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**REGULAR EMBEDDINGS OF A GRAPH**

HIROSHI MAEHARA

## REGULAR EMBEDDINGS OF A GRAPH

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In this paper we study embeddings of a graph  $G$  in Euclidean space  $R^n$  that are 'regular' in the following sense: given any two distinct vertices  $u$  and  $v$  of  $G$ , the distance between the corresponding points in  $R^n$  equals  $\alpha$  if  $u$  and  $v$  are adjacent, and equals  $\beta$  otherwise. It is shown that for any given value of  $s = (\beta^2 - \alpha^2)/\beta^2$ , the minimum dimension of a Euclidean space in which  $G$  is regularly embeddable is determined by the characteristic polynomials of  $G$  and  $\bar{G}$ .

**1. Introduction.** To embed a graph in Euclidean spaces with various restrictions, and to find the minimum dimension of the space for these embeddings, are interesting problems [1], [4], [5]. In this paper we consider a regular embedding of a graph.

An embedding of a graph  $G$  in a Euclidean space  $R^n$  is called a *regular embedding* of  $G$  provided that, for any two distinct vertices  $u$  and  $v$  of  $G$ , the distance between the corresponding points in  $R^n$  equals  $\alpha$  if  $u$  and  $v$  are adjacent, and equals  $\beta$  otherwise. The vertices of  $G$  are mapped onto distinct points of  $R^n$ , but there is no restriction on the crossing of edges. The value  $s = (\beta^2 - \alpha^2)/\beta^2$  is called the *parameter* of the regular embedding. Let  $\dim(G, s)$  denote the minimum number  $n$  such that  $G$  can be regularly embedded in  $R^n$  with parameter  $s$ .

Consider, for example, the circuit graph  $C_5$ . For every regular embedding of  $C_5$ , it is seen that

$$\frac{1}{2}(-\sqrt{5} - 1) \leq s \leq \frac{1}{2}(\sqrt{5} - 1)$$

and

$$\dim(C_5, s) = \begin{cases} 2 & \text{if } s = \frac{1}{2}(\pm\sqrt{5} - 1), \\ 4 & \text{otherwise.} \end{cases}$$

The 'critical' embeddings of  $C_5$  in  $R^2$  with  $s = \frac{1}{2}(\pm\sqrt{5} - 1)$  are illustrated in Fig. 1.

Let  $\phi(G; x)$  denote the characteristic polynomial of a graph  $G$  (that is,  $\phi(G; x) = |x\mathbf{I} - \mathbf{A}(G)|$ ), and put

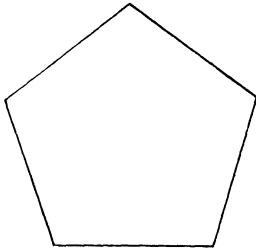
$$\Phi(G; x) = \phi(G; -x) - (-1)^g \phi(\bar{G}; x - 1),$$

where  $g$  is the number of vertices of  $G$ , and  $\bar{G}$  is the complement of  $G$ . Let  $x^-$  and  $x^+$  be, respectively, the minimum root and the maximum root of the polynomial  $\Phi(G; x)$ . Suppose that  $x^- < 0$ , and  $1 < x^+$ . Then our results are stated as follows.

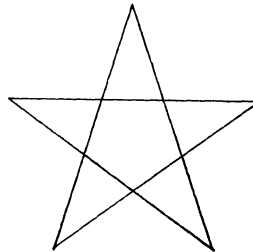
For every regular embedding of  $G$ ,  $1/x^- \leq s \leq 1/x^+$  and

$$\dim(G, 1/x^*) = g - 1 - (\text{the multiplicity of the root } x^*),$$

where  $x^* = x^-$  or  $x^+$ . For other values of  $s$ ,  $\dim(G, s) = g - 1$ .



$$s = \frac{1}{2} \left( +\sqrt{5} - 1 \right)$$



$$s = \frac{1}{2} \left( -\sqrt{5} - 1 \right)$$

FIGURE 1

**2. A theorem for isometric embeddings.** We shall recall a theorem in distance geometry ([2], Ch. IV). Let  $S = \{p_0, \dots, p_k\}$  be a finite semimetric space with distance function  $d$ . The determinant

$$\begin{vmatrix} 0 & 1 & \cdot & \dots & 1 \\ 1 & 0 & d_{01} & \dots & d_{0k} \\ \cdot & d_{10} & 0 & & \cdot \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & d_{k0} & \cdot & \dots & 0 \end{vmatrix} \quad d_{ij} := d(p_i, p_j)^2$$

is called the *Cayley-Menger determinant* of the semimetric space  $(S, d)$ , and is denoted by  $D(S)$  or by  $D(p_0 \cdots p_k)$ . Note that the value of the determinant does not depend on a labeling (ordering)  $p_0, \dots, p_k$  of the points of  $S$ .

If  $S = \{p_0, \dots, p_k\} \subset R^n$ ,  $n \geq k$ , then we denote by  $\text{Vol}(S)$  the  $k$ -dimensional volume of the simplex (perhaps degenerate) spanned by  $S$ . In this case,  $\text{Vol}(S)$  and the Cayley-Menger determinant of  $S$  are related as follows:

$$\text{Vol}(S)^2 = \frac{(-1)^{k+1}}{2^k (k!)^2} D(S).$$

For details, see Blumenthal [2], p. 98.

A semimetric space  $S$  is said to be irreducibly embeddable in  $R^n$  provided that it is isometric to a subset of  $R^n$  but not isometric to any subset of  $R^{n-1}$ .

**THEOREM (Blumenthal [2]).** *A semimetric space  $S$  is irreducibly embeddable in  $R^n$  if and only if*

(i)  *$S$  contains an  $(n + 1)$ -tuple  $p_0, \dots, p_n$  such that*

$$\text{sign } D(p_0 \cdots p_j) = (-1)^{j+1} \quad (j = 1, \dots, n);$$

(ii) *for every pair  $x, y$  of points of  $S$ ,*

$$D(p_0 \cdots p_n, x) = D(p_0 \cdots p_n, y) = D(p_0 \cdots p_n, x, y) = 0.$$

**3. The Cayley-Menger polynomial of a graph.** A regular embedding of a graph  $G$  with parameter  $s$  is called, briefly, an  $s$ -embedding of  $G$ . To apply Blumenthal's theorem let us define a distance function  $d_s$  on the vertex set  $V(G)$  of  $G$  by

$$d_s(u, v) = \begin{cases} 0 & \text{if } u = v, \\ (1 - s)^{1/2} & \text{if } u \text{ and } v \text{ are adjacent,} \\ 1 & \text{otherwise.} \end{cases}$$

Then the Cayley-Menger determinant of the semimetric space  $(V(G), d_s)$  is a polynomial in  $s$ , which we shall call the *Cayley-Menger polynomial* of  $G$  and denote by  $\text{CM}(G; s)$ . For example,  $\text{CM}(K_n; s) = (-1)^n n(1 - s)^{n-1}$ , and  $\text{CM}(\overline{K}_n; s) = (-1)^n n$ , where  $K_n$  denotes the complete graph of order  $n$ .

Since there is a 0-embedding of  $G$  in a Euclidean space as a regular simplex of side-length 1, we can restate Blumenthal's theorem in the following way. For any two graphs  $G$  and  $H$ , let  $H \subset G$  mean that  $H$  is an induced subgraph of  $G$ .

**THEOREM 1.** *There exists a  $t$ -embedding ( $t < 1$ ) of a graph  $G$  in  $R^n$  if and only if there is a  $G_0 \subset G$  with  $g_0 (\leq n + 1)$  vertices such that*

(i) *for any  $F \subset G_0$ ,  $\text{sign CM}(F; t) = \text{sign CM}(F; 0)$ ;*

(ii) *for any  $G_0 \subsetneq H \subset G$ ,  $\text{CM}(H; t) = 0$ .*

*In this case,  $\dim(G, t) = g_0 - 1$ .*

Let  $s^+(G)$  be the minimum positive root of the polynomial  $\text{CM}(G; s)$ , if it exists, and  $\infty$  otherwise. For example,  $s^+(K_2) = 1$ , and  $s^+(\overline{K}_n) = \infty$ . Let  $s^-(G)$  be the maximum negative root of  $\text{CM}(G; s)$ , if it exists, and  $-\infty$  otherwise.

LEMMA 1. For  $H \subset G$ ,  $s^-(H) \leq s^-(G) < s^+(G) \leq s^+(H)$ .

*Proof.* We shall only show that  $s^+(G) \leq s^+(H)$ . Let  $s_0$  be the minimum value of  $s^+(F)$  for  $F \subset G$ . It is sufficient to show that  $s_0 \geq s^+(G)$ . If  $s_0 = \infty$  then clearly  $s_0 = s^+(G) = \infty$ . Suppose  $s_0 < \infty$  and  $\text{CM}(F_0; s_0) = 0$  for some  $F_0 \subset G$ . In this case,  $s_0 \leq 1$ , because  $s^+(K_2) = 1$ . Since  $\text{sign CM}(F; s) = \text{sign CM}(F; 0)$  for  $F \subset G$  and for  $0 \leq s < s_0$ , it follows from Theorem 1 that for every  $0 \leq s < s_0$ , there is an  $s$ -embedding  $f_s: G \rightarrow R^n$  of  $G$  where  $n + 1 \geq g := |V(G)|$ , the cardinality of the vertex set  $V(G)$  of  $G$ . Since  $\text{Vol}(f_s(V(F_0)))^2$  is the product of  $\text{CM}(F_0; s)$  by a constant, and  $\text{CM}(F_0; s_0) = 0$ , we have

$$\text{Vol}(f_s(V(F_0))) \rightarrow 0 \quad \text{as } s \rightarrow s_0.$$

Hence we have

$$\text{Vol}(f_s(V(G))) \rightarrow 0 \quad \text{as } s \rightarrow s_0.$$

Then by the continuity,  $\text{CM}(G; s_0) = 0$ , and hence  $s_0 \geq s^+(G)$ .

Note that if  $G$  contains at least one edge, then  $s^+(G) \leq 1$ .

THEOREM 2. For every  $s^-(G) < s < \min(s^+(G), 1)$ , there is an  $s$ -embedding of  $G$ , and  $\dim(G, s) = g - 1$ . If  $-\infty < s^*(G) < 1$  then there is an  $s^*(G)$ -embedding of  $G$ , where  $s^*(G) = s^-(G)$  or  $s^+(G)$ .

*Proof.* We shall only prove the existence of an  $s^+(G)$ -embedding of  $G$ , provided that  $s^+(G) < 1$ . Let  $H$  be a maximal induced subgraph of  $G$  such that  $\text{CM}(H; s^+(G)) \neq 0$ . Then

(i) if  $F \subset H$  then  $s^+(G) < s^+(H) \leq s^+(F)$ , and hence

$$\text{sign CM}(F; s^+(G)) = \text{sign CM}(F; 0);$$

(ii) if  $H \subsetneq F \subset G$ , then  $\text{CM}(F; s^+(G)) = 0$  by the maximality of  $H$ . Hence there is an  $s^+(G)$ -embedding of  $G$ , by Theorem 1.

4. Calculation of  $\text{CM}(G; s)$ . Let  $\mathbf{I}_r$  and  $\mathbf{J}_r$  denote, respectively, the identity  $r \times r$  matrix and  $r \times r$  matrix each entry of which is 1. (In the following, the subscripts are often omitted.) Put  $\mathbf{K}_r = \mathbf{J}_r - \mathbf{I}_r$  and

$$\mathbf{B}(G) = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \mathbf{A}(G) & \\ 0 & & \end{bmatrix},$$

where  $\mathbf{A}(G)$  is the adjacency matrix of  $G$ , and put  $g = |V(G)|$ . Then, by

the definition of  $\text{CM}(G; s)$ ,

$$\text{CM}(G; s) = |\mathbf{K}_{g+1} - s\mathbf{B}(G)| = |s\mathbf{K}| |(1/s)\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)|.$$

Since  $\mathbf{K}^{-1} = (1/g)\mathbf{J} - \mathbf{I}$ ,

$$\begin{aligned} \mathbf{K}^{-1}\mathbf{B}(G) &= (1/g)\mathbf{JB}(G) - \mathbf{B}(G) \\ &= \begin{bmatrix} 0 & d_1/g & \cdots & d_g/g \\ \vdots & & & \vdots \\ 0 & d_1/g & \cdots & d_g/g \end{bmatrix} - \mathbf{B}(G), \end{aligned}$$

where  $d_i$  is the sum of entries in the  $i$ th column of  $\mathbf{A}(G)$ . In the matrix  $x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)$ , by subtracting the top row from other rows, we have

$$|x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)| = \begin{vmatrix} x & -d_1/g & \cdots & -d_g/g \\ -x & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ -x & & & \end{vmatrix}.$$

On the right-hand side, adding to the top row the product of the  $i$ th row by  $1/g, i = 2, \dots, g + 1$ , we have

$$\begin{aligned} |x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)| &= \begin{vmatrix} 0 & x/g & \cdots & x/g \\ -x & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ -x & & & \end{vmatrix} \\ &= -x^2/g \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ 1 & & & \end{vmatrix} \\ &= -x^2/g \left\{ \begin{vmatrix} x & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ 1 & & & \end{vmatrix} - x |x\mathbf{I} + \mathbf{A}(G)| \right\} \\ &= -x^2/g \{ |x\mathbf{I}_{g+1} + \mathbf{A}(G + K_1)| - x |x\mathbf{I} + \mathbf{A}(G)| \} \\ &\text{(where } G + K_1 \text{ is the join of } G \text{ and } K_1 \text{, defined by } \overline{G + K_1} = \overline{G} \cup \overline{K_1}\text{)} \\ &= -x^2/g \{ (-1)^{g+1} |(-x)\mathbf{I} - \mathbf{A}(G + K_1)| \\ &\quad + (-1)^{g+1} x |(-x)\mathbf{I} - \mathbf{A}(G)| \} \\ &= (-1)^g x^2/g \{ \phi(G + K_1; -x) + x\phi(G; -x) \}. \end{aligned}$$

Using Cvetković's theorem ([3], p. 57):

$$\begin{aligned} \phi(G_1 + G_2; x) &= (-1)^{g_2} \phi(G_1; x) \phi(\overline{G}_2; -x - 1) \\ &\quad + (-1)^{g_1} \phi(G_2; x) \phi(\overline{G}_1; -x - 1) \\ &\quad - (-1)^{g_1 + g_2} \phi(\overline{G}_1; -x - 1) \phi(\overline{G}_2; -x - 1), \quad g_i = |V(G_i)|. \end{aligned}$$

After a brief calculation, we have

$$|x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)| = (-1)^g x^2 / g \{ \phi(G; -x) - (-1)^g \phi(\overline{G}; x - 1) \}.$$

Since  $|(1/x)\mathbf{K}_{g+1}| = (-1)^g g(1/x)^{g+1}$ , we have the following:

**THEOREM 3.**

$$\text{CM}(G; 1/x) = (1/x)^{g-1} \{ \phi(G; -x) - (-1)^g \phi(\overline{G}; x - 1) \}.$$

**5. Bounds on the parameter  $s$ . Put**

$$\Phi(G; x) = \phi(G; -x) - (-1)^g \phi(\overline{G}; x - 1),$$

where  $g$  is the number of vertices of  $G$ . Then Theorem 3 says

$$\text{CM}(G; s) = s^{g-1} \Phi(G; 1/s).$$

Note that  $s_0 \neq 0$  is a root of  $\text{CM}(G; s)$  if and only if  $1/s_0$  is a root of  $\Phi(G; x)$ . Thus we have the following theorem:

**THEOREM 4.** *The polynomial  $\Phi(G; x)$  has a positive root if and only if  $s^+(G) < \infty$ . In this case,  $1/s^+(G)$  is the maximum root of  $\Phi(G; x)$ . The polynomial  $\Phi(G; x)$  has a negative root if and only if  $s^-(G) > -\infty$ . In this case,  $1/s^-(G)$  is the minimum root of  $\Phi(G; x)$ .*

Now let  $V(G) = \{v_1, \dots, v_g\}$ ,  $g \geq 2$ , and put

$$G_{i_1 \dots i_k} = G - v_{i_1} - \dots - v_{i_k}, \quad k \leq g - 1.$$

**LEMMA 2.**

$$\frac{d^k}{dx^k} \Phi(G; x) = (-1)^k k! \sum_{\{i_1 \dots i_k\}} \Phi(G_{i_1 \dots i_k}; x),$$

where the summation extends over all  $k$ -subsets of  $\{1, \dots, g\}$ .

*Proof.* Since

$$\frac{d}{dx} \phi(G; x) = \sum_{i=1}^{i=g} \phi(G_i; x)$$

(see [6], p. 331), we have

$$\begin{aligned} \frac{d}{dx} \Phi(G; x) &= \frac{d}{dx} \{ \phi(G; -x) - (-1)^g \phi(\bar{G}; x - 1) \} \\ &= - \sum \{ \phi(G_i; -x) - (-1)^g \phi(\bar{G}_i; x - 1) \} \\ &= - \sum \Phi(G_i; x). \end{aligned}$$

Differentiating repeatedly, we have the lemma.

**LEMMA 3.** *If  $-\infty < s^*(G) < \infty$ , and the multiplicity of the root  $x^* = 1/s^*(G)$  of  $\Phi(G; x)$  equals  $k + 1$ , then*

$$\Phi(G_{i_1, \dots, i_j}; x^*) = 0 \quad \text{for } j \leq k, \{i_1, \dots, i_j\} \subset \{1, \dots, g\},$$

where  $s^*(G) = s^-(G)$  or  $s^+(G)$ .

*Proof.* Since  $\Phi^{(j)}(G; x^*) = 0$  for  $j \leq k$  it follows from the above lemma that

$$(\#) \quad (-1)^j j! \sum \Phi(G_{i_1, \dots, i_j}; x^*) = 0.$$

Since there is an  $s$ -embedding of  $G$  for every  $s^-(G) < s < s^+(G)$ , it follows by the continuity that

$$\text{sign CM}(G_{i_1, \dots, i_j}; s^*(G)) = (-1)^{g-j} \quad \text{or } 0.$$

Since  $\text{CM}(G_{i_1, \dots, i_j}; s^*(G)) = s^*(G)^{g-j-1} \Phi(G_{i_1, \dots, i_j}; x^*)$ , it follows that the non-zero term of the left-hand side of (#) must have the same sign, which is impossible. Hence  $\Phi(G_{i_1, \dots, i_j}; x^*) = 0$ .

**THEOREM 5.** *If there is a  $t$ -embedding of  $G$  then*

$$s^-(G) \leq t \leq s^+(G).$$

*Proof.* It is clear that the theorem holds true for graphs with fewer vertices than three. Assume that there exists a graph for which the theorem does not hold, and let  $H$  be one of such graphs which is minimal in the number of vertices. Then there is a  $t$ -embedding of  $H$  such that



$t < s^-(H)$  or  $s^+(H) < t$ . Suppose that  $s^+(H) < t$ . (The case  $t < s^-(H)$  is similar, and is omitted.) Let  $V(H) = \{v_1, \dots, v_h\}$ , and put  $H_i = H - v_i$ ,  $i = 1, \dots, h$ ;  $x^+ = 1/s^+(H)$ . Then  $x^+$  is the maximum root of  $\Phi(H; x)$  and  $1/t < x^+$ . By the minimality of  $H$ ,  $t \leq s^+(H_i)$ ,  $i = 1, \dots, h$ .

Now we show that  $x^+$  is a simple root of  $\Phi(H; x)$ . If  $x^+$  is a multiple root, then  $\Phi(H_i; x^+) = 0$  by Lemma 3, which implies that  $s^+(H_i) = s^+(H) < t$ , a contradiction. Thus  $x^+$  must be a simple root of  $\Phi(H; x)$ .

Since  $\Phi(H; x)$  changes sign when  $x$  passes through  $x^+$ , a simple root,  $\text{CM}(H; s)$  also changes sign when  $s$  passes through  $s^+(H)$ . Since  $\text{sign CM}(H; t) = \text{sign CM}(H; 0)$  or  $\text{CM}(H; t) = 0$  (because there is a  $t$ -embedding of  $H$ ), and  $s^+(H) < t$ , there must be a root  $s_1$  of  $\text{CM}(H; s)$  such that  $s^+(H) < s_1 \leq t$ . Thus  $\Phi(H; x)$  has a root  $x_1 = 1/s_1$  such that  $1/t \leq x_1 < x^+$ . Then, by Rolle's theorem, there is a  $\xi$ ,  $x_1 < \xi < x^+$ , such that  $\Phi'(H; \xi) = 0$ . But since  $1/\xi < 1/x_1 \leq t \leq s^+(H_i)$ , there is a  $(1/\xi)$ -embedding of  $H_i$ , and  $\Phi(H_i; \xi)$  is non-zero and has the same sign for every  $i$ . This contradicts the fact that  $0 = \Phi'(H; \xi) = -\sum \Phi(H_i; \xi)$ .

**6. The dimension of a critical embedding.** Let  $G$  be a graph with vertex set  $V(G) = \{v_1, \dots, v_g\}$ , and put

$$G_{i_1 \dots i_j} = G - v_{i_1} - \dots - v_{i_j}, \quad j < g.$$

**THEOREM 6.** *If  $-\infty < s^*(G) < 1$ , and the multiplicity of the root  $x^* = 1/s^*(G)$  of  $\Phi(G; x)$  equals  $k$ , then*

$$\dim(G, s^*(G)) = g - k - 1,$$

where  $s^*(G) = s^-(G)$  or  $s^+(G)$ .

*Proof.* Since

$$0 \neq \Phi^{(k)}(G; x^*) = (-1)^k k! \sum_{\{i_1 \dots i_k\}} \Phi(G_{i_1 \dots i_k}; x^*),$$

there is a  $\{j_1, \dots, j_k\}$  such that  $\Phi(G_{j_1 \dots j_k}; x^*) \neq 0$ . By Lemma 1, it follows easily that if  $F \subset G_{j_1 \dots j_k}$  then

$$\text{sign CM}(F; s^*(G)) = \text{sign CM}(F; 0).$$

Using Lemma 3, it follows that if  $G_{j_1 \dots j_k} \subsetneq H \subset G$  then  $\text{CM}(H; s^*(G)) = 0$ . Hence  $\dim(G; s^*(G)) = g - k - 1$ , by Theorem 1.

7. **On regular graphs.** If  $G$  is a regular  $\rho$ -valent graph with  $g$  vertices, then by Sachs' theorem ([6], p. 56),

$$\phi(\bar{G}; x) = (-1)^g \frac{x + \rho + 1 - g}{x + \rho + 1} \phi(G; -x - 1).$$

Hence we have

$$\Phi(G; x) = \frac{g}{x + \rho} \phi(G; -x).$$

Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g$  be the eigenvalues of  $A(G)$ . Then  $\lambda_1 = \rho$  is a simple root of  $\phi(G; x)$ , and  $\lambda_g < 0$ . Therefore

$$s^+(G) = -1/\lambda_g$$

and

$$s^-(G) = \begin{cases} -1/\lambda_2 & \text{if } \lambda_2 > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

**EXAMPLE.** Let  $G$  be the Petersen graph. The characteristic polynomial of  $G$  is  $(x - 3)(x - 1)^5(x + 2)^4$ . Hence  $s^+(G) = 1/2$ ,  $s^-(G) = -1$ , and

$$\dim(G, s) = \begin{cases} 4, & s = -1, \\ 5, & s = 1/2, \\ 9, & -1 < s < 1/2. \end{cases}$$

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