

# Pacific Journal of Mathematics

***q*-KONHAUSER POLYNOMIALS**

WALEED A. AL-SALAM AND A. VERMA

## $q$ -KONHAUSER POLYNOMIALS

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**A pair of biorthogonal sets of polynomials suggested by the  $q$ -Laguerre polynomials are constructed. These are biorthogonal on  $(0, \infty)$  with respect to a continuous or discrete distribution function. Several properties are also given.**

**1. Introduction.** Let  $\alpha(x)$  be a distribution function on the interval (finite or infinite)  $[a, b]$  with infinitely many points of increase and such that  $\int_a^b x^n d\alpha(x) < \infty$  for all  $n = 0, 1, 2, \dots$

The set of polynomials in  $x$ ,  $\{P_n(x)\}$ , and the set of polynomials  $\{Q_n(x)\}$ ,  $\deg Q_n(x) = n$  for  $n = 0, 1, 2, \dots$  are said to be biorthogonal with respect to  $d\alpha(x)$  on  $(a, b)$  if

$$(1.1) \quad \int_a^b P_n(x)Q_m(x)d\alpha(x) \begin{cases} = 0 & (n \neq m) \\ \neq 0 & (n = m) \end{cases}.$$

Didon [4] and Deruyts [3] considered this concept in some detail. For example for a given  $\{P_n(x)\}$  the set  $\{Q_n(x)\}$  is uniquely determined and conversely.

Both Didon and Deruyts paid special attention to the situation in which  $P_n(x)$  is a polynomial of degree  $n$  in  $x^k$  ( $k$  fixed). In this case (1.1) is equivalent to

$$(1.2) \quad \int_a^b x^i P_n(x) d\alpha(x) = 0 \quad \text{and} \quad \int_a^b x^{ik} Q_n(x) d\alpha(x) = 0$$

( $0 \leq i < n$ ).

and both integrals are  $\neq 0$  for  $i = n$ .

Thus if  $k = 1$ ,  $\{P_n(x)\}$  and  $\{Q_n(x)\}$  collapse to the set of orthogonal polynomials associated with  $\alpha(x)$  on  $(a, b)$ .

Both Didon and Deruyts gave as examples the case in which  $d\alpha(x) = x^{\alpha-1}(1-x)^{\beta-1} dx$ , the distribution for the Jacobi polynomials on  $(0, 1)$ . Deruyts also gave the case in which  $d\alpha(x) = x^\alpha e^{-x} dx$  on  $(0, \infty)$ , the distribution for the Laguerre polynomials.

More recently these polynomials gained a sudden popularity with the interesting work of Konhauser [7, 8] and Preiser [10] (see also [2]). In particular the biorthogonal system related to the Laguerre distribution is now known as the Konhauser polynomials.

With the recent interest in orthogonal  $q$ -polynomials it has become of interest to look for a  $q$ -generalization of the Konhauser polynomials.

Our starting point would naturally be the  $q$ -Laguerre polynomials which were introduced by Hahn [5]. The polynomials belong to an indeterminate moment problem and thus there is more than one distribution function with respect to which the  $q$ -Laguerre polynomials are orthogonal. In particular there is a discrete distribution and a continuous one [9]. This is not a problem in our case since, as one might expect, it is the moments that really determine the orthogonal as well as the biorthogonal sets of polynomials.

**2. Preliminaries.** In this paper we shall use the following notation. For  $|q| < 1$ ,

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)$$

and, for arbitrary complex  $n$ ,

$$(a; q)_n = (a; q)_\infty / (aq^n; q)_\infty,$$

so that in particular if  $n = 1, 2, \dots$  we have

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

in which case the restriction  $|q| < 1$  is not necessary.

For writing economy we shall write  $[a]_n$  to mean  $(a; q)_n$ . If the base is not  $q$  but, say  $p$ , then we shall mention it explicitly as  $(a; p)_n$ .

The  $q$ -derivative (base  $q$ ) is  $D_q f(x) = \{(x) - f(qx)\}/x$ . Its  $n$ th iterate is [5]

$$(2.1) \quad D_q^n f(x) = x^{-n} \sum_{j=0}^n \frac{[q^{-n}]_j}{[q]_j} q^j f(xq^j).$$

The  $q$ -gamma function may be defined (see Askey [1] for an interesting treatment) by

$$\Gamma_q(x) = \frac{[q]_\infty}{[q^x]_\infty} (1 - q)^{1-x}, \quad 0 < q < 1.$$

The  $q$ -Laguerre polynomials

$$L_n^{(\alpha)}(x | q) = \frac{[q^{1+\alpha}]_n}{[q]_n} \sum_{j=0}^n \frac{[q^{-n}]_j q^{\frac{1}{2}j(j+1) + j(\alpha+n)}}{[q]_j [q^{1+\alpha}]_j} x^j$$

are orthogonal on  $(0, \infty)$  with respect to the continuous distribution

$$(2.2) \quad d\Omega(\alpha, x) = \frac{Ax^\alpha}{[-x]_\infty} dx, \quad (\alpha > -1)$$

where  $A = \Gamma_q(-\alpha)/\Gamma(-\alpha)\Gamma(1 + \alpha)(1 - q)^{1+\alpha}$  or the discrete distribution  $d\beta(\alpha, x)$  which has jumps  $Bx^{\alpha+1}/[-x]_\infty$  at  $x = q^k$ ,  $k = 0, \pm 1, \pm 2, \dots$ . Where

$$B = \frac{2[q^{1+\alpha}]_\infty \{[-q]_\infty\}^2}{[-q^{1+\alpha}]_\infty [-q^{-\alpha}]_\infty [q]_\infty}.$$

The moments in either case are (see Moak [9])

$$(2.3) \quad \mu_n = [q^{1+\alpha}]_n q^{-\frac{1}{2}n(2\alpha+n+1)}, \quad n = 0, 1, 2, \dots$$

The  $q$ -binomial theorem is

$$(2.4) \quad \sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} x^n = \frac{[ax]_\infty}{[x]_\infty}, \quad (|x| < 1).$$

**3. The  $q$ -Konhauser polynomials.** We define for  $n = 0, 1, 2, \dots$

$$(3.1) \quad Z_n^{(\alpha)}(x, k | q) = \frac{[q^{1+\alpha}]_{nk}}{(q^k; q^k)_n} \sum_{j=0}^n \frac{(q^{-nk}; q^k)_j q^{\frac{1}{2}k_j(k_j-1)+k_j(n+\alpha+1)}}{(q^k; q^k)_j [q^{1+\alpha}]_{jk}} x^{kj}$$

and

$$(3.2) \quad Y_n^{(\alpha)}(x, k | q) = \frac{1}{[q]_n} \sum_{r=0}^n \frac{x^r q^{\frac{1}{2}r(r-1)}}{[q]_r} \times \sum_{j=0}^r \frac{[q^{-r}]_j (q^{1+\alpha+j}; q^k)_n}{[q]_j} q^j$$

and prove that

$$(3.3) \quad \int_0^\infty Z_n^{(\alpha)}(x, k | q) Y_m^{(\alpha)}(x, k | q) d\Omega(\alpha, x) = k_n \delta_{nm},$$

where

$$K_n = \frac{[q^{1+\alpha}]_{nk}}{[q]_n} q^{-nk}.$$

Formula (3.1), (3.2) and (3.3) reduce for  $k = 1$  to the  $q$ -Laguerre polynomials  $L_n^{(\alpha)}(x | q)$  and its orthogonality relation (2.2). To prove (3.3) it is necessary and sufficient to show

$$(3.3(a)) \quad I_{n,m} \equiv \int_0^\infty x^m Z_n^{(\alpha)}(x, k | q) d\Omega(\alpha, x) \begin{cases} = 0, & 0 \leq m < n, \\ \neq 0, & m = n, \end{cases}$$

and

$$(3.3(b)) \quad J_{n,m} \equiv \int_0^\infty x^{km} Y_n^{(\alpha)}(x, k | q) d\Omega(\alpha, x) \begin{cases} = 0, & 0 \leq m < n, \\ \neq 0, & m = n. \end{cases}$$

*Proof of (3.3(a)).* in the left hand side of (3.3(a)) substituting for  $Z_n^{(\alpha)}(x, k | q)$  from (3.1) and integrating term by term and using (2.3), we get

$$(3.4) \quad \begin{aligned} I_{n,m} &= \frac{[q^{1+\alpha}]_{nk} q^{-\frac{1}{2}m(m+2\alpha+1)}}{(q^k; q^k)_n} \\ &\quad \times \sum_{j=0}^n \frac{(q^{-nk}; q^k)_j [q^{1+\alpha+kj}]_m}{(q^k; q^k)_j} q^{kj(n-m)} \\ &= \frac{(-1)^m [q^{1+\alpha}]_{nk}}{(q^k; q^k)_n} [D_{p^k}^n(xp^{1+\alpha}; p)_m]_{x=1} \end{aligned}$$

where  $p = q^{-1}$ . The last equality is obtained by replacing  $p = 1/q$  in the summation that appears in (3.4), simplifying and then comparing with (2.1).

Now  $(xp^{1+\alpha}; p)_m$  is a polynomial in  $x$  of degree  $m$ . Hence for  $m = 0, 1, \dots, n-1$  its  $q$ -difference is zero, whereas for  $m = n$  we get

$$(3.5) \quad I_{n,n} = (-1)^n [q^{1+\alpha}]_{nk} q^{-\frac{1}{2}nk(n+1) - \frac{1}{2}n(2\alpha+n+1)}.$$

This completes the proof of (3.3(a)).

To prove (3.3(b)) we require the following formula which is a  $q$ -analog of a result of Carlitz [2].

$$(3.6) \quad \begin{aligned} (q^{-kl}; q^k)_m &= \sum_{r=0}^m \frac{(q^{1+\alpha+ki})_r}{[q]_r} q^{-r(1+\alpha+ki)} \\ &\quad \times \sum_{s=0}^r q^s \frac{[q^{-r}]_s (q^{1+\alpha+s}; q^k)_m}{[q]_s}. \end{aligned}$$

Formula (3.6) can be proved by using Jackson's  $q$ -analog of Taylor's theorem [6] for polynomials of degree  $\leq m$ ,

$$(3.7) \quad f(x) = \sum_{r=0}^m D_q^r f(x) \Big|_{x=1} \frac{x^r [1/x]_r}{[q]_r}.$$

Put  $f(x) = (xq^{1+\alpha}; q^k)_m$  in (3.7) to get

$$(3.8) \quad (xq^{1+\alpha}; q^k)_m = \sum_{r=0}^m \frac{x^r [1/x]_r}{[q]_r} \sum_{s=0}^r \frac{[q^{-r}]_s}{[q]_s} q^s (q^{1+\alpha+s}; q^k)_m$$

which for  $x = q^{-1-\alpha-ki}$  reduces to (3.6).

*Proof of (3.3(b)).* Substitute for  $Y_n^{(\alpha)}(x, k | q)$  from (3.2) in the left hand side of (3.3(b)), integrating term by term, then using (3.6) we get

$$(3.9) \quad J_{n,m} = \frac{[q^{1+\alpha}]_{km}}{[q]_n} q^{-\frac{1}{2}km(2\alpha+1+km)} (q^{-km}; q^k)_n.$$

Since  $(q^{-km}; q^k)_n = 0$  for  $m = 0, 1, 2, \dots, n-1$  and

$$(3.10) \quad J_{n,n} = (-)^n \frac{[q^{1+\alpha}]_{kn}}{[q]_n} (q^{-kn}; q^k)_n q^{-\frac{1}{2}nk(nk+2\alpha+n+2)},$$

hence the proof of (3.3(b)) is complete.

Furthermore (3.10), put together with the fact that the leading coefficient of  $Z_n^{(\alpha)}(x; k | q)$  is  $(-)^n q^{\frac{1}{2}kn(kn+2\alpha+n)}$ , yields (3.3).

**4. Properties of  $Z_n^{(\alpha)}(x, k | q)$  and  $Y_n^{(\alpha)}(x, k | q)$ .** We devote this section to some of the interesting properties of the polynomials  $Z_n^{(\alpha)}(x, k | q)$  and  $Y_n^{(\alpha)}(x, k | q)$  introduced in §3. We mention below first some of the properties of  $Z_n^{(\alpha)}(x, k | q)$

$$(4.1) \quad \sum_{n=0}^{\infty} \frac{Z_n^{(\alpha)}(x, k | q)}{[q^{1+\alpha}]_{nk}} t^n = \frac{f(tx^k)}{(t; q^k)_{\infty}},$$

where

$$(4.2) \quad \begin{aligned} f(u) &= \sum_{j=0}^{\infty} \frac{q^{\frac{1}{2}kj(kj+j+2\alpha)}}{(q^k; q^k)_j [q^{1+\alpha}]_{kj}} (-u)^j. \\ Z_n^{(\alpha)}(xy, k | q) &= \sum_{j=0}^n \frac{[q^{1+\alpha}]_{kn}}{[q^{1+\alpha}]_{kn-kj}} \frac{(1/y^k; q^k)_j}{(q^k; q^k)_j} y^{kj} Z_{n-j}^{(\alpha)}(x, k | q). \end{aligned}$$

If  $Z_n^{(\alpha)}(x, k | q) = \sum_{m=0}^n c(n, m) Z_m^{(\beta)}(x, k | q)$  then

$$(4.3) \quad c(n, m) = \frac{[q^{1+\alpha}]_{nk} q^{km(\alpha-\beta)}}{[q^{1+\alpha}]_{mk} (q^k; q^k)_{n-m}} \\ \times \sum_{j=0}^{n-m} \frac{(q^{-nk+mk}; q^k)_j [q^{1+\beta+km}]_{jk}}{(q^k; q^k)_j [q^{1+\alpha+km}]_{kj}} \cdot q^{kj(n-m+\alpha-\beta)}$$

for  $k = 1$  this reduces to the connection coefficient for the  $q$ -Laguerre polynomials.

$$(4.4) \quad \{D_p^k x^{\alpha+1} D_p\} Z_n^{(\alpha)}(x, k | q) \\ = (-)^k \frac{[q^{1+\alpha}]_{nk}}{[q^{1+\alpha}]_{nk-k}} x^\alpha Z_{n-1}^{(\alpha)}(x, k | q),$$

$$(4.5) \quad q^{\frac{1}{2}k(k+2\alpha+1)} x^k Z_n^{(\alpha+k)}(x, k | q) \\ = [q^{1+\alpha+kn}]_k Z_n^{(\alpha)}(x, k | q) - Z_{n+1}^{(\alpha)}(x, k | q) (1 - q^{k(n+1)}).$$

If  $x^{kn} = \sum_{m=0}^n D(n, m) Z_m^{(\alpha)}(x, k | q)$  then

$$(4.6) \quad D(n, m) = \frac{[q^{1+\alpha}]_{kn} (q^{-kn}; q^k)_m}{[q^{1+\alpha}]_{km}} q^{\frac{1}{2}kn(kn+2\alpha+1)}.$$

*Proof of (4.1).* Substituting from (3.1) in the left hand side of (4.1), changing the order of summations and summing the resulting inner series by  $q$ -binomial theorem, we get the right hand side of (4.1).

*Proof of (4.2).* In (4.1) replacing  $x$  by  $xy$  and in the right hand side of the resulting identity expanding  $(ty^k; q^k)_\infty / (t; q^k)_\infty$  by  $q$ -binomial theorem and equating the coefficients of  $t^n$  on both sides we get (4.2).

*Proof of (4.3).* Multiplying both sides of (4.3) by  $Y_i^{(\beta)}(x, k | q) d\Omega(\beta, x)$  where  $0 \leq i \leq n$  and integrating from 0 to  $\infty$ , we get the desired value of  $c(n, m)$  on using (3.3), (3.3(b)) and (3.9).

*Proof of (4.4)–(4.6)* follow by routine methods hence the details are omitted. In a similar manner one can obtain the following properties of the polynomials  $Y_n^{(\alpha)}(x, k | q)$

$$(4.7) \quad Y_n^{(\alpha)}(x; k | q) \\ = \sum_{m=0}^n \frac{(q^k; q^k)_n [q]_m (q^{\alpha-\beta}; q^k)_{n-m}}{(q^k; q^k)_m [q]_n (q^k; q^k)_{n-m}} q^{m(\alpha-\beta)} Y_m^{(\beta)}(x, k | q)$$

If  $x^n = \sum_{m=0}^n D(n, m) Y_m^{(\alpha)}(x, k | q)$  then

$$(4.8) \quad D(n, m) = (-1)^{n+m} q^{-m\alpha - \frac{1}{2}km(m-1)} [q^{-n}]_m \\ \times \sum_{j=0}^{n-m} \frac{[q^{-n+m}]_j (q^{k+k m}; q^k)_j}{[q^{1+m}]_j (q^k; q^k)_j} q^{-j(km+\alpha)},$$

$$(4.9) \quad Y_n^{(\alpha)}(x; k | q) = [-x]_{\infty} x^{k-\alpha-1} \left[ D_{q^k}^n \left\{ \frac{x^{\beta+n}}{[-x^{1/k}]_{\infty}} \right\} \right]_{x^k}$$

where  $\beta = (1 + \alpha - k)/k$ .

Once more we remark that (4.7), (4.8), (4.9) reduce, when  $k = 1$ , to corresponding properties for the  $q$ -Laguerre polynomials.

Other formulas which are  $q$ -analogs of known results on the Konhauser polynomials can be easily obtained.

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UNIVERSITY OF ALBERTA  
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AND

ARIZONA STATE UNIVERSITY  
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