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Given a sequence of Turing degrees $\langle a_i \rangle_{i<\omega}$, $a_i < a_{i+1}$, is there a function of f such that (i) $\deg(f)$ is a minimal upper bound on $\langle a_i \rangle_{i<\omega}$, and (ii) $\{\deg((f)_n) \mid n < \omega\} = \{a_i \mid i < \omega\}$? In this note we show that the most natural minimal upper bound on $\langle a_i \rangle_{i<\omega}$ is of the form $\deg(f)$ for such an f.

Because there seem to be a cluster of interesting notions and question related to this problem, we start with some definitions. Fix a recursive pairing function $(x, y) \mapsto \langle x, y \rangle$; $(f)_x(y) = f(\langle x, y \rangle)$. Where I is a set of Turing degrees and $f \in {}^\omega\omega$, f represents (subrepresents) I iff $I = \{\deg((f)_n) \mid n < \omega\}$ ($I \subseteq \{\deg((f)_n) \mid n < \omega\}$). For $I' \subseteq I$, I' is cofinal in I iff for every $a \in I$ there is a $b \in I'$ with $a \leq b$. f weakly represents (weakly subrepresents) I iff f represents (subrepresents) some I' cofinal in I. A degree **a** represents (subrepresents, weakly represents, weakly subrepresents) I iff some $f \in$ **a** does so. I is an ideal iff I is non-empty closed downward and under join.

Terminology. A tree T is a total function from $2^{<\omega} = \text{Str}$ into Str so that for any $\delta \in \text{Str}$, $T(\delta\hat{\ }0)$ and $T(\delta\hat{\ }1)$ are incompatible extensions of $T(\delta)$. $\delta \in \text{Str}(s)$ iff $\delta \in \text{Str}$ and $\text{dom}(\delta) = s$. A pre-tree of height s is a function T: $\text{Str}(s) \to \text{Str}$ where for all $\delta \in \text{Str}(s-1)$, $T(\delta\hat{\ }\langle 0 \rangle)$ and $T(\delta\hat{\ }\langle 1 \rangle)$ are incompatible extensions of $T(\delta)$. For $\delta \in \text{Str}$ and $A \in {}^\omega 2$, $\delta \subseteq A$ iff for all $i \in \text{dom}(\delta)$, $\delta(i) = A(i)$. Where T is a tree, $B \in [T]$ iff for some $A \in {}^\omega 2$; for all n, $T(A \upharpoonright n) \subset B$; (i.e. B is a path through T). Where T is a pre-tree of height s, $B \in [T]$ iff for some $\delta \in \text{Str}$, $\text{dom}(\delta) = s$ and $T(\delta) \subset B$.

Where T is a tree and $A \in {}^\omega 2$, let

$$\text{Code}(T, A)(\delta) = T(\langle A(0), \delta(0), \ldots, \delta(n-1), A(n) \rangle),$$

where $n = \text{dom}(\delta) - 1$. Notice: $\text{Code}(T, A)(\langle\,\rangle) \supsetneqq T(\langle\,\rangle)$. Where T is a pre-tree of height $\leq 2n+1$ and $\tau \in \text{Str}$, $\text{dom}(\tau) \geq n$, $\text{Code}(T, \tau)$ is defined similarly. For T a tree (pre-tree) and $B \in [T]$, let $\text{Coded}(B, T)$ be the real $A \in {}^\omega 2$ (string τ) such that $A(e) = i$ ($\tau(e) = i$) iff for some δ, $T(\delta) \subseteq B$ and $\delta(2e) = i$. If T is a pre-tree of height $2n$ or $2n+1$, dom(Coded(B, T)) $= n$; so if T is a pre-tree, $B \in [T]$ and $\tau =$ Coded(B, T), Code(T, τ) is well defined.

We'll say that τ is on T iff $\tau \in$ Range(T). Let τ_0, τ_1 be an e-splitting of τ iff $\tau_0, \tau_1 \supseteq \tau$ and for some x and t, $\{e\}_t^{\tau_0}(x)$ and $\{e\}_t^{\tau_1}(x)$ are defined and different. By "the least e-splitting of τ", we mean that $\langle \tau_0, \tau_1, x, t \rangle$ is minimal. Where T is a tree, let e-Split(T)($\langle\, \rangle$) $=$ T($\langle\, \rangle$); if e-Split(T)(δ) is defined, e-Split(T)($\delta\hat{}\langle 0 \rangle$), e-Split(T)($\delta\hat{}\langle 1 \rangle$) is the least e-splitting of e-Split(T)(δ) on T, if such there be; otherwise they are undefined. Clearly e-Split(T) is partial-recursive in T.

Where T is a pre-tree, e-Split$_s$(T) is defined like e-Split(T), except that (1) all searches for e-splittings on T are bounded by s; (2) e-Split(T)(δ) is defined iff for all τ with dom(τ) $=$ dom(δ), e-Split(T)(τ) is defined. (2) insures that e-Split$_s$(T) is a pre-tree. For T a tree or pre-tree, Full(T, δ)(τ) $= T(\delta\hat{}\tau)$. (If $\delta \notin$ dom(T), Full(T, δ) $= \varnothing$, which is still a pre-tree.)

THEOREM. *Suppose* $I = \{\mathbf{a}_i \mid i < \omega\}$ *is a sequence of Turing degrees, and for all* i, $\mathbf{a}_i < \mathbf{a}_{i+1}$. *Then some minimal upper-bound on* I *represents* I.

To prove this, we use the simplest construction of a minimal upper bound on I. Fix $\langle A_i \rangle_{i<\omega}$ so that for all i, $A_i \in \mathbf{a}_i$. Let $T_{-1} = \mathrm{Id} \upharpoonright \mathrm{Str}$.

$$T_{2e} = \begin{cases} e\text{-Split}(T_{2e-1}) & \text{if } e\text{-Split}(T_{2e-1}) \text{ is total;} \\ \mathrm{Full}(T_{2e-1}, \tau_e) & \text{otherwise,} \end{cases}$$

where τ_e is the least τ such that $T_{2e-1}(\tau)$ is on e-Split(T_{2e-1})(τ) and has no e-splitting on T_{2e-1}.

$$T_{2e+1} = \mathrm{Code}(T_{2e}, A_e).$$

A tree T is uniformly recursively pointed iff for some e, $T = \{e\}^B$ for all $B \in [T]$. All T_e are uniformly recursively pointed, and so $T_{2e-1} \equiv_T T_{2e} \leq_T T_{2e+1} \leq_T A_e$. Let $\{B\} = \bigcap_{e<\omega}[T_e]$; where $\mathbf{b} = \deg(B)$, $\mathbf{b}$ is a minimal upper bound on I. We must show that B computes a g which represents I.

Let

$$f(e) = \begin{cases} 0 & \text{if } T_{2e} \text{ was defined by the first case;} \\ \tau_e + 1 & \text{otherwise.} \end{cases}$$

$$f^-(e) = 0 \quad \text{if } f(e) = 0; \qquad f^-(e) = 1 \quad \text{otherwise.}$$

We'll let $\delta \in$ Str represent the hypothesis that $\delta \subset f^-$. Assuming this hypothesis, for dom(δ) $= n + 1$, B tries to recover $\langle T_e \rangle_{-1 \leq e \leq 2n}$ and A_n.

If $\delta \subset f^-$, eventually B will have this right. If $\delta \not\subset f^-$, B will not be so fortunate. Where e is least so that $\delta(e) \neq f^-(e)$, e curses δ iff $f^-(e) = 1$ and $\delta(e) = 0$; e disrupts δ iff $f^-(e) = 0$ and $\delta(e) = 1$. If δ is cursed, by assuming δ B eventually finds himself waiting eternally for a splitting which never comes; if δ is disrupted, constant changes in B's guesses at a node beyond which there are no splits will prevent B's guesses from settling down.

At each stage s, on hypohtesis δ B constructs the sequence of pre-trees $T_e^{\delta,s}$, $-1 \leq e \leq 2n$, as follows: $T_{-1}^{\delta,s} = \mathrm{Id} \upharpoonright \mathrm{Str}(s+1)$;

$$T_{2e}^{\delta,s} = \begin{cases} e\text{-Split}_s\left(T_{2e-1}^{\delta,s}\right) & \text{if } \delta(e) = 0, \\ \mathrm{Full}\left(T_{2e-1}^{\delta,s}, \tau_e^{\delta,s}\right) & \text{if } \delta(e) = 1, \end{cases}$$

where $\tau_e^{\delta,s}$ is the longest τ such that e-$\mathrm{Split}_s(T_{2e-1}^{\delta,s})(\tau)$ is defined, $\subset B$, and has no e-splitting on $T_{2e-1}^{\delta,s}$ after s steps of searching. Let $F(e,\delta,s) = \mathrm{Coded}(B, T_{2e}^{\delta,s})$. $F(e,\delta,s)$ is B's stage s guess at $A_e \upharpoonright k$, where $k = \mathrm{dom}(F(e,\delta,s))$, based on hypothesis δ.

$$T_{2e+1}^{\delta,s} = \mathrm{Code}\left(T_{2e}^{\delta,s}, F(e,\delta,s)\right).$$

By remarks after the definitions of Code and Coded, this is well-defined.

Let $\mathrm{dom}(\delta) = n+1$. If $T_{2n}^{\delta,s} \neq \varnothing$, for all e with $-1 \leq e \leq 2n$, $T_e^{\delta,s} \neq \varnothing$; let $f^{\delta,s}\colon n+1 \to \omega$ be given by:

$$f^{\delta,s}(e) = \begin{cases} 0 & \text{if } \delta(e) = 0 \\ \tau_e^{\delta,s} + 1 & \text{if } \delta(e) = 1. \end{cases}$$

$f^{\delta,s}$ is B's guess at $f \upharpoonright n+1$ at stage s, assuming δ. If $T_{2n}^{\delta,s} = \varnothing$, at stage s B hasn't enough information to make a guess. If $\delta \not\subseteq \delta'$, $T_e^{\delta,s} = T_e^{\delta',s}$ for $e \leq 2n$, and $f^{\delta,s} = f^{\delta',s} \upharpoonright n+1$.

We now consider the possible behavior of $f^{\delta,s}$ as s increases.

(1) If $\delta \subset f^-$ there is an s such that for all $t \geq s$, $f^{\delta,t}$ is defined, $f^{\delta,t} = f^{\delta,s} = f \upharpoonright n+1$, $T_e^{\delta,t} = T_e \upharpoonright \mathrm{Str}(l_e^t)$ for $-1 \leq e \leq 2n$, where l_e^t is nondecreasing in t and approaches ω for $t \geq s$; furthermore for $t \geq s$, $F(n,\delta,t) \subset A_n$, and so $\bigcup_{t \geq s} F(n,\delta,t) = A_n$. All this follows by induction on n.

(2) If δ is cursed, there is an s such that either (a) for all $t \geq s$, $f^{\delta,t}$ is defined and $f^{\delta,t} = f^{\delta,s}$, or (b) for all $t \geq s$, $f^{\delta,t}$ is undefined. Furthermore, in case (a), for all $t \geq s$, $F(n,\delta,t) = F(n,\delta,s)$. To see this, suppose e curses δ; by (1) there is a stage s_0 by which $f^{\delta \upharpoonright e,t}$ is defined and equal to

$f \upharpoonright e$ for all $t \geq s_0$; furthermore $T_{2e-1}^{\delta,t} = T_{2e-1} \upharpoonright \mathrm{Str}(l_{2e-1}^t)$. Fix the least level l such that for some δ with $\mathrm{dom}(\delta) = l$, e-Split$(T_{2e-1})(\delta)$ is undefined. In building $T_{2e}^{\delta,t}$, B gets stuck at level l; so eventually B is waiting for e-splittings on $T_{2e-1}^{\delta,t}$ of a string with no such e-splittings. So for some $s_1 \geq s_0$, for all $t \geq s_1$, $T_{2e}^{\delta,t} = T_{2e}^{\delta,s_1}$. Clearly for $-1 \leq j < j' \leq 2n$, $\mathrm{Range}(T_j^{\delta,t}) \subseteq \mathrm{Range}(T_{j'}^{\delta,t})$. So by induction we find s so that for all $j \leq 2n$ and $t \geq s$, $T_j^{\delta,t} = T_j^{\delta,s}$. If $T_j^{\delta,s} = \varnothing$, for $t \geq s$, $f^{\delta,t}$ is undefined. Otherwise $f^{\delta,t}(e) = 0$.

(3) If δ is disrupted and $f^{\delta,s}$ is defined, for some $t > s$ either $f^{\delta,t}$ is undefined or $f^{\delta,t} \neq f^{\delta,s}$. To see this, suppose e disrupts δ and select s_0 as above. Once $t \geq s_0$, $\tau_e^{\delta,t}$ goes to ω with t, since e-splittings for e-Split$_t(T_{2e-1}^{\delta,t})(\tau_e^{\delta,t}) = e$-Split$(T_{2e-1})(\tau_e^{\delta,t})$ eventually turn up on T_{2e-1}, and thus on $T_{2e-1}^{\delta,t'}$ for sufficiently large $t' \geq t$; when this happens, $\tau_e^{\delta,t'} \supsetneq \tau_e^{\delta,t}$. Fixing s, for sufficiently large $t \geq s$, if $f^{\delta,t}$ is defined, $f^{\delta,t}(e) > f^{\delta,s}(e)$.

We now view $h \in \omega^{<\omega}$ as a guess at $f \upharpoonright \mathrm{dom}(h)$. Let $h^-(e) = 0$ if $h(e) = 0$, $h^-(e) = 1$ otherwise. An h-block is a maximal interval $[s_0, s_1] = \{t \mid s_0 \leq t \leq s_1\}$ or $[s_0, \infty] = \{t \mid s_0 \leq t\}$ such that for all s in that interval, $h = f^{h^-,s}$. For any h there are finitely many h-blocks. If $h^- \subset f^-$, this follows from (1); if h^- is cursed, this follows from (2). Note that if $h^- \subset f^-$ or if h^- is cursed and (2a) is true, the final h-block is of the form $[s, \infty]$. If h^- is disrupted by e, this follows from (3) and the previous observation that for sufficiently large t, $\tau_e^{h^-,t}$ increases non-decreasingly with t. If s and t belong to one h-block and $s \leq t$, $F(e, h^-, s) \subset F(e, h^-, t)$ for $-1 \leq e < \mathrm{dom}(h)$. For the moment, assume that $\mathbf{a}_0 = \mathbf{0}$. For $h \in \omega^{<\omega}$, $k \in \omega$ and $\mathrm{dom}(h) = n + 1$, let

$$(g)_{\langle h,k\rangle}(s) = \begin{cases} F(n, h^-, s) + 1 & \text{if } s \text{ belongs to the } k\text{th } h\text{-block;} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $g \leq_T B$. If $h \not\subset f$, or if the kth h-block is not of the form $[s, \infty]$, $(g)_{\langle h,k\rangle}$ differs only finitely from $\lambda s.0$. If $h \subset f$ and the kth h-block is of the form $[s, \infty]$, since $A_n = \bigcup_{t \geq s} F(n, h^-, t)$, $A_n \leq_T (g)_{\langle h,k\rangle}$. Furthermore, $\lambda s.F(n, h^-, s) \leq_T A_0 \oplus \cdots \oplus A_n \leq_T A_n$; thus $(g)_{\langle h,k\rangle} \leq_T A_n$. So either $\deg((g)_{\langle h,k\rangle}) = \mathbf{a}_n$ or $= \mathbf{0} = \mathbf{a}_0$. Thus g represents I.

Now suppose $\mathbf{a}_0 \neq \mathbf{0}$. Select $D \in \mathbf{a}_0$. Suppose we revised our definition of $(g)_{\langle h,k\rangle}(s)$ by requiring in the "otherwise" case that $(g)_{\langle h,k\rangle}(s) = D(s)$. If $h^- \subset f^-$ and the kth block is of the form $[s_0, \infty]$, we still have $\deg((g)_{\langle h,k\rangle}) = \mathbf{a}_n$; if otherwise and if h^- is not cursed, $\deg((g)_{\langle h,k\rangle}) = \mathbf{a}_0$. But if h^- is cursed and the kth block is of the form $[s_0, \infty]$,

$\deg((g)_{\langle h,k\rangle}) = \mathbf{0}$. To remedy this, we slightly hair-up the definition of $(g)_{\langle h,k\rangle}$:

$$(g)_{\langle h,k\rangle}(2s) = \begin{cases} F(h, h^-, s) + 1 & \text{if } s \text{ belongs to the } k\text{th } h\text{-block.} \\ D(s) & \text{otherwise} \end{cases}$$

$$(g)_{\langle h,k\rangle}(2s+1) = D(s).$$

g is now as desired.

COROLLARY. *If I is a countable ideal, some minimal upper bound on I weakly represent I.*

Proof. There is an $I' \subseteq I$ cofinal in I and linearly ordered; apply *Theorem* 1 to I' and notice that a minimal upper bound on I' is also one for I.

Questions. Does every ideal have a representing minimal upper bound?

Does a sequence $\langle a_i \rangle_{i<\omega}$ as above have a minimal upper bound which does not represent it?

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