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Necessary conditions are given in order that Tor(A, B) be a direct sum of cyclic groups for abelian groups A and B. Sufficient conditions are also given that compare with the necessary conditions. They are only slightly stronger if at all, and the two are equivalent for groups of cardinality not exceeding $\aleph_2$.

Due to the complexity of the problem, these conditions are not absolute, but constitute a reduction to smaller cardinality. The results generalize earlier results of R. Nunke.

What is known of the structure and properties of Tor(A, B) for abelian groups A and B is primarily due to R. Nunke. The main results appear in [**4**], [**5**], and [**6**]. Contributions of other authors, however, were manifested in [**2**] and [**3**]. This note generalizes the results of Nunke published in [**4**] and [**6**] concerning a basic question: When is Tor(A, B) a direct sum of cyclic groups? Nunke has provided a completely satisfactory answer when neither the cardinality of A nor B exceeds $\aleph_1$ and in certain other cases, as well, allowing A and B to be arbitrarily large. However, the general case where A or B has cardinality greater than $\aleph_1$ was deferred. In this paper, we settle the question for cardinality $\aleph_2$. For the general case, we give necessary conditions in order that Tor(A, B) be a direct sum of cyclic groups, and we also give sufficient conditions that are very close to the necessary ones, but the gap is not bridged completely.

Since Tor(A, B) is not affected by the nontorsion portions of A and B, we can assume without loss of generality that A and B are torsion. Further, we can specialize, as usual, to the case that A and B are both p-primary. Therefore, all groups are assumed to be p-primary. The following three theorems summarize the major known results concerning the question of when Tor(A, B) is a direct sum of cyclic groups. Following [**6**], we say that G is Σ-cyclic if G is a direct sum of cyclic groups.

THEOREM A (*Nunke* [**4**, *Corollary* 3.5]). *If* $p^\omega A \neq 0$, *then* Tor(A, B) *is* Σ-*cyclic only if* B *is* Σ-*cyclic.*

The preceding result also appears in [**6**] as Theorem 12(i).

THEOREM B (*Nunke* [**6**, *Theorem* 15]). *If A and B have the same uncountable cardinality m and each of A and B has the property that every subgroup of cardinality less than m is Σ-cyclic, then* $\mathrm{Tor}(A, B)$ *is Σ-cyclic.*

THEOREM C (*Nunke* [**6**, *Corollary* 16]). *If $p^\omega A = 0 = p^\omega B$ and the cardinality of neither A nor B exceeds $\aleph_1$, then* $\mathrm{Tor}(A, B)$ *is Σ-cyclic.*

In this paper we shall generalize the preceding results. It is perhaps of some interest, too, that the proofs herein are not dependent upon the above results. In particular, an alternate proof of Theorem A is established that does not require, for example, the $p^{\omega+1}$-purity of a p^ω-high subgroup nor anything similar.

Since purification can be accomplished without changing the cardinality of an infinite subgroup, we shall deal almost exclusively with pure subgroups. Repeatedly used will be the result of L. Fuchs [**1**, Theorem 63.2] that if $A \rightarrowtail B \twoheadrightarrow C$ is pure exact then so is

$$\mathrm{Tor}(A, X) \rightarrowtail \mathrm{Tor}(B, X) \twoheadrightarrow \mathrm{Tor}(C, X).$$

This leads to the following theorem, which will be one of our main tools.

THEOREM 1. *Let A_0 and B_0 be pure subgroups of A and B, respectively. Then the following are true.*

(i) $\mathrm{Tor}(A_0, B_0)$ *is a pure subgroup of* $\mathrm{Tor}(A, B)$.

(ii) $\mathrm{Tor}(A_0, B_0)$ *is a direct summand of* $\mathrm{Tor}(A, B)$ *if, in addition to the purity of A_0 and B_0, both* $\mathrm{Tor}(A/A_0, B_0)$ *and* $\mathrm{Tor}(A, B/B_0)$ *are Σ-cyclic.*

(iii) *If, in addition to the preceding hypotheses,* $\mathrm{Tor}(A_0, B_0)$ *is Σ-cyclic, then* $\mathrm{Tor}(A, B)$ *is Σ-cyclic.*

Proof. (i) The purity of $\mathrm{Tor}(A_0, B_0)$ in $\mathrm{Tor}(A, B)$ follows immediately by transitivity of purity since $\mathrm{Tor}(A_0, B_0)$ is pure in $\mathrm{Tor}(A, B_0)$ and since $\mathrm{Tor}(A, B_0)$ is pure in $\mathrm{Tor}(A, B)$. (ii) If we specify that $\mathrm{Tor}(A, B/B_0)$ is Σ-cyclic, the pure-exact sequence

$$\mathrm{Tor}(A, B_0) \rightarrowtail \mathrm{Tor}(A, B) \twoheadrightarrow \mathrm{Tor}(A, B/B_0)$$

must split due to the pure projectivity of the Σ-cyclic group $\mathrm{Tor}(A, B/B_0)$. Therefore, $\mathrm{Tor}(A, B_0)$ is a direct summand of $\mathrm{Tor}(A, B)$. Similarly, if $\mathrm{Tor}(A/A_0, B_0)$ is Σ-cyclic, the pure-exact sequence

$$\mathrm{Tor}(A_0, B_0) \rightarrowtail \mathrm{Tor}(A, B_0) \twoheadrightarrow \mathrm{Tor}(A/A_0, B_0)$$

must split. Thus $\mathrm{Tor}(A_0, B_0)$ is a direct summand of $\mathrm{Tor}(A, B_0)$ and therefore of $\mathrm{Tor}(A, B)$. (iii) The last assertion follows from the observation that, under the still more comprehensive hypothesis,

$$\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A/A_0, B_0) \oplus \mathrm{Tor}(A_0, B_0) \oplus \mathrm{Tor}(A, B/B_0),$$

and is Σ-cyclic.

For the purpose of this paper we employ the following technical definition of a tower of a group G.

DEFINITION. By a tower of a group G we mean a chain of pure subgroups $\{G_\alpha\}_{\alpha<\lambda}$, indexed by some ordinal λ, satisfying the following conditions whenever α and β are less than the index ordinal λ.

(1) $G_\alpha \subseteq G_\beta$ if $\alpha < \beta$.

(2) $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ if β is a limit.

(3) $G = \bigcup_{\alpha<\lambda} G_\alpha$.

It is to be emphasized that herein a tower is always pure, that is, the subgroups G_α must be pure. If G_α is either zero or G for each $\alpha < \lambda$, the tower $\{G_\alpha\}_{\alpha<\lambda}$ of G is said to be *trivial*, whereas if

(4) $|G_\alpha| < |G|$

is satisfied for all $\alpha < \lambda$ the tower is said to be *proper*. For example, a countably infinite group without elements of infinite height has a proper tower consisting of finite subgroups because such a group is Σ-cyclic. But a finite group or a countable group having elements of infinite height can have no proper tower. The next proposition shows that these are the only exceptions; compare Lemma 14 in [**6**].

PROPOSITION 1. *Every uncountable group has a proper tower.*

Proof. Suppose that $|G| = m > \aleph_0$ and let λ denote the cofinality of m, $\mathrm{cof}(m) = \lambda$. By the choice of λ, there are subsets H_α of G having cardinality less than m such that $G = \bigcup_{\alpha<\lambda} H_\alpha$. Assume that $\gamma < \lambda$ and that pure subgroups G_α of G have already been chosen for $\alpha < \gamma$ so that conditions (1), (2), and (4) are satisfied whenever $\alpha, \beta < \gamma$. Further, assume that $G_{\alpha+1} \supseteq H_\alpha$ whenever $\alpha + 1 < \gamma$. If γ is a limit, define $G_\gamma = \bigcup_{\alpha<\gamma} G_\alpha$. Since purity is an inductive property, G_γ is pure in G. Moreover, $|G_\gamma| < m$, for otherwise $m = \sup\{m_\alpha\}_{\alpha<\gamma}$ where $m_\alpha = |G_\alpha| < m$. But this contradicts $\mathrm{cof}(m) = \lambda$ since $\gamma < \lambda$. Now we turn to the case that γ is not a limit. Since $|G_{\gamma-1}| < m$ and since $|H_{\gamma-1}| < m$, there is a pure subgroup G_γ of G containing both $G_{\gamma-1}$ and $H_{\gamma-1}$ and having cardinality less than m; purification can be accomplished without changing the cardinality [**1**, Proposition 26.2]. Hence the induction survives, and the pure subgroups G_α, $\alpha < \lambda$, form a proper tower; condition (3) is satisfied because $G_{\alpha+1} \supseteq H_\alpha$.

We shall now present necessary conditions for $\mathrm{Tor}(A, B)$ to be a direct sum of cyclic groups. The fact that cardinalities are reduced (in the interesting cases) makes the following theorem a valid reduction formula.

THEOREM 2. *Necessary conditions for* $\mathrm{Tor}(A, B)$ *to be* Σ*-cyclic are that* A *and* B *have towers* $\{A_\alpha\}_{\alpha<\lambda}$ *and* $\{B_\alpha\}_{\alpha<\lambda}$, *respectively, such that, for all* $\alpha < \lambda$, *the following hold.*

$$(*) \qquad \begin{aligned} &\text{(a)} \quad \mathrm{Tor}(A_{\alpha+1}/A_\alpha, B_\alpha) \text{ is } \Sigma\text{-cyclic.} \\ &\text{(b)} \quad \mathrm{Tor}(A_\alpha, B_\alpha) \text{ is } \Sigma\text{-cyclic.} \\ &\text{(c)} \quad \mathrm{Tor}(A_\alpha, B_{\alpha+1}/B_\alpha) \text{ is } \Sigma\text{-cyclic.} \end{aligned}$$

If A *and* B *have the same uncountable cardinality, then both towers are to be made proper. If not, but the larger group is uncountable then its tower is to be made proper and the tower of the smaller group is to be made trivial.*

Proof. Assume that $\mathrm{Tor}(A, B)$ is Σ-cyclic; let

$$\mathrm{Tor}(A, B) = \sum_{i \in I} \langle x_i \rangle.$$

Since $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$ and since $(*)$ is symmetrical, we may assume without loss of generality that $|A| \leq |B|$. If $|B| \leq \aleph_0$, the statement of the theorem is essentially vacuous because A and B both are permitted trivial towers, which satisfy the conditions of $(*)$ trivially. Therefore, we shall assume that B is uncountable and that $A \neq 0$, for $A = 0$ leads to a triviality, too. Thus we can assume that $\mathrm{Tor}(A, B)$ is uncountable. Consequently, the index set I is uncountable used in the decomposition $\mathrm{Tor}(A, B) = \Sigma_{i\in I} \langle x_i \rangle$.

We shall first consider the case $|A| = |B| = m$ (where $m > \aleph_0$). Temporarily, let C denote $\mathrm{Tor}(A, B)$. If J is any subset of I, let C_J be the subgroup of C defined by

$$C_J = \sum_{j \in J} \langle x_j \rangle.$$

For the case being considered, each of A, B and C has cardinality m. Let $\mathrm{cof}(m) = \lambda$. According to the proof of Proposition 1, there exist proper towers $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\alpha\}_{\alpha<\lambda}$ and $\{C_\alpha\}_{\alpha<\lambda}$ of A, B and C, respectively.

However, this is not enough. We need proper towers with the following additional property

$$\text{(5)} \qquad \operatorname{Tor}(A_\alpha, B_\alpha) = C_\alpha, \quad \text{where } C_\alpha = C_{J(\alpha)}$$

for some subset $J(\alpha)$ of I. We shall presently argue the existence of proper towers that satisfy the aforementioned condition (5). For convenience of reference, let $A = \cup_{\alpha<\lambda} V_\alpha$ and $B = \cup_{\alpha<\lambda} W_\alpha$ where V_α and W_α are subsets of A and B, respectively, having cardinality less than m. Assume that $\gamma < \lambda$ and that pure subgroups A_α and B_α of A and B, respectively, have already been chosen when $\alpha < \gamma$ so that condition (5) is satisfied and so that each of the chains of subgroups $\{A_\alpha\}_{\alpha<\gamma}$ and $\{B_\alpha\}_{\alpha<\gamma}$ satisfies (upon the appropriate notational change) conditions (1), (2), and (4). Also, as part of the induction hypothesis, assume that $A_{\alpha+1} \supseteq V_\alpha$ and that $B_{\alpha+1} \supseteq W_\alpha$ whenever $\alpha + 1 < \gamma$. As usual, there are two cases to consider in the effort to extend the chains $\{A_\alpha\}_{\alpha<\gamma}$ and $\{B_\alpha\}_{\alpha<\gamma}$.

Case 1. γ is a limit. Let $A_\gamma = \cup_{\alpha<\gamma} A_\alpha$ and $B_\gamma = \cup_{\alpha<\gamma} B_\alpha$. With the exception of condition (5), everything needed to be verified for the extended chain has already been established in the proof of Proposition 1, namely, the purity of A_γ and B_γ as well as condition (4) for each one. Conditions (1) and (2) are obviously still valid for $\alpha, \beta \leq \gamma$. Hence we turn our attention to (5) and observe that if $J(\gamma) = \cup_{\alpha<\gamma} J(\alpha)$ then, in fact, (5) holds for $\alpha = \gamma$.

Case 2. γ is not a limit. Since $A_{\gamma-1}$ and $B_{\gamma-1}$ have cardinality less than m, there exist pure subgroups $A_{\gamma,1}$ and $B_{\gamma,1}$ of A and B having infinite cardinality $k < m$ and satisfying $A_{\gamma,1} \supseteq \langle A_{\gamma-1}, V_{\gamma-1}\rangle$ and $B_{\gamma,1} \supseteq \langle B_{\gamma-1}, W_{\gamma-1}\rangle$. Since $\operatorname{Tor}(A_{\gamma,1}, B_{\gamma,1})$ has cardinality k, there exists a subset $J(\gamma, 1)$ of I having cardinality k such that $\operatorname{Tor}(A_{\gamma,1}, B_{\gamma,1}) \subseteq C_{J(\gamma,1)}$. In turn, there exist pure subgroups $A_{\gamma,2} \supseteq A_{\gamma,1}$ and $B_{\gamma,2} \supseteq B_{\gamma,1}$ having cardinality k such that $\operatorname{Tor}(A_{\gamma,2}, B_{\gamma,2}) \supseteq C_{J(\gamma,1)}$; see, for example, Proposition 8 in **[6]**. Continuing in the familiar manner, we obtain ascending sequences $\{A_{\gamma,n}\}$ and $\{B_{\gamma,n}\}$ of pure subgroups of A and B and an ascending sequence $\{J(\gamma, n)\}$ of subsets of I, all having cardinality k, such that

$$\operatorname{Tor}(A_{\gamma,n}, B_{\gamma,n}) \subseteq C_{J(\gamma,n)} \subseteq \operatorname{Tor}(A_{\gamma,n+1}, B_{\gamma,n+1}).$$

Upon setting $A_\gamma = \cup A_{\gamma,n}$, $B_\gamma = \cup B_{\gamma,n}$, and $J(\gamma) = \cup J(\gamma, n)$, we obtain groups of cardinality $k < m$ that satisfy condition (5).

We have shown that if $|A|=|B|=m$ where $m>\aleph_0$ there exist proper towers $\{A_\alpha\}_{\alpha<\lambda}$ and $\{B_\alpha\}_{\alpha<\lambda}$ of A and B, respectively, satisfying condition (5). If $|A|<|B|=$ m (where $m>\aleph_0$) then we can take $A_\alpha=A$ for each α and focus our attention entirely on B. Everything goes exactly as before as far as B and $C=\operatorname{Tor}(A, B)$ are concerned. Thus, in this case, a trivial tower $\{A_\alpha\}_{\alpha<\lambda}$ of A and a proper tower $\{B_\alpha\}_{\alpha<\lambda}$ of B are constructed so that condition (5) is still satisfied. To finish the proof of the theorem, we show that the conditions of (*) are satisfied whenever $\operatorname{Tor}(A, B)$ is Σ-cyclic and $\{A_\alpha\}_{\alpha<\lambda}$ and $\{B_\alpha\}_{\alpha<\lambda}$ are towers of A and B satisfying condition (5). Condition (b) of (*) is immediate since $\operatorname{Tor}(A_\alpha, B_\alpha)$ is a subgroup of $\operatorname{Tor}(A, B)$. From the inclusion

$$\operatorname{Tor}(A_{\alpha+1}, B_\alpha)/\operatorname{Tor}(A_\alpha, B_\alpha) \subseteq \operatorname{Tor}(A_{\alpha+1}, B_{\alpha+1})/\operatorname{Tor}(A_\alpha, B_\alpha)$$

and the fact that

$$\operatorname{Tor}(A_{\alpha+1}, B_{\alpha+1})/\operatorname{Tor}(A_\alpha, B_\alpha) = \sum_{i\in J(a+1)-J(\alpha)} \langle x_i \rangle,$$

we conclude that $\operatorname{Tor}(A_{\alpha+1}, B_\alpha)/\operatorname{Tor}(A_\alpha, B_\alpha)$ is Σ-cyclic. Hence the exactness of the sequence

$$\operatorname{Tor}(A_\alpha, B_\alpha) \rightarrowtail \operatorname{Tor}(A_{\alpha+1}, B_\alpha) \twoheadrightarrow \operatorname{Tor}(A_{\alpha+1}/A_\alpha, B_\alpha)$$

implies that $\operatorname{Tor}(A_{\alpha+1}/A_\alpha, B_\alpha)$ is Σ-cyclic. Therefore, condition (a) is satisfied. Likewise, by symmetry, condition (c) is satisfied, and Theorem 2 is proved.

COROLLARY 1. *If* $|A|\aleph_0<|B|$, *then* $\operatorname{Tor}(A, B)$ *is* Σ-*cyclic if and only if* B *has a proper tower* $\{B_\alpha\}_{\alpha<\lambda}$ *such that* $\operatorname{Tor}(A, B_\alpha)$ *and* $\operatorname{Tor}(A, B_{\alpha+1}/B_\alpha)$ *are* Σ-*cyclic.*

Proof. The necessity of the tower follows directly from Theorem 2 because the tower for A is trivial and that of B is proper. The sufficiency follows from the fact that the pure-exact sequence

$$\operatorname{Tor}(A, B_\alpha) \rightarrowtail \operatorname{Tor}(A, B_{\alpha+1}) \twoheadrightarrow \operatorname{Tor}(A, B_{\alpha+1}/B_\alpha)$$

splits for each α since $\operatorname{Tor}(A, B_{\alpha+1}/B_\alpha)$ is Σ-cyclic.

Due to the significance of Theorem A we outline here a new and simple proof of it based on the preceding corollary. Suppose that $\operatorname{Tor}(A, B)$ is Σ-cyclic and that A has elements of infinite height. Our aim is to show that B must be Σ-cyclic. We may assume that A is countable since A has a countable (pure) subgroup with elements of infinite height. Since

$$\operatorname{Tor}(p^{\omega}A, p^{\omega}B) = p^{\omega}\operatorname{Tor}(A, B) = 0$$

and $p^{\omega}A \neq 0$, it follows at once that $p^{\omega}B = 0$. Thus if B is countable it is Σ-cyclic. Assume that B is uncountable. Corollary 1 asserts that B has a proper tower $\{B_{\alpha}\}_{\alpha<\lambda}$ such that $\operatorname{Tor}(A, B_{\alpha})$ and $\operatorname{Tor}(A, B_{\alpha+1}/B_{\alpha})$ are Σ-cyclic. Since the tower of B is proper, we conclude by induction on the cardinality of B that both B_{α} and $B_{\alpha+1}/B_{\alpha}$ are Σ-cyclic. This implies that B, itself, is Σ-cyclic since B_{α} splits out of $B_{\alpha+1}$. The proof of the following is similar and is therefore omitted.

COROLLARY 2. *Suppose that A has a subgroup of cardinality $\aleph_1$ that is not Σ-cyclic. If* $\operatorname{Tor}(A, B)$ *is Σ-cyclic, then B has a tower $\{B_{\alpha}\}_{\alpha<\lambda}$ such that $B_0 = 0$ and $B_{\alpha+1}/B_{\alpha}$ is a group of cardinality $\leq \aleph_1$ without elements of infinite height.*

In connection with Corollary 2, we perhaps should remark that a tower of $B_{\alpha+1}/B_{\alpha}$ can be used to refine the original tower $\{B_{\alpha}\}$, if necessary, to obtain $|B_{\alpha+1}/B_{\alpha}| \leq \aleph_1$. We now turn to sufficient conditions for $\operatorname{Tor}(A, B)$ to be Σ-cyclic.

THEOREM 3. *Sufficient conditions for* $\operatorname{Tor}(A, B)$ *to be Σ-cyclic are that A and B have towers $\{A_{\alpha}\}_{\alpha<\lambda}$ and $\{B_{\alpha}\}_{\alpha<\lambda}$, respectively, such that, for all $\alpha < \lambda$, the following hold.*

$$(**)\qquad \begin{aligned} &\text{(a)}\quad \operatorname{Tor}(A_{\alpha+1}/A_{\alpha}, B_{\alpha}) \textit{ is } \Sigma\textit{-cyclic.}\\ &\text{(b)}\quad \operatorname{Tor}(A_{\alpha}, B_{\alpha}) \textit{ is } \Sigma\textit{-cyclic.}\\ &\text{(d)}\quad \operatorname{Tor}(A_{\alpha+1}, B_{\alpha+1}/B_{\alpha}) \textit{ is } \Sigma\textit{-cyclic.}\end{aligned}$$

Proof. Letting $A_{\alpha+1}$ and $B_{\alpha+1}$ play the role of A and B in Theorem 1 and letting A_{α} and B_{α} play the role of A_0 and B_0, we conclude that

$$\operatorname{Tor}(A_{\alpha+1}, B_{\alpha+1}) = \operatorname{Tor}(A_{\alpha}, B_{\alpha}) \oplus C_{\alpha},$$

where

$$C_{\alpha} \cong \operatorname{Tor}(A_{\alpha+1}/A_{\alpha}, B_{\alpha}) \oplus \operatorname{Tor}(A_{\alpha+1}, B_{\alpha+1}/B_{\alpha})$$

is Σ-cyclic. In view of conditions (1)–(3) on a tower, it follows that

$$\operatorname{Tor}(A, B) = \operatorname{Tor}(A_0, B_0) \oplus \sum_{\alpha<\lambda} C_{\alpha}.$$

Therefore, $\operatorname{Tor}(A, B)$ is Σ-cyclic, and Theorem 3 is proved.

Observe that Theorem B (as well as its corollary Theorem C) can be deduced quickly from Theorem 3 as follows. First, according to Proposition 1, both A and B have proper towers $\{A_\alpha\}_{\alpha<\lambda}$ and $\{B_\alpha\}_{\alpha<\lambda}$ (indexed by the same ordinal λ since $|A|=|B|$). Under the hypothesis of Theorem B, the subgroups A_α and B_α must be Σ-cyclic for each α. Hence the sufficient conditions of $(**)$ are satisfied, and $\operatorname{Tor}(A, B)$ is Σ-cyclic.

Notice that there is apparently only a small gap between the necessary conditions $(*)$ and the sufficient conditions $(**)$ under which $\operatorname{Tor}(A, B)$ is Σ-cyclic with the difference between the two being the difference between condition (c) and the slightly stronger condition (d). Our next result eliminates this distinction for groups of cardinality not exceeding $\aleph_2$.

THEOREM 4. *If A and B have cardinality not exceeding $\aleph_2$, condition* $(*)$ *is necessary and sufficient in order that* $\operatorname{Tor}(A, B)$ *be Σ-cyclic.*

Proof. All that is needed is the verification that condition

(c) $\operatorname{Tor}(A_\alpha, B_{\alpha+1}/B_\alpha)$ is Σ-cyclic,

for each α, implies the stronger condition

(d) $\operatorname{Tor}(A_{\alpha+1}, B_{\alpha+1}/B_\alpha)$ is Σ-cyclic.

Crucial to the argument is the simple fact that if $\{A_\alpha\}_{\alpha<\lambda}$ is a tower of A then so is $\{A_\alpha\}_{\beta\leq\alpha<\lambda}$ for any $\beta<\lambda$. Technically speaking, the index ordinal for the new tower is μ, where $\beta+\mu=\lambda$. Now, choose $\beta<\lambda$ so that A_β fails to be Σ-cyclic if any A_α is not Σ-cyclic. In fact, choose β so that A_β has elements of infinite height if A does. The idea is to let A_β and A be as close as possible in regard to certain standard features. Claim: if condition (c) is satisfied, then so is condition (d) whenever $\alpha\geq\beta$. The claim may be verified by examining the different cases. For example, suppose that $B_{\alpha+1}/B_\alpha$ has elements of infinite height. Then A_α must be Σ-cyclic, which implies that $A_{\alpha+1}$ is Σ-cyclic according to the choice of β. If, on the other hand, $B_{\alpha+1}/B_\alpha$ is Σ-cyclic, then (d) is certainly valid no matter what $A_{\alpha+1}$ is. Having disposed of the case where $B_{\alpha+1}/B_\alpha$ has elements of infinite height and the case where $B_{\alpha+1}/B_\alpha$ is Σ-cyclic, we turn to the interesting case where neither of these is true. We shall make strong use of the fact that $|B_{\alpha+1}/B_\alpha|\leq\aleph_1$; either $|B|\leq\aleph_1$ or the tower $\{B_\alpha\}$ of B used in $(*)$ is proper because $|A|\leq\aleph_2$ also. Likewise, $|A_\alpha|\leq|A_{\alpha+1}|\leq\aleph_1$. Therefore, because $p^\omega(B_{\alpha+1}/B_\alpha)=0$ the only way that (d) can fall while (c) stands is for $A_{\alpha+1}$ to have elements of infinite height while A_α has none. This, however, is precluded by the choice of β. Thus the claim is fully supported, and condition (d) holds for the new tower $\{A_\alpha\}_{\beta\leq\alpha<\lambda}$. Therefore, $(*)$ is sufficient.

REMARK. Whether $(*)$, in the context of Theorem 2, is always sufficient or not we leave open. If it is, a more general approach than that used in the proof of Theorem 4 apparently will be required for the proof. Nevertheless, we venture the following.

Conjecture. If A and B are groups of cardinality not exceeding $\aleph_\omega$, then condition $(*)$ of Theorem 2 is necessary and sufficient for $\mathrm{Tor}(A, B)$ to be Σ-cyclic.

We close with the following interesting observation.

PROPOSITION 2. *In the following, the groups A, B and X are all without elements of infinite height,*

$$p^\omega A = p^\omega B = p^\omega X = 0.$$

If $\mathrm{Tor}(A, X)$ *is Σ-cyclic whenever* $|X| < \aleph$ *and if* $|B| \leq \aleph$, *then* $\mathrm{Tor}(\mathrm{Tor}(A, B), X)$ *is Σ-cyclic whenever* $|X| \leq \aleph$.

Proof. If $|B| < \aleph$, the conclusion follows trivially because $\mathrm{Tor}(A, B)$ is Σ-cyclic. Likewise, if $|X| < \aleph$, the conclusion follows immediately from the commutativity and associativity of Tor, which yields

$$\mathrm{Tor}(\mathrm{Tor}(A, B), X) = \mathrm{Tor}(\mathrm{Tor}(A, X), B).$$

Thus we may assume that both B and X have cardinality $\aleph$. Obviously we may assume that $\aleph > \aleph_0$. We shall prove that $\mathrm{Tor}(A, \mathrm{Tor}(B, X))$ is Σ-cyclic. Let $\{B_\alpha\}_{\alpha<\lambda}$ and $\{X_\alpha\}_{\alpha<\lambda}$ be proper towers of B and X. Consider the pure exact sequence

$$\mathrm{Tor}(B_{\alpha+1}/B_\alpha, X_\alpha) \rightarrowtail \mathrm{Tor}(B_{\alpha+1}, X_{\alpha+1})/\mathrm{Tor}(B_\alpha, X_\alpha)$$
$$\twoheadrightarrow \mathrm{Tor}(B_{\alpha+1}, X_{\alpha+1}/X_\alpha).$$

Since $\mathrm{Tor}(B_{\alpha+1}/B_\alpha, X_\alpha)$ and $\mathrm{Tor}(B_{\alpha+1}, X_{\alpha+1}/X_\alpha)$ are without elements of infinite height, $\mathrm{Tor}(B_{\alpha+1}, X_{\alpha+1})/\mathrm{Tor}(B_\alpha, X_\alpha)$ can have no elements of infinite height, and it has cardinality less than $\aleph$. Consequently,

$$\mathrm{Tor}(A, \mathrm{Tor}(B_{\alpha+1}, X_{\alpha+1}))/\mathrm{Tor}(A, \mathrm{Tor}(B_\alpha, X_\alpha))$$

is Σ-cyclic, for each α, and so is $\mathrm{Tor}(A, \mathrm{Tor}(B_\alpha, X_\alpha))$. Therefore, it follows that $\mathrm{Tor}(A, \mathrm{Tor}(B, X))$ is Σ-cyclic.

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