivalence class of X' by $c(X')$, then

$$\mathfrak{N}^* = \left\{ c\left(\begin{bmatrix} -w_1^s & 0 & 0 & 0 \\ -x_1^s & 0 & 0 & 0 \\ y_1^s & 0 & 0 & 0 \\ t_1 & y_1 & x_1 & w_1 \end{bmatrix} \right) : w_1 \in \mathfrak{W},\ t_1 = t_1^s \right\}. \tag{2.2}$$

The adjoint (left) action of N on $\mathfrak{N}$ is given by $\operatorname{Ad} z(X) = zXz^{-1}$, for $z \in N$ and $X \in \mathfrak{N}$. Thus, the coadjoint (right) action of N on $\mathfrak{N}^*$ is

$$\operatorname{Ad}^* z(c(X')) = c(z^t X' z^{t-1}). \tag{2.3}$$

This follows since, for $X \in \mathfrak{N}$, we have

$$\begin{aligned} \langle X \mid \operatorname{Ad}^* z(c(X')) \rangle &= \langle \operatorname{Ad} z(X) \mid c(X') \rangle \\ &= \langle zXz^{-1} \mid X' \rangle = \langle X \mid z^t X' z^{t-1} \rangle = \langle X \mid c(z^t X' z^{t-1}) \rangle, \end{aligned}$$

using properties of the trace. If X' is lower block-triangular then $z^t X' z^{t-1}$ is generally not lower block-triangular and this explains the need for equivalence classes.

For brevity, denote the elements of $\mathfrak{N}^*$ by (t_1, y_1, x_1, w_1), keeping in mind that $(t_1, y_1, x_1, w_1) = (t_2, y_2, x_2, w_2)$ if and only if $t_1 = t_2$, $y_1 = y_2$, $x_1 = x_2$, and w_1 is identical with w_2 below the block-diagonal. By computing the matrix product $z^t X' z^{t-1}$, with $z = (t, y, x, w) \in N$ and $X' = (t_1, y_1, x_1, w_1) \in \mathfrak{N}^*$, we find the formula

$$\begin{aligned} \operatorname{Ad}^* z(X') = \big(& w^t t_1 w^{st},\ w^t(y_1 + t_1 x^{st}), \\ & w^t(x_1 - t_1 y^{st}),\ w^t(w_1 - x_1 x^t - y_1 y^t - t_1 t^{st} w^t) w^{t-1} \big). \end{aligned} \tag{2.4}$$

Letting z vary in N, (2.4) gives the orbit of X' in $\mathfrak{N}^*$ under the coadjoint action of N. In general, the orbit is very complicated but a considerable simplification occurs if t_1 has the property that each submatrix $t_1^{[k]}$, with $n_1 + \cdots + n_k$ rows and columns, in the lower left corner of t_1 $(k = 1, \ldots, r)$ is non-singular, Let us call this property *secondary block-diagonalizable* (SBD) as suggested by the following matrix fact. Let Λ be the set of all $(n - n_0) \times (n - n_0)$ matrices

$$\lambda = \begin{bmatrix} & & & \lambda_r \\ & & \iddots & \\ & \iddots & & \\ \lambda_1 & & & \end{bmatrix}$$

with $\lambda_i = \lambda_i^s \in \mathrm{Gl}(n_i, F)$ $(i = 1,\ldots,r)$ along the second block-diagonal and zero elsewhere.

LEMMA 2.1. *With respect to the blocking schemes indicated earlier, the matrix $t_1 = t_1^s$ is SBD if and only if $t_1 = w^t \lambda w^{st}$ for some $\lambda \in \Lambda$ and $w \in W$. Moreover, λ and w are unique in this case.*

Proof. A statement proved in [5] (p. 59) can be applied to $t_1 J = (t_1 J)^t$ to yield this lemma.

The next result gives the maximal orbits in $\mathfrak{N}^*$ under the coadjoint action of N.

THEOREM 2.2. *For $i = 1, 2$ let $(t_i, y_i, x_i, w_i) \in \mathfrak{N}^*$ and suppose t_1 is SBD, $t_1 = w^t \lambda_1 w^{st}$, $w \in W$, $\lambda_1 \in \Lambda$. Then* (a) *(t_1, y_1, x_1, w_1) is in the same orbit as $(\lambda_1, 0, 0, 0)$, and* (b) *(t_1, y_1, x_1, w_1) is in the same orbit as (t_2, y_2, x_2, w_2) if and only if $t_2 = v^t \lambda_1 v^{st}$ for some $v \in W$.*

Proof. First we show that (t_1, y_1, x_1, w_1) is in the same orbit as $(t_1, 0, 0, 0)$ by constructing a suitable $z \in N$. Recall that $z = (t, y, x, w) \in N$ provided $wt^s - tw^s = yx^s - xy^s$. It follows that $(T - yx^s, y, x, I) \in N$ for any $T = T^s$. According to (2.4), to have

$$\mathrm{Ad}^*(T - yx^s, y, x, I)(t_1, y_1, x_1, w_1) = (t_1, 0, 0, 0),$$

we must set $y_1 + t_1 x^{st} = 0$, $x_1 - t_1 y^{st} = 0$ and find $T = T^s$ so that

$$w_1 - x_1 x^t - y_1 y^t - t_1 (T - yx^s)^{st} \cong 0. \tag{2.5}$$

where $\cong$ means these two matrices agree below the block-diagonal. Thus, let $x = t_1^{t-1} y_1^{st}$ and $y = -t_1^{t-1} x_1^{st}$ (since t_1 is SBD, it is invertible) and note that (2.5) is equivalent to

$$t_1 T^t \cong w_1 + y_1 x_1^s t_1^{-1}. \tag{2.6}$$

We claim that $t_1 A \cong B$ always has a solution $A = A^s$ when t_1 is SBD. The proof is a recursive construction of the block-columns of A. Here, A and B have a blocking scheme with diagonal blocks of dimensions $n_1 \times n_1, \ldots, n_r \times n_r$ from upper left to lower right. Let $A^{(j)}$ denote the jth block-column of A. We will write $A^{(j)} \cong B^{(j)}$ when these block-column submatrices of A and B agree below the block-diagonal position. To begin, let

$$X_1 = \left[\begin{array}{c|c} Y & 0 \\ \hline 0_{n_r \times n_1} & 0 \end{array}\right]$$

be a solution to $t_1 X_1^{(1)} \cong B^{(1)}$. Such a solution exists since t_1 is SBD and hence $t_1^{[r-1]}$ is invertible. Define $A_1 = X_1 + X_1^s$ then $A_1 = A_1^s$ and $t_1 A_1^{(1)} = t_1 X_1^{(1)} \cong B^{(1)}$. Suppose for some k $(1 \le k < r - 1)$ we have $A_k = A_k^s$ and

$t_1 A_k^{(j)} \cong B^{(j)}$ $(j = 1,\ldots,k)$. Let

$$X_{k+1} = \left[\begin{array}{c|c|c} 0 & Y & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Where $X_{k+1}^{(k+1)} = [{}^{Y}_{0}]$ and Y is the $(n_1 + n_2 + \cdots + n_{r-k}) \times n_{k+1}$ submatrix of $X_{k+1}^{(k+1)}$ above the secondary block-diagonal. Since $t_1^{[r-k]}$ is invertible, there is such a matrix X_{k+1} satisfying $t_1 X^{(k+1)} \cong B^{(k+1)} - t_1 A_k^{(k+1)}$. Let

$$A_{k+1} = A_k + X_{k+1} + X_{k+1}^s$$

then

$$A_{k+1} = A_{k+1}^s, \qquad t_1 A_{k+1}^{(k+1)} \cong t_1 A_k^{(k+1)} + t_1 X_{k+1}^{(k+1)} \cong B^{(k+1)},$$

and, for $1 \leq j \leq k$,

$$t_1 A_{k+1}^{(j)} \cong t_1 A_k^{(j)} \cong B^{(j)}.$$

The matrix $A = A_{r-1}$ is the desired solution. We have proved that (2.6) has a solution T with $T = T^s$ and it follows that (t_1, y_1, x_1, w_1) is in the same orbit as $(t_1, 0, 0, 0)$. Clearly, $(0, 0, 0, w) \in N$ and

$$\mathrm{Ad}^*(0,0,0,w)(t_1,0,0,0) = (\lambda_1,0,0,0)$$

by (2.4) so $(t_1, 0, 0, 0)$ is in the same orbit as $(\lambda_1, 0, 0, 0)$ and (a) is proved. The proof of (b) is straightforward.

If t_1 is SBD then we say that the orbit of $(t_1, y_1, x_1, w_1) \in \mathfrak{N}^*$ is an SBD orbit. By Theorem 2.2 the SBD orbits are in 1 : 1 correspondence with the set Λ. Next, we will show that SBD orbits are orbits of maximal dimension. Let m denote the number of entries below the block diagonal in a blocking scheme corresponding to $n - n_0 = n_1 + \cdots + n_r$ so that $\dim W = m$.

THEOREM 2.3. (a) *Each SBD orbit is an algebraic surface in $\mathfrak{N}^*$ having dimension over F equal to $2m + 2n_0(n - n_0)$.* (b) *The SBD orbits are orbits of maximal dimension in $\mathfrak{N}^*$.*

Proof. (a) Let O be an SBD orbit represented by $(\lambda, 0, 0, 0)$ with $\lambda \in \Lambda$. Then O consists of all $(t_1, y_1, x_1, w_1) \in \mathfrak{N}^*$ such that $t_1 = w^t \lambda w^{st}$ for some $w \in W$. By the uniqueness part of Lemma 2.1, t_1 can be parameterized by $w \in W$. Thus, $\dim O = m + n_0(n - n_0) + n_0(n - n_0) + m$. (b) The complement of the union of the SBD orbits consists of all elements $(t_1, y_1, x_1, w_1) \in \mathfrak{N}^*$ with $\det t_1^{[k]} = 0$ for some $k = 1,\ldots,r$. Thus, the union of the SBD orbits is a Zariski open set. It is known that the union of the orbits of maximal dimension is also a Zariski open set. If

the SBD orbits were not of maximal dimension we would have two disjoint Zariski open sets, which is impossible.

REMARK. We have not proved that there could not be an orbit of maximal dimension which is not SBD.

Now that we have isolated the SBD orbits, we will present the corresponding irreducible unitary representations of N. Recall that a subalgebra $\mathfrak{U}$ of the Lie algebra $\mathfrak{N}$ is said to be *subordinate* to $X' \in \mathfrak{N}^*$ provided $\langle [X, Y] \mid X' \rangle = 0$ for every $X, Y \in \mathfrak{U}$. For fixed $X' \in \mathfrak{N}^*$ let $\mathfrak{U}$ be a subordinate subalgebra of maximal dimension and let $U = \exp \mathfrak{U}$. Define a unitary character on U by setting $\chi(u) = \exp 2\pi i \langle \log u \mid X' \rangle$, for $u \in U$. The desired irreducible unitary representation of N is the one induced from χ.

It happens that there is a maximal subalgebra subordinate to every $X' \in \mathfrak{N}^*$. In fact, define $\mathfrak{U} = \{(t_1, y_1, 0, 0) \in \mathfrak{N}\}$, then $\mathfrak{U}$ is an abelian subalgebra of $\mathfrak{N}$ and it can be verified that

$$\dim_{\mathbf{R}} \mathfrak{U} = (\dim_{\mathbf{R}} \mathfrak{N}) - (\dim_{\mathbf{R}} \mathcal{O})/2,$$

for an SBD orbit $\mathcal{O}$, to show that $\mathfrak{U}$ has maximal dimension.

For $X \in \mathfrak{U}$ we have $X^2 = 0$, which implies that $\exp X = I + X$. Let $U = \exp \mathfrak{U}$, then $U = \{(t, y, 0, I) \in N\}$. U is an abelian subgroup of N with group operation

$$(t, y, 0, I)(t', y', 0, I) = (t + t', y + y', 0, I).$$

Also, define $V = \{(0, 0, x, w) \in N\}$, then V is a subgroup of N with group operation given by

$$(0, 0, x, w)(0, 0, x', w') = (0, 0, x + wx', ww').$$

Elements of N can be uniquely expressed in the form $z = uv$, with $u \in U$ and $v \in V$. In fact,

$$(t, y, x, w) = (tw^s + yx^s, y, 0, I)(0, 0, x, w). \tag{2.19}$$

Also, V normalizes U since

$$\begin{aligned}(0, 0, x, w)(t, y, 0, I)(0, 0, x, w)^{-1} \\ = (xy^s w^s + wtw^s + wyx^s, wy, 0, I)\end{aligned} \tag{2.20}$$

It follows that $N = UV$ is a semidirect product and representations of N induced from characters of U can be realized to act in $L^2(V)$.

Fix $\lambda \in \Lambda$, let $X' = (\lambda, 0, 0, 0) \in \mathfrak{N}^*$, and consider the unitary character of U defined for $u = (t, y, 0, I) \in U$ by

$$\chi_\lambda(u) = \exp 2\pi i \langle \log u \mid X' \rangle = \exp 2\pi i \langle t \mid \lambda \rangle, \tag{2.21}$$

where $\langle t \mid \lambda \rangle = \operatorname{Re} \operatorname{tr} t\lambda^t$. The irreducible unitary representation of N, denoted $\tilde{\lambda}$, induced by χ_λ and acting in $L^2(V)$ is then given by $\tilde{\lambda}(u_0 v_0)\phi(v) = \chi_\lambda(vu_0v^{-1})\phi(vv_0)$, for $u_0 \in U$, $v_0 \in V$, and $\phi \in L^2(V)$.

Let X be the additive group $F^{(n-n_0)\times n_0}$. Choose as independent parameters for the elements $(0, 0, x, w) \in V$ the matrices $\xi = w^{-1}x \in X$ and $\mu = w \in W$. The Haar measure of V is $d\xi\, d\mu$, where $d\xi$ and $d\mu$ denote the Haar measures of X and W, so we have an identification of $L^2(V)$ with $L^2(X \times W)$. In terms of ξ and μ, the formula for $\tilde{\lambda}$ is

$$(2.23)\quad \tilde{\lambda}(t, y, x, w)\phi(\xi, \mu) = \exp 2\pi i \langle \lambda \mid \mu(\xi y^s + tw^s + yx^s + y\xi^s)\mu^s \rangle \phi\big(w^{-1}(\xi + x), \mu w\big),$$

for $(t, y, x, w) \in N$ and $\phi \in L^2(X \times W)$.

To summarize this application of the theory of Kirillov we state

THEOREM 2.4 (i) *For* $\lambda \in \Lambda$, (2.23) *defines an irreducible unitary representation* $\tilde{\lambda}$ *of* N. (ii) *The mapping from* Λ *to the dual object of* N *which sends* λ *to the unitary equivalence class of* $\tilde{\lambda}$ *is injective, preserves Borel structure, and the complement of the image is a null set with respect to Plancherel measure.*

3. The action of M on the dual of N. Since M normalizes N, the map $z \to m^{-1}zm$ is an automorphism of N for each $m \in M$. It follows that $z \to \tilde{\lambda}(m^{-1}zm)$, for $\lambda \in \Lambda$, is an irreducible unitary representation of N unitarily equivalent to $(m \cdot \lambda)^{\sim}$ for some $m \cdot \lambda \in \Lambda$. We now investigate this action $(m, \lambda) \to m \cdot \lambda$ of M on Λ.

Denote the elements of M, defined in (1.6), by $m = (s, a)$, with $s \in \operatorname{Sp}(n_0, F)$ and $a \in \operatorname{Gl}(n_r, F) \times \cdots \times \operatorname{Gl}(n_1, F)$. Express s in terms of $n_0 \times n_0$ submatrices by writing

$$s = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}$$

and recall that a is regarded as an $(n - n_0) \times (n - n_0)$ block-diagonal matrix. Let $z = (t, y, x, w) \in N$. Since $y, x \in X = F^{(n-n_0)\times n_0}$, we can form the $(n - n_0) \times 2n_0$ matrix $[yx]$. It will be convenient to define

$$(3.1)\qquad g(y, x) = g([yx]) = yx^s.$$

Matrix multiplication now gives the following expression for $m^{-1}zm$:

$$(3.2)\quad m^{-1}zm = \big(a^{-1}ta^{s-1}, a^{-1}(ys_{11} + xs_{21}), a^{-1}(ys_{12} + xs_{22}), a^{-1}wa\big).$$

Let $\lambda \in \Lambda$ and $\phi \in L^2(X \times W)$, then (3.2) together with (2.23) yields

$$(3.3)\quad \tilde{\lambda}(m^{-1}zm)\phi(\xi, \mu)$$
$$= \exp 2\pi i \big\langle \lambda \mid \mu\big(\xi(ys_{11} + xs_{21})^s a^{s-1}$$
$$+ a^{-1}(tw^s + g([yx]s))a^{s-1} + a^{-1}(ys_{11} + xs_{21})\xi^s\big)\mu^s \big\rangle$$
$$\cdot \phi\big(a^{-1}w^{-1}a\big(\xi + a^{-1}(ys_{12} + xs_{22})\big), \mu a^{-1}wa\big).$$

Since the representation $z \to \tilde{\lambda}(m^{-1}zm)$ is unitarily equivalent to $(m\cdot\lambda)^{\sim}$, there exists a unitary operator $D_\lambda(m)$ on $L^2(X \times W)$, for each $m \in M$ and $\lambda \in \Lambda$, such that

$$(3.4)\qquad \tilde{\lambda}(m^{-1}zm) = D_\lambda(m)^{-1}(m\cdot\lambda)^{\sim}(z)D_\lambda(m),$$

for all $z \in N$. The operator $D_\lambda(m)$ is unique up to a scalar factor in $\mathcal{T} = \{\alpha \in \mathbf{C}: \mid \alpha \mid = 1\}$, because $(m\cdot\lambda)^{\sim}$ is irreducible. Also, for m_1, $m_2 \in M$,

$$\tilde{\lambda}\big(m_2^{-1}\big(m_1^{-1}zm_1\big)m_2\big) = \tilde{\lambda}\big((m_1m_2)^{-1}z(m_1m_2)\big)$$

implies that $m_1\cdot(m_2\cdot\lambda) = (m_1m_2)\cdot\lambda$, from which it follows

$$(3.5)\qquad D_\lambda(m_1m_2) = \alpha(\lambda, m_1, m_2)D_{m_2\cdot\lambda}(m_1)D_\lambda(m_2),$$

for some $\alpha(\lambda, m_1, m_2) \in \mathcal{T}$.

The way to describe the operators $D_\lambda(m)$ and the action $(m, \lambda) \to m\cdot\lambda$ is to work separately with the following obvious subgroups of M:

$$S = \{(s, I): s \in \mathrm{Sp}(n_0, F)\},$$

and

$$A = \{(I, a): a \in \mathrm{Gl}(n_r, F) \times \cdots \times \mathrm{Gl}(n_1, F)\}.$$

Fix $\lambda \in \Lambda$. It is easy to see from (3.2) that A normalizes the subgroup $V = \{(0, 0, x, w) \in N\}$ of N. Therefore, the formula

$$D_\lambda(m)\phi(v) = \gamma(m)^{1/2}\phi(m^{-1}vm)$$

$(m \in A, \phi \in L^2(V))$, where $\gamma(m) = d(m^{-1}vm)/dv$, defines a unitary operator on $L^2(V)$. If $m = (I, a) \in A$ then, in terms of the parameters (ξ, μ), the formula becomes

$$(3.6)\qquad D_\lambda(I, a)\phi(\xi, \mu) = \gamma(I, a)^{1/2}\phi(a\xi, a^{-1}\mu a),$$

for $\phi \in L^2(X \times W)$. A simple computation shows

LEMMA 3.1. *For* $m = (I, a) \in A$ *and* $\lambda \in \Lambda$,

$$\tilde{\lambda}(m^{-1}zm) = D_\lambda(m)^{-1}(a^{t-1}\lambda a^{st-1})\tilde{}(z)D_\lambda(m),$$

for all $z \in N$, $D_\lambda(m)$ *defined by* (3.6). *Thus*, $m \cdot \lambda = a^{t-1}\lambda a^{st-1}$.

If $n_0 = 0$ then $M = A$ and we are done, so assume in the discussion of S that follows that $n_0 \neq 0$. Now let $m = (s, I) \in S$. In order to describe $D_\lambda(s, I)$, we need to parallel Weil's construction of the oscillator representation (cf. [4] 2.3). Fix $\lambda \in \mathrm{Gl}(n - n_0, F)$ and let X be the vector space $F^{(n-n_0)} \times n_0$ over F. The dual of the additive group of X is identified with X using the bicharacter

$$\langle x, y \rangle_\lambda = \exp 2\pi i \langle \lambda \mid xy^s + yx^s \rangle = \chi_\lambda(xy^s + yx^s). \tag{3.7}$$

For each $(y, x) \in X \times X$, let $U'_\lambda(y, x)$ denote the unitary operator on $L^2(X)$ defined by

$$U'_\lambda(y, x)\psi(\xi) = \chi_\lambda(\xi y^s + y\xi^s)\psi(\xi + x). \tag{3.8}$$

The family of operators

$$U_\lambda(y, x, \alpha) = \alpha U'_\lambda(y, x) \qquad (y, x \in X, \alpha \in \mathfrak{T})$$

forms a group, denoted $\overline{A_\lambda(X)}$, whose composition law is

$$U_\lambda(y_1, x_1, \alpha_1)U_\lambda(y_2, x_2, \alpha_2) = U_\lambda(y_1 + y_2, x_1 + x_2, \chi_\lambda(y_2x_1^s + x_1y_2^s)\alpha_1\alpha_2).$$

There is an isomorphic group $A_\lambda(X) = X \times X \times \mathfrak{T}$ with composition law

$$(y_1, x_1, \alpha_1)(y_2, x_2, \alpha_2) = (y_1 + y_2, x_1 + x_2, \chi_\lambda(y_2x_1^s + x_1y_2^s)\alpha_1\alpha_2).$$

U_λ: $A_\lambda(X) \to \overline{A_\lambda(X)}$ is the unique irreducible unitary representation of $A_\lambda(X)$ which leaves the center, $\{0\} \times \{0\} \times \mathfrak{T}$, pointwise fixed, in the sense that $U_\lambda(0, 0, \alpha) = \alpha I$.

Let $B_\lambda(X)$ denote the group of automorphisms of $A_\lambda(X)$ which leave the center pointwise fixed and let $\sigma \in B_\lambda(X)$. Then there exists a unitary operator $r_\lambda(\sigma)$ of $L^2(X)$, unique up to a scalar factor in $\mathfrak{T}$, such that

$$U_\lambda((y, x, \alpha)\sigma) = r_\lambda(\sigma)^{-1}U_\lambda(y, x, \alpha)r_\lambda(\sigma). \tag{3.9}$$

The mapping $\sigma \to r_\lambda(\sigma)$ determines a projective representation of $B_\lambda(X)$ called the Weil representation.

Let $s \in \mathrm{Sp}(n_0, F)$ and define $f_{s,\lambda}$: $X \times X \to \mathfrak{T}$ by

$$f_{s,\lambda}(y, x) = \chi_\lambda(g([yx]s) - g([yx]), \tag{3.10}$$

where $g([yx]) = yx^s$. Then the pair $(s, f_{s,\lambda})$ determines an element of $B_\lambda(X)$ defined by

$$(3.11) \qquad (y, x, \alpha)(s, f_{s,\lambda}) = ([yx]s, f_{s,\lambda}(y, x)\alpha),$$

for $(y, x, \alpha) \in A_\lambda(X)$, as a computation shows. Morever, the set $\{(s, f_{s,\lambda}): s \in \mathrm{Sp}(n_0, F)\}$ together with the operation

$$(s_1, f_{s_1,\lambda})(s_2, f_{s_2,\lambda}) = (s_1 s_2, f_{s_1 s_2,\lambda})$$

forms a subgroup of $B_\lambda(X)$ isomorphic to $\mathrm{Sp}(n_0, F)$. For each $s \in \mathrm{Sp}(n_0, F)$, use $\sigma = (s, f_{s,\lambda})$ in (3.9) to define

$$(3.12) \qquad r_\lambda(s) = r_\lambda(s, f_{s,\lambda}).$$

The mapping $s \to r_\lambda(s)$ determines a projective representation of $\mathrm{Sp}(n_0, F)$; this is the oscillator representation.

In order to make use of r_λ in the context of the representation $\tilde{\lambda}$ of N acting in $L^2(X \times W)$, we identify $L^2(X \times W)$ with $L^2(W, L^2(X))$. That is, the function $\phi \in L^2(X \times W)$ is identified with the $L^2(X)$-valued function

$$\mu \to \phi_\mu = \phi(\cdot, \mu),$$

defined for $\mu \in W$. Recall that an operator Q on $L^2(W, L^2(X))$ is called *decomposable* if $(Q\phi)_\mu = Q(\mu)\phi_\mu$ for $\phi \in L^2(W, L^2(X))$, where $Q(\mu)$ is an operator on $L^2(X)$ for each $\mu \in W$. In this case we say $Q = \int_W Q(\mu)\, d\mu$. If we define $N_1 = \{(t, y, x, I) \in N\}$, then N_1 is a subgroup of N and the operators $\tilde{\lambda}(t, y, x, I)$ are decomposable. For $\lambda \in \Lambda$ and $\mu \in W$, we will use the notation

$$(3.13) \qquad \begin{cases} \lambda \cdot \mu = \mu^t \lambda \mu^{st}, \\ (\tilde{\lambda} \cdot \mu)(t, y, x, I) = \chi_{\lambda \cdot \mu}(t + yx^s) U'_{\lambda \cdot \mu}(y, x). \end{cases}$$

Then (2.23) and (3.8) imply

$$(3.14) \qquad \tilde{\lambda}(t, y, x, I) = \int_W (\tilde{\lambda} \cdot \mu)(t, y, x, I)\, d\mu$$

The relationship between $\tilde{\lambda}$ and the oscillator representation is given by the following sequence of results:

LEMMA 3.2. *Let* $\lambda \in \Lambda$, $\mu \in W$, $(t, y, x, I) \in N_1$, *and* $s \in \mathrm{Sp}(n_0, F)$, *then*

$$(3.15) \quad (\tilde{\lambda} \cdot \mu)(t, [yx]s, I) = r_{\lambda \cdot \mu}(s)^{-1}(\tilde{\lambda} \cdot \mu)(t, y, x, I) r_{\lambda \cdot \mu}(s).$$

Proof. Here is an outline of the proof. (1) Define $\tau_{\lambda\cdot\mu}\colon N_1 \to A_{\lambda\cdot\mu}(X)$ by the formula

$$\tau_{\lambda\cdot\mu}(t, y, x, I) = (y, x, \chi_{\lambda\cdot\mu}(t + yx^s)),$$

then $\tau_{\lambda\cdot\mu}$ is a homomorphism and $\tilde{\lambda}\cdot\mu = U_{\lambda\cdot\mu}\tau_{\lambda\cdot\mu}$. (2) Denote by $\overline{(s, I)}$ the automorphism of N_1 given by

$$(t, y, x, I) \to (t, [yx]s, I).$$

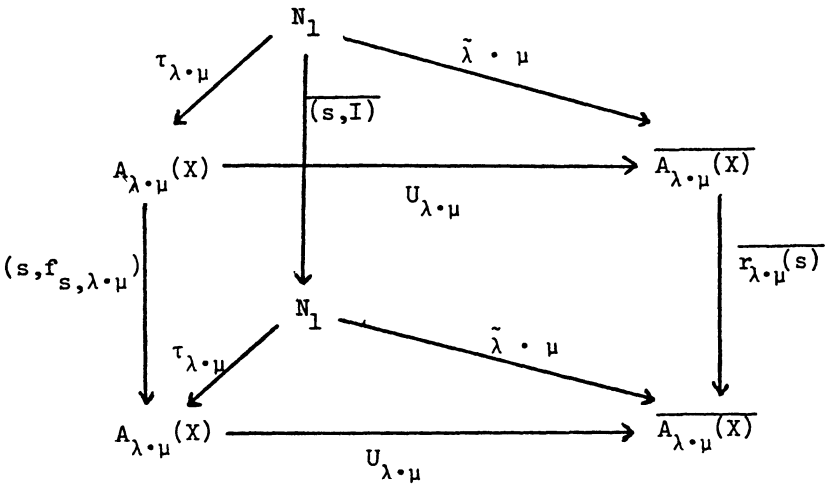
Then the following is a commutative diagram:

$$\begin{array}{ccc} N_1 & \overset{\overline{(s,I)}}{\to} & N_1 \\ {\scriptstyle \tau_{\lambda\cdot\mu}}\downarrow & & \downarrow{\scriptstyle \tau_{\lambda\cdot\mu}} \\ A_{\lambda\cdot\mu}(X) & \underset{(s, f_{s,\lambda\cdot\mu})}{\to} & A_{\lambda\cdot\mu}(X) \end{array}$$

(3) Denote by $\overline{r_{\lambda\cdot\mu}(s)}$ the automorphism of $\overline{A_{\lambda\cdot\mu}(X)}$ given by

$$U_{\lambda\cdot\mu}(y, x, \alpha) \to r_{\lambda\cdot\mu}(s)^{-1}U_{\lambda\cdot\mu}(y, x, \alpha)r_{\lambda\cdot\mu}(s),$$

then the following is a commutative diagram:



which completes the proof.

LEMMA 3.3. *For $\mu \in W$, define a unitary operator $l(\mu)$ on $L^2(X)$ by $l(\mu)\psi(\xi) = \psi(\mu^{-1}\xi)$. Then, for $s \in \mathrm{Sp}(n_0, F)$,*

$$r_{\lambda\cdot\mu}(s) = l(\mu)^{-1}r_\lambda(s)l(\mu),$$

up to a scalar factor in $\mathfrak{T}$.

Proof. First, note that for $\psi \in L^2(X)$ and $(t, y, x, I) \in N_1$,

$$\begin{aligned} &l(\mu)^{-1}\chi_\lambda(\mu(t + yx^s)\mu^s)U'_\lambda(\mu[yx])l(\mu)\psi(\xi) \\ &\quad = \chi_\lambda(\mu(t + yx^s)\mu^s)\chi_\lambda(\mu(\xi y^s + y\xi^s)l(\mu)\psi(\mu\xi + \mu x) \\ &\quad = \chi_{\lambda\cdot\mu}(t + yx^s)\chi_{\lambda\cdot\mu}(\xi y^s + y\xi^s)\chi(\xi + x) \\ &\quad = \chi_{\lambda\cdot\mu}(t + yx^s)U'_{\lambda\cdot\mu}(y, x)\psi(\xi). \end{aligned}$$

That is,

$$l(\mu)^{-1}\tilde{\lambda}\cdot I(\mu t\mu^s, \mu[yx], I)l(\mu) = \tilde{\lambda}\cdot\mu(t, y, x, I). \tag{3.16}$$

It follows that

$$\begin{aligned} &l(\mu)^{-1}r_\lambda(s)^{-1}l(\mu)\tilde{\lambda}\cdot\mu(t, y, x, I)l(\mu)^{-1}r_\lambda(s)l(\mu) \\ &\quad = l(\mu)^{-1}r_\lambda(s)^{-1}\tilde{\lambda}\cdot I(\mu t\mu^s, \mu[yx], I)r_\lambda(s)l(\mu) \\ &\quad = l(\mu)^{-1}\tilde{\lambda}\cdot I(\mu t\mu^s, \mu[yx]s, I)l(\mu) \\ &\quad = \tilde{\lambda}\cdot\mu(t, [yx]s, I), \end{aligned}$$

using (3.16) and (3.15). Since $r_{\lambda\cdot\mu}(s)$ is the unique unitary operator (up to scalar factor in $\mathfrak{T}$) with this property, we have the desired result.

THEOREM 3.3. *For $(s, I) \in S$, define the unitary operator $D_\lambda(s)$ on $L^2(W, L^2(X))$ by setting*

$$D_\lambda(s) = \int_W l(\mu)^{-1}r_\lambda(s)l(\mu)\, d\mu.$$

Then,

$$\tilde{\lambda}\big((s, I)^{-1}(t, y, x, w)(s, I)\big) = D_\lambda(s)^{-1}\tilde{\lambda}(t, y, x, w)D_\lambda(s),$$

for every $(t, y, x, w) \in N$.

Proof. For $(t, y, x, w) \in N$, we have

$$\tilde{\lambda}(t, y, x, w) = \tilde{\lambda}(tw^s, y, x, I)\tilde{\lambda}(0, 0, 0, w)$$

and, hence,

$$\begin{aligned} &D_\lambda(s)^{-1}\tilde{\lambda}(t, y, x, w)D_\lambda(s) \\ &\quad = D_\lambda(s)^{-1}\tilde{\lambda}(tw^s, y, x, I)D_\lambda(s)D_\lambda(s)^{-1}\tilde{\lambda}(0, 0, 0, w)D_\lambda(s). \end{aligned}$$

But, for $\phi \in L^2(W, L^2(X))$,

$$\begin{aligned}
&\big(D_\lambda(s)^{-1}\tilde{\lambda}(tw^s, y, x, I)D_\lambda(s)\phi\big)_\mu(\xi) \\
&\quad = l(\mu)^{-1}r_\lambda(s)^{-1}l(\mu)\tilde{\lambda}\cdot\mu(tw^s, y, x, I)l(\mu)^{-1}r_\lambda(s)l(\mu)\phi_\mu(\xi) \\
&\quad = r_{\lambda\cdot\mu}(s)^{-1}\tilde{\lambda}\cdot\mu(tw^s, y, x, I)r_{\lambda\cdot\mu}(s)\phi_\mu(\xi) \\
&\quad = \tilde{\lambda}\cdot\mu\big((s, I)^{-1}(tw^s, y, x, I)(s, I)\big)\phi_\mu(\xi) \\
&\quad = \big(\tilde{\lambda}\big((s, I)^{-1}(tw^s, y, x, I)(x, I)\big)\phi\big)_\mu(\xi),
\end{aligned}$$

by Lemma 3.3 and (3.15). Also, from (2.23), we have

$$\big(\tilde{\lambda}(0,0,0,w)\phi\big)_\mu(\xi) = \phi_{\mu w}(w^{-1}\xi) = l(w)\phi_{\mu w}(\xi). \tag{3.17}$$

Thus,

$$\begin{aligned}
&\big(D_\lambda(s)^{-1}\tilde{\lambda}(0,0,0,w)D_\lambda(s)\phi\big)_\mu(\xi) \\
&\quad = l(\mu)^{-1}r_\lambda(s)^{-1}l(\mu)\big(\tilde{\lambda}(0,0,0,w)D_\lambda(s)\phi\big)_\mu(\xi) \\
&\quad = l(\mu)^{-1}r_\lambda(s)^{-1}l(\mu)l(w)\big(D_\lambda(s)\phi\big)_{\mu w}(\xi) \\
&\quad = l(\mu)^{-1}r_\lambda(s)^{-1}l(\mu)l(w)l(\mu w)^{-1}r_\lambda(s)l(\mu w)\phi_{\mu w}(\xi) \\
&\quad = l(w)\phi_{\mu w}(\xi) = \big(\tilde{\lambda}(0,0,0,w)\phi\big)_\mu(\xi);
\end{aligned}$$

that is, $\tilde{\lambda}(0,0,0,w)$ commutes with $D_\lambda(s)$. Combining these facts yields

$$\begin{aligned}
&D_\lambda(s)^{-1}\tilde{\lambda}(t, y, x, w)D_\lambda(s) \\
&\quad = D_\lambda(s)^{-1}\tilde{\lambda}(tw^s, y, x, I)\tilde{\lambda}(0,0,0,w)D_\lambda(s) \\
&\quad = D_\lambda(s)^{-1}\tilde{\lambda}(tw^s, y, x, I)D_\lambda(s)\tilde{\lambda}(0,0,0,w) \\
&\quad = \tilde{\lambda}\big((s, I)^{-1}(tw^s, y, x, I)(s, I)\big)\tilde{\lambda}(0,0,0,w) \\
&\quad = \tilde{\lambda}(tw^s, [yx]s, I)\tilde{\lambda}(0,0,0,w) \\
&\quad = \tilde{\lambda}(t, [yx]s, w) = \tilde{\lambda}\big((s, I)^{-1}(t, y, x, w)(s, I)\big).
\end{aligned}$$

In summary, we have the result:

THEOREM 3.4 (a) *The action of M on Λ is given by*

$$(s, a)\cdot\lambda = a^{t^{-1}}\lambda a^{st^{-1}} \quad \textit{for } (s, a) \in M, \lambda \in \Lambda. \tag{3.19}$$

(b) *With $D_\lambda(s, a) = D_\lambda(s, I)D_\lambda(I, a)$ for $\lambda \in \Lambda$ and $(s, a) \in M$, we have*

$$\tilde{\lambda}\big((s, a)^{-1}z(s, a)\big) = D_\lambda(s, a)^{-1}(a^{t^{-1}}\lambda a^{st^{-1}})^{\sim}(z)D_\lambda(s, a),$$

for all $z \in N$.

Next, we investigate the orbits in Λ under the action of M. Since (3.19) is correct for $n_0 = 0$ as well as for $n_0 \neq 0$, the results here hold in both cases. For any $\lambda \in \Lambda$, let $\lambda = (\lambda^{(1)},\ldots,\lambda^{(r)})$, where $\lambda^{(k)} = (\lambda^{(k)})^s$ is the non-singular, $n_k \times n_k$ submatrix of λ in the kth position ($k = 1,\ldots,r$) along the secondary block-diagonal of λ (counting from lower left to upper right). For $(s, a) \in M$, let $a = (a^{(1)},\ldots,a^{(r)})$, where $a^{(k)} \in \mathrm{Gl}(n_k, F)$ is the submatrix of a in the kth position along the block-diagonal (counting from lower right to upper left). In terms of this notation, the action of M on Λ is given by

$$m \cdot \lambda = \left(a^{(1)t-1}\lambda^{(1)}a^{(1)st-1},\ldots,a^{(r)t-1}\lambda^{(r)}a^{(r)st-1}\right). \tag{3.20}$$

A variant of Sylvestor's Law can be applied to each component to prove the following result:

THEOREM 3.5. *For $F = \mathbf{C}$, the action of M on Λ is transitive. For $F = \mathbf{R}$, there are $(n_1 + 1)(n_2 + 1)\cdots(n_r + 1)$ orbits in Λ under the action of M. Each orbit is represented by an element $(\lambda^{(1)},\ldots,\lambda^{(r)})$ such that, for $k = 1,\ldots,r$, $\lambda^{(k)}$ has the secondary diagonal form*

$$\begin{bmatrix} & & & & -1 \\ & & & \iddots & \\ & & -1 & & \\ & 1 & & & \\ \iddots & & & & \\ 1 & & & & \end{bmatrix} = J_{n_k,m_k},$$

with m_k entries of 1 in the lower left positions and $n_k - m_k$ entries of -1 in the remaining positions ($m_k = 0, 1,\ldots,n_k$).

4. The commuting algebra of T restricted to NM. Corresponding to a certain blocking scheme for $2n \times 2n$ matrices, we have defined a parabolic subgroup P of $\mathrm{Sp}(n, F)$, the opposite parabolic NM, and the principal series representation T induced from a unitary character on P and acting in $L^2(N)$. Next, we obtain theorems concerning the commuting algebra of $T|_{NM}$ generalizing results in [**6**], [**2**], and [**1**]. Proofs will be omitted here; they are similar to what can be found in the papers just mentioned.

Let $\Lambda_1,\ldots,\Lambda_l$ denote the orbits in Λ under the action of A and let $\lambda_1,\ldots,\lambda_l$ be orbit representatives (see Theorem 3.5). Define the stability subgroup of λ_i in A:

$$A_i = \{(I, a) \in A : (I, a)\cdot\lambda_i = \lambda_i\},$$

for $i = 1,\ldots,l$. The subgroup A_i is uniquely determined by the orbit, up to conjugate subgroups in A. We are also interested in the stability subgroup

of λ_i in M. This turns out to be SA_i, as is easily seen. Consider, now, the restriction of D_{λ_i} to SA_i. By (3.5), this is a multiplier representation of SA_i acting in $L^2(X \times W)$. The two theorems below describe the stability subgroups and $D_{\lambda_i|SA_i}$.

Analogous to the notation $J_{n,m}$ of Theorem 3.5, we have

$$I_{n,m} = \begin{bmatrix} -I_{n-m} & 0 \\ 0 & I_m \end{bmatrix} \qquad (m = 0, 1, \ldots, n),$$

where I_m and I_{n-m} denote identity matrices. The (pseudo) orthogonal groups are defined by

$$O(m, n-m, F) = \{a \in \mathrm{Gl}(n, F) : aI_{n,m}a^t = I_{n,m}\}.$$

If $m = 0$ then $O(0, n-0, F) = O(n, F)$ is the usual orthogonal group.

THEOREM 4.1. *Let $F = \mathbf{C}$. The single orbit $\Lambda_1 = \Lambda$ is represented by $\lambda_1 = (J_{n_1}, \ldots, J_{n_r}) = J_{n-n_0}$ and A_1 is isomorphic to*

$$O(n_1, \mathbf{C}) \times \cdots \times O(n_r, \mathbf{C}).$$

If $n_0 = 0$, then $SA_1 = A_1$ and $D_{\lambda_1|SA_1}$ is given by

$$D_{\lambda_1}(a)\phi(\mu) = \phi(a^{-1}\mu a),$$

for $\phi \in L^2(W)$ and $a \in O(n_1, \mathbf{C}) \times \cdots \times O(n_r, \mathbf{C})$. If $n_0 \neq 0$, then SA_1 is isomorphic to

$$\mathrm{Sp}(n_0, \mathbf{C}) \times O(n_1, \mathbf{C}) \times \cdots \times O(n_r, \mathbf{C})$$

and $D_{\lambda_1|SA_1}$ is given by

$$\big(D_{\lambda_1}(s, a)\phi\big)_\mu(\xi) = l(\mu)^{-1} r_{\lambda_1}(s) l(\mu) l(a) \phi_{a^{-1}\mu a}(\xi),$$

for $\phi \in L^2(W, L^2(X))$, $s \in \mathrm{Sp}(n_0, \mathbf{C})$, and $a \in O(n_1, \mathbf{C}) \times \cdots \times O(n_r, \mathbf{C})$.

REMARK. The description of A_1 is due to the following: for $a = (a^{(1)}, \ldots, a^{(r)}) \in \mathrm{Gl}(n_1, \mathbf{C}) \times \cdots \times \mathrm{Gl}(n_r, \mathbf{C})$,

$$a^{t-1} J a^{st-1} = J \Leftrightarrow a^{(k)t-1} J_{n_k} a^{(k)st-1} = J_{n_k} \quad \text{for all } k = 1, \ldots, r.$$

But multiplying both sides of this equation by J_{n_k} on the right results in $a^{(k)t-1}a^{(k)-1} = I_{n_k}$. Thus, $(I, a) \in A_1$ if and only if $a^{(k)t-1} \in O(n_k, \mathbf{C})$ for all $k = 1, \ldots, r$. This means $a^{(k)} \in O(n_k, \mathbf{C})$ for all k.

THEOREM 4.2. *Let $F = \mathbf{R}$. Let Λ_1 be the orbit represented by $\lambda_1 = (J_{n_1,m_1}, \ldots, J_{n_r,m_r})$. Then A_1 is isomorphic to*

$$O(m_1, n_1 - m_1, \mathbf{R}) \times \cdots \times O(m_r, n_r - m_r, \mathbf{R}).$$

If $n_0 = 0$, *then* $SA_1 = A_1$ *and* $D_{\lambda_1|SA_1}$ is given by

$$D_{\lambda_1}(a)\phi(\mu) = \phi(a^{-1}\mu a),$$

for $\phi \in L^2(W)$ *and* $a \in O(m_1, n_1 - m_1, \mathbf{R}) \times \cdots \times O(m_r, n_r - m_r, \mathbf{R})$. *If* $n_0 \neq 0$, *then* SA_1 *is isomorphic to*

$$\mathrm{Sp}(n_0, \mathbf{R}) \times O(m_1, n_1 - m_1, \mathbf{R}) \times \cdots \times O(m_r, n_r - m_r, \mathbf{R})$$

and $D_{\lambda_1|SA_1}$ *is given by*

$$\left(D_{\lambda_1}(s, a)\phi\right)_\mu(\xi) = l(\mu)^{-1} r_{\lambda_1}(s) l(\mu) l(a) \phi_{a^{-1}\mu a}(\xi),$$

for $\phi \in L^2(W, L^2(X))$.

Corresponding to the orbit representatives $\lambda_1, \ldots, \lambda_l$, let $\mathcal{A}_i' = \mathcal{A}'(D_{\lambda_i|SA_i})$ denote the commuting algebra of the multiplier representation of $SA_i (i = 1, \ldots, l)$. Let $L_i \in \mathcal{A}_i'$, then

$$(I, a)\cdot\lambda_i \to D_{\lambda_i}(I, a) L_i D_{\lambda_i}(I, a)^{-1},$$

for $(I, a) \in A$, is a well-defined map from Λ_i into $\mathcal{L}(L^2(V))$. For suppose $(I, a)\cdot\lambda_i = (I, a')\cdot\lambda_i$, then $(I, a')^{-1}(I, a) = (I, a'') \in A_i$ and $(I, a) = (I, a')(I, a'')$. Thus,

$$\begin{aligned} D_{\lambda_i}(I, a) L_i D_{\lambda_i}(I, a)^{-1} &= D_{\lambda_i}(I, a') D_{\lambda_i}(I, a'') L_i D_{\lambda_i}(I, a'')^{-1} D_{\lambda_i}(I, a')^{-1} \\ &= D_{\lambda_i}(I, a') L_i D_{\lambda_i}(I, a')^{-1} \end{aligned}$$

since L_i commutes with $D_{\lambda_i}(I, a'')$.

We can now state the main results regarding the commuting algebra $\mathcal{A}'(T|_{NM})$. Let $\mathcal{P}$: $L^2(N) \to L^2(\Lambda, HS(L^2(V)), dm(\lambda))$ be the Plancherel transform of $L^2(N)$ defined as in [**6**]. Let $\hat{T} = \mathcal{P} T \mathcal{P}^{-1}$.

THEOREM 4.3. *Let* $F = \mathbf{C}$. *Let* λ_1, A_1, *and* D_{λ_1} *be as in Theorem* 4.2. *The mapping from* $\mathcal{A}'(D_{\lambda_1|SA_1})$ *to* $\mathcal{L}(L^2(\Lambda, HS(L^2(V)), dm(\lambda)))$ defined by

$$L_1 \to \int_\Lambda D_{\lambda_1}(I, a) L_1 D_{\lambda_1}(I, a)^{-1} \otimes I \, dm((I, a)\cdot\lambda_1)$$

$(a \in A)$ *is a* **-isomorphism between the von Neumann algebras* $\mathcal{A}'(D_{\lambda_1|SA_1})$ *and* $\mathcal{A}'(\hat{T}|_{NM})$. *Thus,* $\mathcal{A}'(T|_{NM})$ *is* **-isomorphic with* $\mathcal{A}'(D_{\lambda_1|SA_1})$.

Proof. This follows from [**3**] (Theorem 6) along with Theorem 4.1.

Similarly, we have

THEOREM 4.4. *Let* $F = \mathbf{R}$. *Let* $\lambda_1,\ldots,\lambda_l$ *be the orbit representatives given in Theorem* 3.5, *where* $l = (n_1 + 1)\cdots(n_r + 1)$. *Let* A_i *and* D_{λ_i} $(i = 1,\ldots,l)$ *correspond to* λ_i *as in Theorem* 4.2. *The mapping from* $\mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_l$ *to* $\mathcal{L}(L^2(\Lambda, HS(L^2(V)), dm(\lambda)))$ defined by

$$(L_1,\ldots,L_l) \to \bigoplus_{i=1}^{l} \int_{\Lambda_i} D_{\lambda_i}(I,a) L_i D_{\lambda_i}(I,a)^{-1} \otimes I \, dm((I,a)\cdot\lambda_i),$$

$(a \in A)$ *is a* **-isomorphism between the von Neumann algebras* $\mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'$ *and* $\mathcal{C}'(\hat{T}|_{NM})$. *Thus,* $\mathcal{C}'(T|_{NM})$ *is* **-isomorphic with* $\mathcal{C}'(D_{\lambda_1|SA_1}) \oplus \cdots \oplus \mathcal{C}'(D_{\lambda_l|SA_l})$.

REFERENCES

[1] T. Farmer, *On the reduction of certain degenerate principal series representations of* Sp(n, **C**), Pacific. J. Math., **84**, No. 2, (1979), 291–303.

[2] _____, *Irreducibility of certain degenerate principal series representations of* Sp(n, **R**), Proc. Amer. Math. Soc., **83**, No. 2, (1981), 411–420.

[3] S. Gaal, *Linear Analysis and Representation Theory*, New York, Heidelberg, Berlin, Springer, 1973.

[4] S. Gelbart, *Weil's Representation and the Spectrum of the Metaplectic Group*, Springer Lecture Notes, No. 530, 1976.

[5] I. M. Gelfand and M. A. Naimark, *Unitäre Darstellungen der Klassischen Gruppen*, Berlin, Akadamie 1957.

[6] K. I. Gross, *The dual of a parabolic subgroup and a degenerate principal series of* Sp(n, **C**), Amer. J. Math., **93** (1971), 398–428.

[7] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys, **17** (1962), 53–104.

[8] J. A. Wolf, *Unitary representations of maximal parabolic subgroups of the classical groups*, Amer. Math. Soc. Memoir, **180** (1976).

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