TOPOLOGICAL EXTENSIONS OF PRODUCT SPACES

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The Stone-Čech compactification $\beta X$ and the Hewitt real-compactification $\nu X$ [6] of a completely regular $T_1$-space $X$ can be obtained as certain spaces of ultrafilters from the collection of zero sets of members of $C^*(X)$ [4]. With the appropriate structure $\beta X$ is the space of all ultrafilters and $\nu X$ those with the countable intersection property. In this framework we give a necessary and sufficient condition for $\beta X \times \beta Y \approx \beta(X \times Y)$.

Glicksberg [5], and then Frolik [3], established for infinite spaces $X$ and $Y$ that $\beta X \times \beta Y \approx \beta(X \times Y)$ if and only if $X \times Y$ is pseudocompact. Our condition is in terms of the zero sets of $X \times Y$ and we do not insist that $X$ and $Y$ be infinite. This result extends to arbitrary products. We give some sufficient conditions for $\nu X \times \nu Y \approx \nu(X \times Y)$ and in case $\nu X \times \nu Y$ (or $\nu(X \times Y)$) is Lindelöf give a condition that is both sufficient and necessary.

1. For $Z$ a normal base [2] for the closed sets of $X$ and $F \in Z$ define $F^* \equiv \{\text{ultrafilters from } Z \text{ that contain } F\}$. $\{F^*: F \in Z\}$ is a base for the closed sets of the ultrafilter space $\omega(Z)$ which is a Hausdorff compactification of $X$. The normality property of $Z$ is not needed to construct the $T_1$-compact space $\omega(Z)$. However, $\omega(Z)$ is a Hausdorff space if and only if $Z$ is a normal family. If $Z$ is the zero sets from $X$ then $\omega(Z) \approx \beta X$. Extensions of this kind are called Wallman-type. Say a base $Z_1$ separates a base $Z_2$ if disjoint members of $Z_1$ are contained in disjoint members of $Z_2$.

**Theorem 1.1.** Let $Z_1 \subset Z_2$ be normal bases for $X$. Then $\omega(Z_1) \approx \omega(Z_2)$ if and only if $Z_1$ separates $Z_2$.

Let $Z_1$ and $Z_2$ be normal bases for the closed sets of $X$ and $Y$.

**Theorem 1.2.** $\omega(Z_1) \times \omega(Z_2)$ is a Wallman-type compactification of $X \times Y$.

**Proof** (Sketch). Let $Z_1 \times Z_2 = \{F \times G: F \in Z_1, G \in Z_2\}$ and $Z_1 \times Z_{2_2}$ be all finite unions from $Z_1 \times Z_2$. $Z_1 \times Z_{2_2}$ is the needed normal base, i.e., $\omega(Z_1) \times \omega(Z_2) \approx \omega(Z_1 \times Z_{2_2})$. The mapping $(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{A} \times \mathcal{B}$
is one-one from $\omega(Z_1) \times \omega(Z_2)$ onto the ultrafilters from $Z_1 \times Z_2$ which are in one-one correspondence with those from $Z_1 \times Z_{2_\omega}$. We take $(\mathcal{U}, \mathcal{B})$ → the ultrafilter from $Z_1 \times Z_{2_\omega}$ that contains $\mathcal{U} \times \mathcal{B}$. This is a homeomorphism. $\beta X \times \beta Y$ is, then, a Wallman-compactification of $X \times Y$.

Let $Z_1$ be the zero sets from $X$ and $Z_2$ those from $Y$. Denote the zero sets from $X \times Y$ by $Z(X \times Y)$. It is evident that $Z_1 \times Z_{2_\omega} \subset Z(X \times Y)$.

Our main result is

**Theorem 1.3.** $\beta X \times \beta Y \approx \beta(X \times Y)$ if and only if $Z_1 \times Z_{2_\omega}$ separates the zero sets of $X \times Y$.

**Proof.** Assume that $Z_1 \times Z_{2_\omega}$ separates $Z(X \times Y)$. By Theorem 1.1, $\omega(Z_1 \times Z_{2_\omega}) \approx \beta(X \times Y)$. Using Theorem 1.2 we have $\beta X \times \beta Y \approx \beta(X \times Y)$.

If $\beta X \times \beta Y \approx \beta(X \times Y)$ then Theorem 1.2 implies that $\omega(Z_1 \times Z_{2_\omega}) \approx \omega(Z(X \times Y))$ and by Theorem 1.1, $Z_1 \times Z_{2_\omega}$ separates $Z(X \times Y)$.

Let $N$ be the positive integers with the discrete topology. In $N \times N$, $F_1 =$ all points below the diagonal and $F_2 =$ all points above the diagonal belong to $Z(N \times N)$ but cannot be separated by $Z_1 \times Z_{2_\omega}$. In $R \times R$, where $R$ is the real line, $Z_1 \times Z_{2_\omega}$ fails to separate the $y$-axis and $y = 1/x$.

**Remark.** From Theorem 1.3 and Theorem 1 of [5] it is seen that, for $X$ and $Y$ infinite spaces, $X \times Y$ is pseudocompact if and only if $Z_1 \times Z_{2_\omega}$ separates $Z(X \times Y)$.

Let $\{X_\alpha\}$ be a collection of completely regular $T_1$-spaces and $Z_\alpha$ the zero sets from $X_\alpha$.

**Theorem 1.4.** $\prod \beta X_\alpha$ is a Wallman compactification of $\prod X_\alpha$.

**Proof (Sketch).** Let $\prod Z_\alpha \equiv \{\prod F_\alpha: F_\alpha \in Z_\alpha, F_\alpha = X_\alpha$ for all but finitely many $\alpha\}$ and $Z$ be all finite unions from $\prod Z_\alpha$. $Z$ has sufficient properties to construct the compact $T_1$-space $\omega(Z)$. We show $\prod \beta X_\alpha \approx \omega(Z)$ and it follows that $\omega(Z)$ is a Hausdorff space and that $Z$ is a normal base.

**Remark.** The Tychonoff Product Theorem can be obtained as a corollary to Theorem 1.4. In this case $\beta X_\alpha \approx X_\alpha$ and the homeomorphism gives $\prod X_\alpha$ compact.

Let $Z$ be as above. Using Theorems 1.1 and 1.4 we arrive at an extension of our main result.
**Theorem 1.5.** \( \prod \beta X_a \approx \beta(\prod X_a) \) if and only if \( Z \) separates the zero sets of \( \prod X_a \).

2. For \( Z \) a normal base for \( X \) let \( p(Z) \) be the subspace of \( \omega(Z) \) consisting of those points that have the countable intersection property (C.I.P.). \( p(Z) \) is called a real-extension of \( X \). Again, if \( Z \) is the zero sets from \( X \) then \( p(X) \approx vX \). If \( Z \) is a normal base, the family of countable intersections from \( Z \), denoted \( Z_\cap \), is a normal base and \( p(Z) \approx p(Z_\cap) \). Although \( Z_\cap \) may introduce "new" ultrafilters none of these will have the C.I.P. e.g. \( Z = \{ F \subset N : F \text{ or } N \setminus F \text{ is finite} \} \). \( Z_\cap \) is all subsets of \( N \) and \( \omega(Z_\cap) \approx \beta N \). \( \omega(Z) \) is the one-point compactification of \( N \). Clearly \( \omega(Z_\cap) \neq \omega(Z) \) yet \( p(Z_\cap) \approx N \approx p(Z) \).

**Theorem 2.1.** Let \( Z_1 \subset Z_2 \) be normal bases for \( X \) each closed under formation of countable intersections. In case \( p(Z_2) \) (or \( p(Z_1) \)) is Lindelöf it follows that \( p(Z_1) \approx p(Z_2) \) if and only if \( Z_1 \) separates \( Z_2 \).

**Remark.** We insist on the Lindelöf property to show the condition is necessary.

Let \( Z_1 \) and \( Z_2 \) be normal bases for \( X \) and \( Y \).

**Theorem 2.2.** \( p(Z_1) \times p(Z_2) \) is a real extension of \( X \times Y \).

*Proof (Sketch).* \( \omega(Z_1) \times \omega(Z_2) \approx \omega(Z_1 \times Z_2) \) by Theorem 1.2. Under the mapping the image of \( (\emptyset, \emptyset) \) has the C.I.P. if and only if both \( \emptyset \) and \( \emptyset \) do. Therefore \( p(Z_1) \times p(Z_2) \approx p(Z_1 \times Z_2) \).

Let \( Z_1, Z_2 \) be the zero sets of \( X, Y \).

**Theorem 2.3.** If \( Z_1 \times Z_2 \) separates \( Z(X \times Y) \) then \( vX \times vY \approx v(X \times Y) \).

*Proof. \( \omega(Z_1 \times Z_2) \approx \omega(Z(X \times Y)) \) by Theorem 1.1. If follows that \( p(Z_1 \times Z_2) \approx v(X \times Y) \). We have \( vX \times vY \approx v(X \times Y) \) from Theorem 2.2.*

**Theorem 2.4.** Assume that \( vX \times vY \) (or \( v(X \times Y) \)) is Lindelöf. Then \( vX \times vY \approx v(X \times Y) \) if and only if \( Z_1 \times Z_2 \) separates \( Z(X \times Y) \).

*Proof. Note that \( Z_1 \times Z_2 \subset Z(X \times Y) \). Theorems 2.1 and 2.2 establish sufficiency.*
If $vX \times vY \approx v(X \times Y)$ then $p(Z_1 \times Z_{2\Sigma_n}) \approx p(Z(X \times Y))$ by Theorem 2.2 and the remarks preceding Theorem 2.1. From Theorem 2.1 we have $Z_1 \times Z_{2\Sigma_n}$ separates $Z(X \times Y)$.

There certainly are spaces $X, Y$ with $vX \times vY$ Lindelöf and $vX \times vY \approx (X \times Y)$. Take a pseudocompact space $X$ [4] with $X \times X$ not pseudocompact. $vX \times vX$ is compact, hence Lindelöf. However $v(X \times X)$ is not compact.

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