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**A MINIMAL UPPER BOUND ON A SEQUENCE OF TURING
DEGREES WHICH REPRESENTS THAT SEQUENCE**

HAROLD T. HODES

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Given a sequence of Turing degrees $\langle a_i \rangle_{i < \omega}$, $a_i < a_{i+1}$, is there a function of f such that (i) $\text{deg}(f)$ is a minimal upper bound on $\langle a_i \rangle_{i < \omega}$, and (ii) $\{\text{deg}((f)_n) \mid n < \omega\} = \{a_i \mid i < \omega\}$? In this note we show that the most natural minimal upper bound on $\langle a_i \rangle_{i < \omega}$ is of the form $\text{deg}(f)$ for such an f .

Because there seem to be a cluster of interesting notions and question related to this problem, we start with some definitions. Fix a recursive pairing function $(x, y) \mapsto \langle x, y \rangle$; $(f)_x(y) = f(\langle x, y \rangle)$. Where I is a set of Turing degrees and $f \in {}^\omega\omega$, f represents (subrepresents) I iff $I = \{\text{deg}((f)_n) \mid n < \omega\}$ ($I \subseteq \{\text{deg}((f)_n) \mid n < \omega\}$). For $I' \subseteq I$, I' is cofinal in I iff for every $a \in I$ there is a $b \in I'$ with $a \leq b$. f weakly represents (weakly subrepresents) I iff f represents (subrepresents) some I' cofinal in I . A degree a represents (subrepresents, weakly represents, weakly subrepresents) I iff some $f \in a$ does so. I is an ideal iff I is non-empty closed downward and under join.

Terminology. A tree T is a total function from $2^{<\omega} = \text{Str}$ into Str so that for any $\delta \in \text{Str}$, $T(\delta \hat{\ } 0)$ and $T(\delta \hat{\ } 1)$ are incompatible extensions of $T(\delta)$. $\delta \in \text{Str}(s)$ iff $\delta \in \text{Str}$ and $\text{dom}(\delta) = s$. A pre-tree of height s is a function $T: \text{Str}(s) \rightarrow \text{Str}$ where for all $\delta \in \text{Str}(s-1)$, $T(\delta \hat{\ } \langle 0 \rangle)$ and $T(\delta \hat{\ } \langle 1 \rangle)$ are incompatible extensions of $T(\delta)$. For $\delta \in \text{Str}$ and $A \in {}^\omega 2$, $\delta \subseteq A$ iff for all $i \in \text{dom}(\delta)$, $\delta(i) = A(i)$. Where T is a tree, $B \in [T]$ iff for some $A \in {}^\omega 2$; for all n , $T(A \upharpoonright n) \subseteq B$; (i.e. B is a path through T). Where T is a pre-tree of height s , $B \in [T]$ iff for some $\delta \in \text{Str}$, $\text{dom}(\delta) = s$ and $T(\delta) \subseteq B$.

Where T is a tree and $A \in {}^\omega 2$, let

$$\text{Code}(T, A)(\delta) = T(\langle A(0), \delta(0), \dots, \delta(n-1), A(n) \rangle),$$

where $n = \text{dom}(\delta) - 1$. Notice: $\text{Code}(T, A)(\langle \rangle) \supseteq T(\langle \rangle)$. Where T is a pre-tree of height $\leq 2n + 1$ and $\tau \in \text{Str}$, $\text{dom}(\tau) \geq n$, $\text{Code}(T, \tau)$ is defined similarly. For T a tree (pre-tree) and $B \in [T]$, let $\text{Coded}(B, T)$ be the real $A \in {}^\omega 2$ (string τ) such that $A(e) = i$ ($\tau(e) = i$) iff for some δ , $T(\delta) \subseteq B$ and $\delta(2e) = i$. If T is a pre-tree of height $2n$ or $2n + 1$,

$\text{dom}(\text{Coded}(B, T)) = n$; so if T is a pre-tree, $B \in [T]$ and $\tau = \text{Coded}(B, T)$, $\text{Code}(T, \tau)$ is well defined.

We'll say that τ is on T iff $\tau \in \text{Range}(T)$. Let τ_0, τ_1 be an e -splitting of τ iff $\tau_0, \tau_1 \supseteq \tau$ and for some x and t , $\{e\}_i^{\tau_0}(x)$ and $\{e\}_i^{\tau_1}(x)$ are defined and different. By "the least e -splitting of τ ", we mean that $\langle \tau_0, \tau_1, x, t \rangle$ is minimal. Where T is a tree, let $e\text{-Split}(T)(\langle \rangle) = T(\langle \rangle)$; if $e\text{-Split}(T)(\delta)$ is defined, $e\text{-Split}(T)(\delta \hat{\langle 0 \rangle})$, $e\text{-Split}(T)(\delta \hat{\langle 1 \rangle})$ is the least e -splitting of $e\text{-Split}(T)(\delta)$ on T , if such there be; otherwise they are undefined. Clearly $e\text{-Split}(T)$ is partial-recursive in T .

Where T is a pre-tree, $e\text{-Split}_s(T)$ is defined like $e\text{-Split}(T)$, except that (1) all searches for e -splittings on T are bounded by s ; (2) $e\text{-Split}(T)(\delta)$ is defined iff for all τ with $\text{dom}(\tau) = \text{dom}(\delta)$, $e\text{-Split}(T)(\tau)$ is defined. (2) insures that $e\text{-Split}_s(T)$ is a pre-tree. For T a tree or pre-tree, $\text{Full}(T, \delta)(\tau) = T(\delta \hat{\tau})$. (If $\delta \notin \text{dom}(T)$, $\text{Full}(T, \delta) = \emptyset$, which is still a pre-tree.)

THEOREM. *Suppose $I = \{\mathbf{a}_i \mid i < \omega\}$ is a sequence of Turing degrees, and for all i , $\mathbf{a}_i < \mathbf{a}_{i+1}$. Then some minimal upper-bound on I represents I .*

To prove this, we use the simplest construction of a minimal upper bound on I . Fix $\langle A_i \rangle_{i < \omega}$ so that for all i , $A_i \in \mathbf{a}_i$. Let $T_{-1} = \text{Id} \upharpoonright \text{Str}$.

$$T_{2e} = \begin{cases} e\text{-Split}(T_{2e-1}) & \text{if } e\text{-Split}(T_{2e-1}) \text{ is total;} \\ \text{Full}(T_{2e-1}, \tau_e) & \text{otherwise,} \end{cases}$$

where τ_e is the least τ such that $T_{2e-1}(\tau)$ is on $e\text{-Split}(T_{2e-1})(\tau)$ and has no e -splitting on T_{2e-1} .

$$T_{2e+1} = \text{Code}(T_{2e}, A_e).$$

A tree T is uniformly recursively pointed iff for some e , $T = \{e\}^B$ for all $B \in [T]$. All T_e are uniformly recursively pointed, and so $T_{2e-1} \equiv_T T_{2e} \leq_T T_{2e+1} \leq_T A_e$. Let $\{B\} = \bigcap_{e < \omega} [T_e]$; where $\mathbf{b} = \text{deg}(B)$, \mathbf{b} is a minimal upper bound on I . We must show that B computes a g which represents I .

Let

$$f(e) = \begin{cases} 0 & \text{if } T_{2e} \text{ was defined by the first case;} \\ \tau_e + 1 & \text{otherwise.} \end{cases}$$

$$f^-(e) = 0 \quad \text{if } f(e) = 0; \quad f^-(e) = 1 \quad \text{otherwise.}$$

We'll let $\delta \in \text{Str}$ represent the hypothesis that $\delta \subset f^-$. Assuming this hypothesis, for $\text{dom}(\delta) = n + 1$, B tries to recover $\langle T_e \rangle_{-1 \leq e \leq 2n}$ and A_n .

If $\delta \subset f^-$, eventually B will have this right. If $\delta \not\subset f^-$, B will not be so fortunate. Where e is least so that $\delta(e) \neq f^-(e)$, e curses δ iff $f^-(e) = 1$ and $\delta(e) = 0$; e disrupts δ iff $f^-(e) = 0$ and $\delta(e) = 1$. If δ is cursed, by assuming $\delta \subset B$ eventually finds himself waiting eternally for a splitting which never comes; if δ is disrupted, constant changes in B 's guesses at a node beyond which there are no splits will prevent B 's guesses from settling down.

At each stage s , on hypothesis δ B constructs the sequence of pre-trees $T_e^{\delta,s}$, $-1 \leq e \leq 2n$, as follows: $T_{-1}^{\delta,s} = \text{Id} \uparrow \text{Str}(s+1)$;

$$T_{2e}^{\delta,s} = \begin{cases} e\text{-Split}_s(T_{2e-1}^{\delta,s}) & \text{if } \delta(e) = 0, \\ \text{Full}(T_{2e-1}^{\delta,s}, \tau_e^{\delta,s}) & \text{if } \delta(e) = 1, \end{cases}$$

where $\tau_e^{\delta,s}$ is the longest τ such that $e\text{-Split}_s(T_{2e-1}^{\delta,s})(\tau)$ is defined, $\subset B$, and has no e -splitting on $T_{2e-1}^{\delta,s}$ after s steps of searching. Let $F(e, \delta, s) = \text{Coded}(B, T_{2e}^{\delta,s})$. $F(e, \delta, s)$ is B 's stage s guess at $A_e \uparrow k$, where $k = \text{dom}(F(e, \delta, s))$, based on hypothesis δ .

$$T_{2e+1}^{\delta,s} = \text{Code}(T_{2e}^{\delta,s}, F(e, \delta, s)).$$

By remarks after the definitions of Code and Coded, this is well-defined.

Let $\text{dom}(\delta) = n+1$. If $T_{2n}^{\delta,s} \neq \emptyset$, for all e with $-1 \leq e \leq 2n$, $T_e^{\delta,s} \neq \emptyset$; let $f^{\delta,s}: n+1 \rightarrow \omega$ be given by:

$$f^{\delta,s}(e) = \begin{cases} 0 & \text{if } \delta(e) = 0 \\ \tau_e^{\delta,s} + 1 & \text{if } \delta(e) = 1. \end{cases}$$

$f^{\delta,s}$ is B 's guess at $f \uparrow n+1$ at stage s , assuming δ . If $T_{2n}^{\delta,s} = \emptyset$, at stage s B hasn't enough information to make a guess. If $\delta \not\subset \delta'$, $T_e^{\delta,s} = T_e^{\delta',s}$ for $e \leq 2n$, and $f^{\delta,s} = f^{\delta',s} \uparrow n+1$.

We now consider the possible behavior of $f^{\delta,s}$ as s increases.

(1) If $\delta \subset f^-$ there is an s such that for all $t \geq s$, $f^{\delta,t}$ is defined, $f^{\delta,t} = f^{\delta,s} = f \uparrow n+1$, $T_e^{\delta,t} = T_e \uparrow \text{Str}(t'_e)$ for $-1 \leq e \leq 2n$, where l'_e is nondecreasing in t and approaches ω for $t \geq s$; furthermore for $t \geq s$, $F(n, \delta, t) \subset A_n$, and so $\bigcup_{t \geq s} F(n, \delta, t) = A_n$. All this follows by induction on n .

(2) If δ is cursed, there is an s such that either (a) for all $t \geq s$, $f^{\delta,t}$ is defined and $f^{\delta,t} = f^{\delta,s}$, or (b) for all $t \geq s$, $f^{\delta,t}$ is undefined. Furthermore, in case (a), for all $t \geq s$, $F(n, \delta, t) = F(n, \delta, s)$. To see this, suppose e curses δ ; by (1) there is a stage s_0 by which $f^{\delta^{e,t}}$ is defined and equal to

$f \uparrow e$ for all $t \geq s_0$; furthermore $T_{2e-1}^{\delta,t} = T_{2e-1} \uparrow \text{Str}(I_{2e-1}^t)$. Fix the least level l such that for some δ with $\text{dom}(\delta) = l$, $e\text{-Split}(T_{2e-1})(\delta)$ is undefined. In building $T_{2e}^{\delta,t}$, B gets stuck at level l ; so eventually B is waiting for e -splittings on $T_{2e-1}^{\delta,t}$ of a string with no such e -splittings. So for some $s_1 \geq s_0$, for all $t \geq s_1$, $T_{2e}^{\delta,t} = T_{2e}^{\delta,s_1}$. Clearly for $-1 \leq j < j' \leq 2n$, $\text{Range}(T_j^{\delta,t}) \subseteq \text{Range}(T_{j'}^{\delta,t})$. So by induction we find s so that for all $j \leq 2n$ and $t \geq s$, $T_j^{\delta,t} = T_j^{\delta,s}$. If $T_j^{\delta,s} = \emptyset$, for $t \geq s$, $f^{\delta,t}$ is undefined. Otherwise $f^{\delta,t}(e) = 0$.

(3) If δ is disrupted and $f^{\delta,s}$ is defined, for some $t > s$ either $f^{\delta,t}$ is undefined or $f^{\delta,t} \neq f^{\delta,s}$. To see this, suppose e disrupts δ and select s_0 as above. Once $t \geq s_0$, $\tau_e^{\delta,t}$ goes to ω with t , since e -splittings for $e\text{-Split}_t(T_{2e-1}^{\delta,t})(\tau_e^{\delta,t}) = e\text{-Split}(T_{2e-1})(\tau_e^{\delta,t})$ eventually turn up on T_{2e-1} , and thus on $T_{2e-1}^{\delta,t'}$ for sufficiently large $t' \geq t$; when this happens, $\tau_e^{\delta,t'} \supseteq \tau_e^{\delta,t}$. Fixing s , for sufficiently large $t \geq s$, if $f^{\delta,t}$ is defined, $f^{\delta,t}(e) > f^{\delta,s}(e)$.

We now view $h \in \omega^{<\omega}$ as a guess at $f \uparrow \text{dom}(h)$. Let $h^-(e) = 0$ if $h(e) = 0$, $h^-(e) = 1$ otherwise. An h -block is a maximal interval $[s_0, s_1] = \{t \mid s_0 \leq t \leq s_1\}$ or $[s_0, \infty] = \{t \mid s_0 \leq t\}$ such that for all s in that interval, $h = f^{h^-,s}$. For any h there are finitely many h -blocks. If $h^- \subset f^-$, this follows from (1); if h^- is cursed, this follows from (2). Note that if $h^- \subset f^-$ or if h^- is cursed and (2a) is true, the final h -block is of the form $[s, \infty]$. If h^- is disrupted by e , this follows from (3) and the previous observation that for sufficiently large t , $\tau_e^{h^-,t}$ increases non-decreasingly with t . If s and t belong to one h -block and $s \leq t$, $F(e, h^-, s) \subset F(e, h^-, t)$ for $-1 \leq e < \text{dom}(h)$. For the moment, assume that $\mathbf{a}_0 = \mathbf{0}$. For $h \in \omega^{<\omega}$, $k \in \omega$ and $\text{dom}(h) = n + 1$, let

$$(g)_{\langle h,k \rangle}(s) = \begin{cases} F(n, h^-, s) + 1 & \text{if } s \text{ belongs to the } k \text{th } h\text{-block;} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $g \leq_T B$. If $h \not\subset f$, or if the k th h -block is not of the form $[s, \infty]$, $(g)_{\langle h,k \rangle}$ differs only finitely from $\lambda s.0$. If $h \subset f$ and the k th h -block is of the form $[s, \infty]$, since $A_n = \bigcup_{t \geq s} F(n, h^-, t)$, $A_n \leq_T (g)_{\langle h,k \rangle}$. Furthermore, $\lambda s.F(n, h^-, s) \leq_T A_0 \oplus \dots \oplus A_n \leq_T A_n$; thus $(g)_{\langle h,k \rangle} \leq_T A_n$. So either $\text{deg}((g)_{\langle h,k \rangle}) = \mathbf{a}_n$ or $\mathbf{0} = \mathbf{a}_0$. Thus g represents I .

Now suppose $\mathbf{a}_0 \neq \mathbf{0}$. Select $D \in \mathbf{a}_0$. Suppose we revised our definition of $(g)_{\langle h,k \rangle}(s)$ by requiring in the “otherwise” case that $(g)_{\langle h,k \rangle}(s) = D(s)$. If $h^- \subset f^-$ and the k th block is of the form $[s_0, \infty]$, we still have $\text{deg}((g)_{\langle h,k \rangle}) = \mathbf{a}_n$; if otherwise and if h^- is not cursed, $\text{deg}((g)_{\langle h,k \rangle}) = \mathbf{a}_0$. But if h^- is cursed and the k th block is of the form $[s_0, \infty]$,

$\text{deg}((g)_{\langle h,k \rangle}) = 0$. To remedy this, we slightly hair-up the definition of $(g)_{\langle h,k \rangle}$:

$$(g)_{\langle h,k \rangle}(2s) = \begin{cases} F(h, h^-, s) + 1 & \text{if } s \text{ belongs to the } k \text{ th } h\text{-block.} \\ D(s) & \text{otherwise} \end{cases}$$

$$(g)_{\langle h,k \rangle}(2s + 1) = D(s).$$

g is now as desired.

COROLLARY. *If I is a countable ideal, some minimal upper bound on I weakly represent I .*

Proof. There is an $I' \subseteq I$ cofinal in I and linearly ordered; apply *Theorem 1* to I' and notice that a minimal upper bound on I' is also one for I .

Questions. Does every ideal have a representing minimal upper bound?

Does a sequence $\langle a_i \rangle_{i < \omega}$ as above have a minimal upper bound which does not represent it?

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